Online Appendix of "Do Dark Pools Harm Price Discovery?"

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1 Institutional Features of Dark Pools

In this section, I discuss institutional details of dark pools and dark liquidity in addition to that in Zhu (2013).

Dark pools differ from each other in many ways. We can categorize them, roughly, into the three groups shown in the top panel of Table 1. As described in Zhu (2013), the first group match customer orders by acting as agents (as opposed to trading on their own accounts). Transaction prices are derived from lit venues, such as the midpoint of the National Best Bid and Offer (NBBO) and the volume-weighted average price (VWAP).

Within the second group, dark pools operate as continuous nondisplayed limit order books, accepting market, limit, or "pegged" orders.[1] This group includes many of the dark pools owned by major broker-dealers, including Credit Suisse Crossfinder, Goldman Sachs Sigma X, Citi Match, Barclays LX, Morgan Stanley MS Pool, and UBS PIN. Unlike Group-1 dark pools that execute orders at the market midpoint or VWAP, Group-2 dark pools derive their own execution prices from the limit prices of submitted orders. Price discovery can therefore take place. Another difference is that Group-2 dark pools may contain orders from proprietary trading desks of the broker-dealers that operate the dark pools. In this sense, these dark pools are not necessarily "agency only."

Dark pools in the third group act like fast electronic market makers that immediately accept or reject incoming orders. Examples include Getco and Knight. Like the second group, transaction

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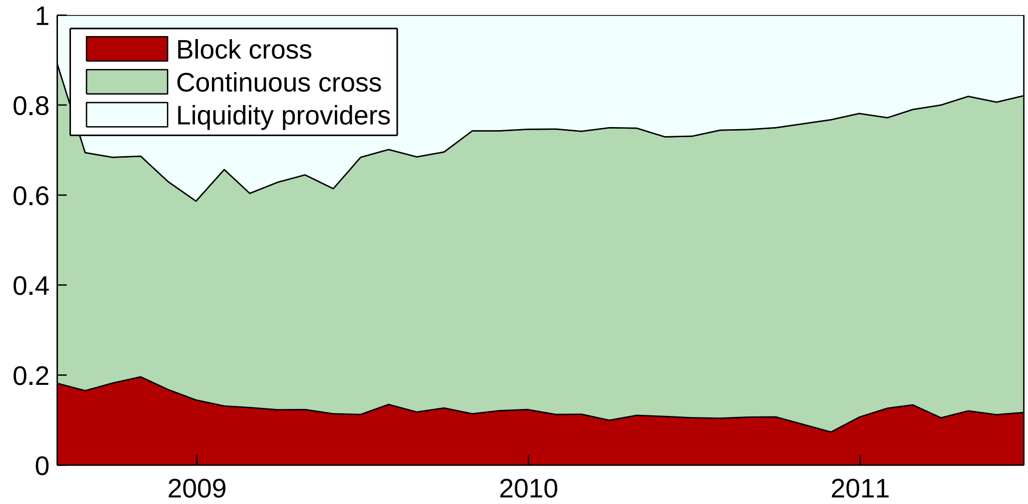
[1] Pegged orders are limit orders with the limit price set relative to an observable market price, such as the bid, the offer, or the midpoint. As the market moves, the limit price of a pegged order moves accordingly.

Figure 1: Market shares of three types of U.S. dark pools as fractions of total U.S. dark pool volume, estimated by Tabb Group. The three types are summarized in the middle panel of Table 1.



prices on these platforms are not necessarily calculated from the national best bid and offer using a transparent rule. In contrast with dark pools in Groups 1 and 2, Group-3 dark pools typically trade on their own accounts as principals (as opposed to agents or marketplaces).

