SENSING AND ESTIMATION

- Obviously various sensing options
  
  Position - GPS, metrology, vision
  
  Velocity - geophone
  
  Accel - accelerometer
  
  Attitude - magnetic compass, gyroscope, horizon sensor
  
  Att rate - rate gyro

- For each type there are many options
  
  - Inertial gyros
  
  - Interferometric gyros
  
  - Ring laser gyros

  => Broad spectrum of costs and capabilities

  => Study the spec sheets well to

  make sure that it meets all criteria
  with exceeding by too much.
TYPICAL ISSUES

- **LINEARITY** - NL SENSORS IMPACT FEEDBACK GAIN → COULD BE A PROBLEM

- **BIAS** - OUTPUT WHEN INPUT IS ZERO
  - TYPICAL VARIES SLOWLY IN INERTIAL SENSORS ⇒ DRIFT
  - CAN BE CALIBRATED OUT
  - BIG PROBLEM WHEN INTEGRATED.

- **ACCURACY** - OR INACCURACY
  - MEASURE OF A DEVIATION FROM TRUE OUTPUT VALUE
  - INCLUDES ALL SOURCES OF ERROR
  - IF SYSTEMATIC, CAN CALIBRATE OUT, BUT NEED AN ACCURATE TRUTH

- **BANDWIDTH** - MOST PHYSICAL DEVICES HAVE A LIMIT TO THE FREQUENCY AT WHICH THEY CAN PROVIDE GOOD MEASUREMENTS

VISION: LIMITED BY PROCESSOR SPEED

ACCEL: ACCELERATION \( \ddot{y} \) ⇒ SENSOR OUTPUT \( y \)

\[
\frac{\ddot{y}}{y} = \frac{1}{ms^2 + ds + k}
\]
**COMPLEMENTARY FILTERING**

- Typical sensing problem: have two separate measurements $y_1$ and $y_2$ of the variable $x$.

\[
y_1 = x + n_1 \\
y_2 = x + n_2
\]

Key point is that $n_1$ and $n_2$ have different properties.

E.g., could have $n_1$ narrowband, $n_2$ wideband.

- How combine $y_1$ and $y_2$ to obtain the best "supersensor"?

\[
x \quad \xrightarrow{n_1} \quad y_1 \quad \xrightarrow{F_1(s)} \quad y \\
\quad \xrightarrow{n_2} \quad y_2 \quad \xrightarrow{F_2(s)} \quad y
\]

Choose $F_1(s) + F_2(s) = 1$.

\[
\lim_{s \to 0} F_1(s) = 0 \quad \Rightarrow \quad F_1 \sim \text{high pass} \\
\lim_{s \to 0} F_2(s) = 0 \quad \Rightarrow \quad F_2 \sim \text{low pass}
\]
This gives
\[ y = f_1(x + n_1) + f_2(x + n_2) = (f_1 + f_2) x + f_1 n_1 + f_2 n_2 \]
\[ \therefore y = x + (1 - f_2) n_1 + f_2 n_2 \]

• If \( n_1 \) is a low freq. narrow band signal
  it is attenuated by \((1 - f_2)\) [HP]

• and \( n_2 \) attenuated at high frequency
  by \( f_2 \) [LP]

 Called **complementary filtering** or **Aiding**.

⇒ key is choosing the cross-over frequency of when to switch from
\( y_1 \) to \( y_2 \).
CAN EXTEND THIS TO THE CASE WITH

A COMBINATION OF A POSITION AND

VELOCITY SENSOR

\[ y_1 = x + n_1 \]
\[ y_2 = x + n_2 \quad \Rightarrow \quad y_3 = \int y_2 = x + \frac{1}{3} n_2 \]

SO NOW THE SENSORS HAVE VERY DIFFERENT

NOISE SPECTRA

\[ |n_1| \quad \text{and} \quad |n_2| \]

\[ \frac{\omega}{3} \quad \text{and} \quad \omega \]

\( \Rightarrow \) SHOULD USE \( y_1 \) AT LOW FREQUENCIES, SWITCHING

OVER TO \( y_2 \)

\[ y = F_1(x + n_1) + F_2(x + n_2) \]

\[ = (F_1 + SF_2)x + F_1n_1 + F_2n_2 \]

WITH \( F_1 + SF_2 = 1 \)

SET \( F_1 \sim \text{LP} \Rightarrow F_1 = \frac{x}{S+\alpha} \Rightarrow F_2 = \frac{1}{S}(1-F_1) \]

\[ = \frac{1}{S+\alpha} \]

\[ \therefore F_2n_2 = \frac{S}{S+\alpha} \left( \frac{\omega}{3} \right) \Rightarrow \text{HP ON LF NOISE} \]

\[ = \frac{1}{S} \frac{S}{\alpha+\alpha} \]
* Given the measurements, can then focus on extracting the information ⇒ filtering problem.

- **Simplest form of filtering**: average.
  - Multiple measures of \( x \Rightarrow \hat{y}_i = x + v_i \)
  - \( \hat{y}_i = x + v_i \Rightarrow y = Hx + v \) (\( M \) times)
  - \( H = [1 \ 1 \ldots 1]^T \)

"Given \( y \) solve for \( x \)" ⇒ least-squares problem

\[
\hat{x} = (H^TH)^{-1}H^Ty \quad \text{[Ignore} \ v] \]

But \( H^TH = M \)

\( H^Ty = \sum_i y_i \)

\[
x = \frac{1}{M} \sum_i y_i \]

- Static cases are interesting, but typically we are trying to estimate a dynamic variable - e.g. the state of your UAV

⇒ model \( \dot{x} = Ax + Bu \)

Want to estimate \( x \) at \( t \) ⇒ \( \hat{x}(t) \).
Estimation Schemes

1. Knowing the plant matrices and the inputs, we could just perform a (digital or analog) simulation

\[ \dot{x} = A \hat{x} + Bu \]

Then \( \dot{x}(t) = x(t) \) provided \( \dot{x}(0) = x_0 \) for all \( t \)

- But we do not know \( x_0 \)!!
- Process sensitive to knowledge of \( A, B \)

Rightarrow called an **open-loop estimator**.

**Analysis of this case:**

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
\dot{\hat{x}} &= A\hat{x} + Bu \\
\text{ERROR} \quad \tilde{x} &= x - \hat{x} \\
\text{SUBTRACT:} \quad (x - \hat{x}) &= A(x - \hat{x}) \\
\text{or} \quad \dot{\tilde{x}} &= A\tilde{x} \quad \Rightarrow \quad \tilde{x}(t) = e^{At}\tilde{x}(0)
\end{align*} \]
\[ \text{ESTIMATION ERROR } \hat{x}(t) \text{ IN TERMS OF} \]
\[ \text{INITIAL ERROR } \hat{x}(0). \]

- **DOES THIS GUARANTEE** \( \hat{x}(t) = 0 \ \forall \ t? \)
  - OR EVEN \( \hat{x}(t) \to 0 \ \text{AS} \ t \to \infty? \)

  - **O.K. IF** \( \hat{x}(0) = 0 \)
  - **WHAT IF** \( \hat{x}(0) \neq 0 \)
    - A STABLE? 
    - A UNSTABLE? \( \{ \text{YIKES} \} \)

- **IF A STABLE, \( \hat{x}(t) \to 0 \), BUT THE DYNAMICS OF THE**
  - **ESTIMATION ERROR ARE THE SAME AS THE**
  - **ORIGINAL SYSTEM (A, "OPEN-LOOP")**
    - COULD BE SLOW
    - NO OBVIOUS WAY TO MODIFY THE
      - **ESTIMATION ERROR DYNAMICS.**

2. **USE ADDITIONAL INFORMATION, WHICH IS**
   - **HOW WELL OUR ESTIMATED OUTPUT TRACKS THE MEASURED OUTPUT**
     \[ \hat{y} = C \hat{x}, \quad \hat{y} = y - \hat{y}, \quad y = C x = C \hat{x} \]
   - **FEEDBACK \( \tilde{y} \) TO IMPROVE OUR ESTIMATE.**
• SELECTED FORM:
  \[ \dot{x} = A \dot{x} + Bu + L \tilde{y} \]
  SELECTABLE GAIN L

SETUP:

![System diagram]

⇒ USE KNOWLEDGE OF MEASURABLE DIFFERENCE IN THE OUTPUTS (\(\tilde{y} \neq 0\)) TO "CLOSE THE LOOP" ON THE ESTIMATOR.