Another classification of dark pools is provided by Tabb Group (2011). They categorize dark pools into block-cross platforms, continuous-cross platforms, and liquidity-provider platforms. The main features of these three groups are summarized in the middle panel of Table 1, and their respective market shares are plotted in Figure 1. As we can see, the market share of block-cross dark pools has declined from nearly 20% in 2008 to just above 10% in 2011. Continuous-cross dark pools have gained market share during the same period, from around 50% to around 70%. The market share of liquidity-providing dark pools increased to about 40% around 2009, but then declined to about 20% in mid 2011. Tabb Group's data, however, do not cover the entire universe of dark pools, and the components of each category can vary over time. For this reason, these statistics are noisy and should be interpreted with caution.

Dark pools are also commonly classified by their crossing frequencies and by how they search for matching counterparties, as illustrated in the bottom panel of Table 1. Aside from mechanisms such as midpoint-matching and limit order books, advertisement is sometimes used to send selected information about orders resting in the dark pool to potential counterparties, in order to facilitate a match.

Characteristics that distinguish dark pools also include ownership structure and order size. Today, most dark pools are owned by broker-dealers (with or without proprietary order flows), whereas a small fraction is owned by consortiums of broker-dealers or exchanges. Order sizes can

Table 1: Dark pools classification. The top panel shows the classification by trading mechanisms. The middle panel shows the classification by Tabb Group. The bottom panel shows a classification by the crossing frequencies and the methods of finding counterparties.

Classification by trading mechanism

Types	Examples	Typical features
Matching at exchange prices	ITG Posit, Liquidnet, Instinet	Mostly owned by agency brokers and exchanges; typically execute orders at midpoint or VWAP, and customer-to-customer
Nondisplayed limit order books	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Citi Match, Barclays LX, Morgan Stanley MS Pool, UBS PIN	Most broker-dealer dark pools; may offer some price discovery and contain proprietary order flow
Electronic market makers	Getco and Knight	High-speed systems handling immediate-or-cancel orders; typically trade as principal

Classification by Tabb Group

Types	Examples	Notes
Block cross	Liquidnet, BIDS, Instinet Cross	Similar to the first group of the top panel
Continuous cross	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Barclays LX, Morgan Stanley MS Pool, LeveL, Deutsche Bank SuperX	Similar to the second group of the top panel; LeveL is owned by a consortium of broker-dealers
Liquidity provider	Getco and Knight	Same as the third group of the top panel

Classification by trading frequency and counterparty search

Types		Examples	Typical features
Scheduled		ITG POSIT Match, Instinet US Crossing	Cross at fixed clock times, with some randomization
Continuous	Matching	ITG POSIT Now, Instinet CBX, Direct Edge MidPoint Match	
	Advertized	POSIT Alert, Liquidnet	Electronic messages sent to potential matched counterparties
	Negotiated	Liquidnet	
	Internal	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Knight Link, Getco	Owned by broker-dealers, run as nondisplayed limit order books or electronic market makers

also vary substantially across dark pools. According to Rosenblatt Securities (2011), Liquidnet has an order size of around 50,000 shares, and Posit has an average transaction size of 6,000 shares. Most broker dark pools have an average transaction size of about 300 shares. This sharp contrast in order sizes can be attributed to the use of algorithms that split "parent" orders into smaller "children" orders, as observed by the Securities and Exchange Commission (2010).

There are at least two reasons why high-quality data are lacking on dark pool trading in the United States. First, in the United States, dark pool trades are reported to "trade reporting facilities," or TRFs, which aggregate trades executed by all off-exchange venues—including dark pools, ECNs, and broker-dealer internalization—into a single category. Thus, it is generally not possible to assign a TRF trade to a specific off-exchange venue that executes the trade.[2] Second, dark pools often do not have their own identification numbers (MPID) for trade reporting. For example, a broker-dealer may report customer-to-customer trades in its dark pool *together with* the broker's own over-the-counter trades with institutions, all under the same MPID. Similarly, trades in an exchange-owned dark pool can be reported together with trades conducted on the exchange's open limit order book, all under the exchange's MPID. Because different trading mechanisms share the same MPID, knowing the MPID that executes a trade is insufficient to determine whether that trade occurred in a dark pool.[3]

Finally, there are two sources of nondisplayed liquidity that are usually not referred to as dark pools. One is broker-dealer internalization, by which a broker-dealer handles customer orders as a principal or an agent (Securities and Exchange Commission, 2010). A crude way of distinguishing dark pools from broker-dealer internalization is that the former are often marketplaces that allow direct customer-to-customer trades, whereas the latter typically involves broker-dealers as intermediaries.[4] The other source of nondisplayed liquidity is the use of hidden orders on exchanges. Examples include reserve ("iceberg") orders and pegged orders, which are limit orders that are partially or fully hidden from the public view.[5] For example, Nasdaq reports that more than 15% of its order flow is nondisplayed.[6] In particular, midpoint-pegged orders on exchanges are similar to dark pool orders waiting to be matched at the midpoint.

[2] The Securities and Exchange Commission (2009) has recently proposed a rule requiring that alternative trading systems (ATS), including dark pools, provide real-time disclosure of their identities on their trade reports.

[3] For example, Ye (2010) finds that only eight U.S. dark pools can be uniquely identified by MPIDs from their Rule 605 reports to the SEC. The majority of dark pools cannot.

[4] There are exceptions. For example, dark pools acting like electronic market makers (like Getco and Knight) also provide liquidity by trading on their own accounts. Nonetheless, they are highly automated systems and rely less on human intervention than, say, dealers arranging trades over the telephone.

[5] A reserve order consists of a displayed part, say 200 shares, and a hidden part, say 1,800 shares. Once the displayed part is executed, the same amount, taken from the hidden part, becomes displayed, until the entire order is executed or canceled. Pegged orders are often fully hidden. Typically, pegged orders and hidden portions of reserve orders have lower execution priority than displayed orders with the same limit price.

[6] See http://www.nasdaqtrader.com/Trader.aspx?id=DarkLiquidity

2 An Example with Binomial Distribution of Delay Costs

In the current model of Zhu (2013), the delay costs of liquidity traders have a continuous distribution. This implies that once the dark pool is added, a positive fraction of liquidity traders move to the dark pool. Indeed, this migration of liquidity traders is the key reason why adding a dark pool can improve price discovery. As natural as this intuition is, one may still ask: what happens if this key channel, i.e. the migration of liquidity traders to the dark pool, is shut down? Obviously, if the introduction of a dark pool only diverts informed traders off the exchange but leaves liquidity traders where they were, then adding a dark pool can harm price discovery. The key question is whether this situation is incentive-compatible. This appendix provides such a "corner solution" example using a discontinuous distribution of delay costs.

The main purposes of this example is to qualify and sharpen my theory by clarifying the limits of the self-selection mechanism. Despite this caveat, to the extent that evidence from recent empirical studies is consistent with the model of Zhu (2013), the self-selection mechanism is relevant and important in reality.

The construction of the example

Suppose that the probability distribution of liquidity traders' types is binomial, high (H) and low (L). Specifically, with probability α_H the type is γ_H, and with probability $\alpha_L = 1 - \alpha_H$ the type is γ_L, where $0 < \gamma_L < \gamma_H$. All other features of the model are the same as the main model of Zhu (2013). I look for an equilibrium that has the following properties:

- With a dark pool, informed traders split between the two venues, all liquidity traders with low delay costs use the dark pool, and all liquidity traders with high delay costs use the exchange.
- Without a dark pool, all liquidity traders with low delay costs delay trading, whereas informed traders and liquidity traders with high delay costs use the exchange.

At a technical level, constructing this example is almost equivalent to "endogenizing" the key assumption of Ye (2011) that liquidity traders are not allowed to choose where to trade. That assumption underlies Ye's result that adding a dark pool harms price discovery.