• ANALYSIS:
  \[ \dot{\hat{x}} = \dot{x} - \dot{\hat{x}} = \left[ A \dot{x} + Bu \right] - \left[ A \dot{\hat{x}} + Bu + L(y - \tilde{y}) \right] \]
  \[ = A(x - \hat{x}) - L(Cx - C\hat{x}) \]
  \[ = A \tilde{x} - LC(x - \hat{x}) = (A - LC)\tilde{x} \]

⇒ NEW ESTIMATION ERROR DYNAMICS
  \[ \dot{\tilde{x}} = (A - LC)\tilde{x} \]
  CLOSED-LOOP ESTIMATOR

\[ \tilde{x}(t) = e^{(A - LC)t}\tilde{x}(0) \]
ESTIMATION ERROR DYNAMICS ARE GOVERNED BY THE MATRIX \((A-\text{LC})\) - CAN CHOOSE \(L\) TO MAKE \(\dot{x} \rightarrow 0\) QUICKLY, OR CAN \(K\)?

NOTE SIMILARITY

1. **REGULATOR PROBLEM**: PICK \(K\) FOR \(A-BK\)
2. **ESTIMATION PROBLEM**: PICK \(L\) FOR \(A-\text{LC}\)

1. **CHOOSE** \(K \in \mathbb{R}^{nxn}\) (SISO) SUCH THAT
   \[
   \det(sI - A + BK) = \chi_c(s) \quad \text{DESIRED LOCATION OF REGULATOR POLES}
   \]

2. **CHOOSE** \(L \in \mathbb{R}^{nx1}\) (SISO) SUCH THAT
   \[
   \det(sI - A + LC) = \chi_0(s)
   \]

**THESE TWO PROBLEMS ARE DUAL**
FREQUENCY DOMAIN INTERPRETATION

- ESTIMATION ERROR DYNAMICS
  \[
  \dot{x} = Ax + Bu u + Bw w \\
  y = Cy x + v
  \]
  \[
  w \sim N(0, R_w) \quad B_w R_w B_w^T > 0 \quad [A, Cy] \text{ observable}
  \]
  \[
  v \sim N(0, R_v) \quad R_v > 0
  \]
  
  \(\Rightarrow\) STRONG FORM OF ASSUMPTIONS
  
  \(\Rightarrow\) STEADY STATE KALMAN FILTER
  \[
  \dot{\hat{x}} = A \hat{x} + Bu u + L(y - Cy \hat{x})
  \]

- GIVING ESTIMATION ERROR DYNAMICS:
  \[
  \dot{\hat{x}} = \dot{x} - \hat{x} = Ax + Bw u + Bw w - \left\{ (A - LCy) \hat{x} + Bu u + L(Cy \hat{x}) \right\}
  \]
  \[
  = (A - LCy) \hat{x} + Bw u + L v
  \]

  1) FILTER STABILITY GOVERNED BY \(\lambda; (A - LCy)\)

  2) EQUATION MAKES EXPLICIT THE CONFLICT BETWEEN
  
  (A) SPEED OF ESTIMATOR ERROR DECAY \(L \uparrow \text{so } \lambda; (A - LC) \text{ far into LHP}\)

  (B) SUSCEPTIBILITY OF ESTIMATION ERROR
  
  BEING CORRUPTED BY SENSOR NOISE.

- KALMAN FILTER SELCTS THE OPTIMAL BALANCE BETWEEN THESE TWO GOALS.
- Have \( \dot{x} = A \dot{x} + L(y - c_v \hat{x}) \) - steady state filter

Laplace TF both sides

\[ \Rightarrow \frac{\dot{x}(s)}{Y(s)} = \frac{L}{sI - A + LC_v} \]

- This is the transfer function of the filter applied to the measurements to form the estimate \( \hat{x} \). (Low pass)

- Increasing \( L \) pushes filter TF up/out

\[ \begin{array}{c|c}
\text{Increasing } L \\
\hline
\|x\| & f \\
\end{array} \]

\[ \Rightarrow \text{Eventually estimate would be too corrupted by the noise in the measurements} \]

- Note that balancing of sensor noise impact done with respect to process noise (\(BW\)).

\[ \Rightarrow \text{Turns out that ratio } \frac{R_w}{R_v} \text{ plays a key role in selection of } L. \]