As before and without loss of generality, we consider the choices of an informed buyer and a liquidity buyer. With a dark pool, the exchange spread is given by

$$S = \frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + \alpha_H \mu_z}\sigma. \tag{1}$$

The dark-pool crossing probabilities on the informed side and the opposite side are, respectively,

$$r^- = \mathbb{E}\left[\min\left(1, \frac{\alpha_L Z^-}{\alpha_L Z^+ + \beta\mu_I}\right)\right], \tag{2}$$

$$r^+ = \mathbb{E}\left[\min\left(1, \frac{\alpha_L Z^- + \beta\mu_I}{\alpha_L Z^+}\right)\right]. \tag{3}$$

Since informed traders split between the two venues, they must be indifferent, that is,

$$r^- = 1 - \frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + \alpha_H\mu_z}. \tag{4}$$

Recall that the expected cost of a liquidity trader of type $\gamma \in \{\gamma_H, \gamma_L\}$ in the dark pool is

$$\frac{r^+ - r^-}{2}\sigma + \gamma\sigma\left(1 - \frac{r^+ + r^-}{2}\right). \tag{5}$$

For type-L liquidity traders to prefer the dark pool to the exchange and to delaying trade, we must have

$$\frac{r^+ - r^-}{2}\sigma + \gamma_L\sigma\left(1 - \frac{r^+ + r^-}{2}\right) \leq \min\left(\gamma_L\sigma, \frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + \alpha_H\mu_z}\sigma\right) = \min(\gamma_L\sigma, (1-r^-)\sigma), \tag{6}$$

where the last equality follows from (4). This inequality reduces to

$$\frac{r^+ - r^-}{r^+ + r^-} \leq \gamma_L \leq 1. \tag{7}$$

For type-H liquidity traders to prefer the exchange to the dark pool and to delaying trade, we must have

$$\frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + \alpha_H\mu_z}\sigma \leq \min\left(\frac{r^+ - r^-}{2}\sigma + \gamma_H\sigma\left(1 - \frac{r^+ + r^-}{2}\right), \gamma_H\sigma.\right). \tag{8}$$

Using (4) again, we can reduce the above inequality to

$$\gamma_H \geq 1. \tag{9}$$

Lastly, the mass of informed traders satisfies:

$$\mu_I = \bar{\mu}F\left(\frac{\alpha_H\mu_z}{(1-\beta)\mu_I + \alpha_H\mu_z}\sigma\right). \tag{10}$$

The above calculations lead to the following proposition.

Proposition 1. *Let β^* and μ_I^* be the solutions to equations (4) and (10). If the incentive-compatibility conditions (7) and (9) hold at the solution (β^*, μ_I^*), then there exists an equilibrium in which, in period 1, (i) a fraction β (resp. $1-\beta$) of informed traders send orders to the dark pool (resp. the exchange), (ii) all type-L liquidity traders send orders to the dark pool, and (iii) all type-H liquidity traders send orders to the exchange immediately.*

Proof. It directly follows from the calculations proceeding the proposition. □

We can characterize a similar equilibrium in a market with only an exchange. Without a dark pool, all informed traders use the exchange. For L-type liquidity traders to strictly prefer delaying trade to using the exchange, we must have

$$\frac{\mu_I}{\mu_I + \alpha_H \mu_z}\sigma \geq \gamma_L \sigma \Longrightarrow \gamma_L \leq \frac{\mu_I}{\mu_I + \alpha_H \mu_z}. \tag{11}$$

For H-type liquidity traders to strictly prefer using the exchange to delaying trades, we must have

$$\frac{\mu_I}{\mu_I + \alpha_H \mu_z}\sigma \leq \gamma_H \sigma \Longrightarrow \gamma_H \geq \frac{\mu_I}{\mu_I + \alpha_H \mu_z}. \tag{12}$$

Lastly, the mass of informed traders satisfies:

$$\mu_I = \bar{\mu} F\left(\frac{\alpha_H \mu_z}{\mu_I + \alpha_H \mu_z}\sigma\right). \tag{13}$$

Corollary 1. *Let μ_I^* be the solutions to equation (13). If the incentive-compatibility conditions (11) and (12) hold at the solution μ_I^*, then there exists an equilibrium in which, in period 1, (i) all informed traders use the exchange, (ii) all type-L liquidity traders delay trading, and (iii) all type-H liquidity traders send orders to the exchange immediately.*

After deriving Proposition 1 and Corollary 1, I now attempt to construct an example in which adding a dark pool harms price discovery. Since my objective is not to provide general characterization, it is helpful to fix some parameters and simplify the incentive conditions. In particular, we can take a sufficiently large $\gamma_H > 1$. We can also take F such that $F(c) \to 1$ for all $c > 0$, that is, information acquisition is almost free. Free information acquisition implies $\mu_I = \bar{\mu}$, a constant. Then, adding a dark pool harms price discovery if two conditions hold:

- The dark pool attracts some informed traders, i.e., $\beta > 0$ in Proposition 1. A sufficient condition is that
$$\bar{r} \equiv \mathbb{E}\left[\min\left(1, \frac{Z^-}{Z^+}\right)\right] > \frac{\alpha_H \mu_z}{\bar{\mu} + \alpha_H \mu_z}. \tag{14}$$
That is, if all informed traders were to send orders to the exchange, then some of them would deviate and use the dark pool.

- The type-L liquidity traders stay in the dark pool if there is one, but delay trading if there is not. That is,

$$\frac{r^+ - r^-}{r^+ + r^-} \leq \gamma_L \leq \frac{\bar{\mu}}{\bar{\mu} + \alpha_H \sigma_z}, \tag{15}$$

where r^+ and r^- are evaluated at the solution β^* of Proposition 1.

For a sufficiently small positive real number $\epsilon > 0$, it is clear that we can find an $\alpha_H(\epsilon)$ and a $\gamma_L(\epsilon)$ such that

$$\bar{r} = \frac{\alpha_H(\epsilon)}{\bar{\mu_I} + \alpha_H(\epsilon)\mu_z} + \epsilon, \tag{16}$$

$$\gamma_L(\epsilon) = \frac{\bar{\mu}}{\bar{\mu_I} + \alpha_H(\epsilon)\mu_z} - \epsilon. \tag{17}$$

As $\epsilon \to 0$, we have $\beta^* \to 0$ and $r^+ - r^- \to 0$, and the two constraints (14) and (15) hold. By continuity, the constraints also hold in a sufficiently small neighbourhood of 0. That is, we can find a sufficiently small but positive ϵ, as well as associated parameters $\alpha_H(\epsilon)$ and $\gamma_L(\epsilon)$, such that the equilibrium of Proposition 1 exists with $\beta^* > 0$ and the equilibrium of Corollary 1 exists. Thus, adding a dark pool strictly decreases the signal-to-noise ratio on the exchange from $\frac{\bar{\mu}}{\alpha_H(\epsilon)\sigma_z}$ to $\frac{(1-\beta^*)\bar{\mu}}{\alpha_H(\epsilon)\sigma_z}$.

3 Constant Delay Costs

In the current model of Zhu (2013), the cost of delay for liquidity trader i is $\gamma_i \sigma$. Increasing the volatility σ therefore also increases the delay costs of liquidity traders. This delay cost can effectively come from collateral or margin requirement, as in Brunnermeier and Pedersen (2009).

In this appendix, I consider the alternative modeling approach by assuming that liquidity trader i's delay cost is γ_i, invariant of volatility, where $\{\gamma_i\}$ are heterogeneous across traders with a cdf of G. (The cdf G here can be different from the G in the main text.) The purpose of this calculation is not to redo the paper, but to demonstrate that the effect of dark pool for price discovery is robust to this perturbation of assumption on delay costs.

Proposition 2. *There are two possible cases of equilibrium in which traders use the dark pool.*

1. *An equilibrium that has $\beta = 0$, $\alpha_d > 0$ and $\alpha_0 = 0$ is characterized by the solution (α_d^*, μ_I^*) to the following equations:*

$$G^{-1}(\alpha_d)(1-\bar{r}) = \frac{\mu_I}{\mu_I + (1-\alpha_d)\mu_z}\sigma, \tag{18}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-\alpha_d)\mu_z}{\mu_I + (1-\alpha_d)\mu_z}\sigma\right). \tag{19}$$

In this case of equilibrium, adding a dark pool increases dark-pool participate rate α_d of liquidity traders, the exchange spread S, and the signal-to-noise ratio $I(\beta, \alpha_d)$; but decreases the mass μ_I of informed traders.

2. *An equilibrium that has $\beta > 0$, $\alpha_d > 0$ and $\alpha_0 > 0$ is characterized by the solution $(\beta^*, \alpha_d^*, \mu_I^*)$ to the following equations:*

$$r^- = 1 - \frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + (1-G(\sigma))\mu_z}, \tag{20}$$

$$\alpha_d = G(\sigma) - G\left(\frac{r^+ - r^-}{r^+ + r^-}\,\sigma\right), \tag{21}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-G(\sigma))\mu_z}{(1-\beta)\mu_I + (1-G(\sigma))\mu_z}\,\sigma\right). \tag{22}$$

In this case of equilibrium, adding a dark pool increases the spread S and the signal-to-noise ratio $I(\beta, \alpha_d)$ if and only if

$$\beta < \frac{G(\sigma) - G((1-r^-)\sigma)}{1 - G((1-r^-)\sigma)}. \tag{23}$$

In both cases, we use an equilibrium selection criterion similar to that in the text, namely that as α_d varies from 0 to $G(\sigma)$, the left-hand side of (18) and (21) cross the right-hand side from below.

Proof. **Case 1.** In this case, the marginal liquidity trader's indifference condition is (18), whose left-hand side is his trading cost in the dark pool, and the right-hand side is his trading cost (spread) on the exchange. The indifference condition of a marginal for-profit trader is (19).

Adding a dark pool is equivalent to increasing $\bar{r}$ from zero to some positive number. To calculate the comparative statics regarding $\bar{r}$, we take total derivative on both sides of (18) and (19) and obtain

$$\underbrace{\left(\frac{dG^{-1}(\alpha_d)}{d\alpha_d}(1-\bar{r}) - \frac{\partial RHS(18)}{\partial \alpha_d}\right)}_{>0}\frac{d\alpha_d}{d\bar{r}} = \underbrace{G^{-1}(\alpha_d)}_{>0} + \underbrace{\frac{\partial RHS(18)}{\partial \mu_I}}_{>0}\frac{d\mu_I}{d\bar{r}}, \tag{24}$$

$$\underbrace{\left(1 - \bar{\mu}\frac{\partial F(\sigma - S)}{\partial \mu_I}\right)}_{>0}\frac{d\mu_I}{d\bar{r}} = \bar{\mu}\underbrace{\frac{\partial F(\sigma - S)}{\partial \alpha_d}}_{<0}\frac{d\alpha_d}{d\bar{r}}. \tag{25}$$

These total derivatives imply that $\partial \mu_I/\partial \bar{r} < 0$ and $\partial \alpha_d/\partial \bar{r} > 0$. (If $\partial \mu_I/\partial \bar{r} \geq 0$, then the above two equations would not be consistent with each other.) Since the term in the bracket of the right-hand side of (19) is $\sigma - S$ and it decreases in $\bar{r}$, we have S increasing in $\bar{r}$. The signal-to-noise ratio $I(\beta, \alpha_e) = (1-\beta)\mu_I/(\alpha_e \sigma_z)$ is also increasing in S. Therefore, adding the dark pool increases

α_d, S and I but decreases μ_I.

Case 2. In this case, informed traders must all be indifferent between the two venues, which implies (20), whose left-hand side is the expected informed profit in the dark pool, scaled by σ, and the right-hand side is the expected informed profit on the exchange, also scaled by σ. Since informed traders have an effective delay cost of σ, the marginal liquidity trader who is indifferent between the two venues must also have a delay cost of σ. Thus, $\alpha_e = 1 - G(\sigma)$. The same argument as in the main text shows that the marginal liquidity trader who is indifferent between dark-pool trading and waiting has a delay cost of $\frac{r^+ - r^-}{r^+ + r^-}\sigma$ (instead of $\frac{r^+ - r^-}{r^+ + r^-}$). Thus, we have (21). Finally, the indifferent condition of a marginal for-profit trader is (22).

Now I show the effect of the dark pool. I use the "tilde" notation ($\tilde{\alpha}_e$ and $\tilde{\mu}_I$) to denote equilibrium variables if the dark pool is absent. Without a dark pool, all informed traders use the exchange, and $\tilde{\alpha}_e \in (1 - G(\sigma), 1)$ and $\tilde{\mu}_I$ solve

$$G^{-1}(1 - \tilde{\alpha}_e) = \frac{\tilde{\mu}_I}{\tilde{\mu}_I + \tilde{\alpha}_e \mu_z}\sigma. \tag{26}$$

$$\tilde{\mu}_I = \bar{\mu} F\left(\frac{\tilde{\alpha}_e \mu_z}{\tilde{\mu}_I + \tilde{\alpha}_e \mu_z}\,\sigma\right). \tag{27}$$

As in the main text, we keep the same selection criterion that as $\tilde{\alpha}_e$ varies, the left-hand side of (26) crosses the right-hand side from above. With a dark pool, the exchange spread is $\frac{(1-\beta)\mu_I \sigma}{(1-\beta)\mu_I + \alpha_e \mu_z}$.

Now we show that if $\tilde{\alpha}_e > (1 - G(\sigma))/(1 - \beta)$, then adding a dark pool widens the exchange spread S and increases the signal-to-noise ratio I. If $\tilde{\alpha}_e$ is evaluated at $(1 - G(\sigma))/(1 - \beta)$, then (22) and (27) imply that $\tilde{\mu}_I$ (without dark pool) is equal to μ_I (with dark pool). By the equilibrium selection criterion, $\tilde{\alpha}_e > (1 - G(\sigma))/(1 - \beta)$ is equivalent to

$$G^{-1}\left(1 - \frac{1 - G(\sigma)}{1 - \beta}\right) > \frac{\tilde{\mu}_I}{\tilde{\mu}_I + \frac{1 - G(\sigma)}{1 - \beta}\mu_z}\sigma = \frac{\mu_I}{\mu_I + \frac{1 - G(\sigma)}{1 - \beta}\mu_z}\sigma = (1 - r^-)\sigma. \tag{28}$$

The above inequality simplifies to (23). Thus, $\tilde{\alpha}_e > (1 - G(\sigma))/(1 - \beta)$ is equivalent to (23).

□

Proposition 1 shows that under volatility-independent delay costs, the equilibrium characterization and the effect of dark pools on market quality are similar to those under volatility-linear delay costs. The difference is that volatility-independent delay costs slightly complicate the conditions under which the two cases of equilibria exist, as I show in the following result.

Proposition 3. *Let $\mu_I(\sigma)$ be the solution to the implicit function*

$$\hat{\mu}_I(\sigma) = \bar{\mu} F\left(\frac{(1 - G(\sigma))\mu_z}{\hat{\mu}_I(\sigma) + (1 - G(\sigma))\mu_z}\,\sigma\right). \tag{29}$$

If $\hat{\mu}_I(\sigma)$ is increasing in σ, then there exists some threshold volatility $\bar{\sigma} > 0$ such that Case 1 of Proposition 1 applies if $\sigma \leq \bar{\sigma}$, and Case 2 of Proposition 1 applies if $\sigma > \bar{\sigma}$.

A sufficient condition for $\hat{\mu}_I(\sigma)$ to be increasing in σ is that $1 - G(\sigma) \geq G'(\sigma)\sigma$.

Proof. Because an informed buyer takes the same action as a liquidity buyer who has a delay cost of σ, we need to find solution $\alpha_d \leq G(\sigma)$. A similar argument as in the main text shows that a Case-1 equilibrium exists if

$$\bar{r} < 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(\sigma))\mu_z}, \tag{30}$$

and a Case-2 equilibrium exists if and only if

$$\bar{r} > 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(\sigma))\mu_z}. \tag{31}$$

To write the conditions (30) and (31) in terms of the primitive parameter σ, a sufficient condition is that $\hat{\mu}_I(\sigma)$ is increasing in σ. Indeed, if $\hat{\mu}_I'(\sigma) \geq 0$, then the right-hand side of (30) would be decreasing in σ. As σ increase beyond some $\bar{\sigma} > 0$, (30) eventually cannot hold and (31) eventually holds. Thus, Case 1 applies if $\sigma \leq \bar{\sigma}$ and Case 2 applies if $\sigma > \bar{\sigma}$. Again, this simple partition of the two cases is possible if, but not necessarily only if, $\hat{\mu}_I(\sigma)$ is (weakly) increasing in σ.

To see under what conditions $\hat{\mu}_I(\sigma)$ is monotone in σ, we take the total derivative of (29) with respect to σ, and obtain

$$\frac{d\hat{\mu}_I(\sigma)}{d\sigma}\left(1 - \bar{\mu}\frac{\partial F(\sigma - S)}{\partial \mu_I}\right) = \bar{\mu}F'(\sigma - S)\left[\frac{(1 - G(\sigma))\mu_z}{\hat{\mu}_I(\sigma) + (1 - G(\sigma))\mu_z} - \frac{G'(\sigma)\sigma\hat{\mu}_I(\sigma)\mu_z}{(\hat{\mu}_I(\sigma) + (1 - G(\sigma))\mu_z)^2}\right]. \tag{32}$$

In the expression above, the term within the bracket on the left-hand side is positive, but the right-hand side has an undetermined sign. The right-hand side is nonnegative if and only if

$$1 - G(\sigma) \geq G'(\sigma)\sigma\frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(\sigma))\mu_z}, \tag{33}$$

a sufficient condition of which is $1 - G(\sigma) \geq G'(\sigma)\sigma$. This sufficient condition is met, for example, by the cdf $G(\sigma) = 1 - A/\sigma$, $\sigma \in [A, \infty)$, for all $A > 0$.

□

4 Dark Pools as Nondisplayed Limit Order Books

This section models a dark pool that operates as a nondisplayed limit order book. In addition to complementing the analysis of a midpoint dark pool by Zhu (2013), this appendix also offers additional insights regarding the impact of dark pool mechanisms on the participation incentives of informed traders.

As discussed in Appendix A of Zhu (2013), the key feature of a limit-order dark pool, relative to a midpoint dark pool, is that the transaction price in the limit-order dark pool is determined by submitted limit orders. Dark pools operating as nondisplayed limit order books therefore provide some price discovery. This limit-order mechanism is commonly used in dark pools operated by broker-dealers.

While limit-order dark pools may execute orders at prices other than the midpoint, such price discretion is often limited by "best-execution" regulations. In the United States, the Order Protection Rule, also known as the "trade through" rule, stipulates that transaction prices in any market center—including dark pools, ECN, and broker-dealer internalization—cannot be strictly worse than the prevailing national best bid and offer (NBBO).[7] For example, if the current best bid is \$10 and the best ask is \$10.50, then the transaction price in any market center must be in the interval $[\$10, \$10.50]$. More recently, regulators have also proposed a stricter "trade-at" rule. Under a trade-at rule, execution prices in dark pools must be strictly better than the best bid or offer on all displayed venues, including exchanges. For example, the Joint CFTC-SEC Advisory Committee (2011) recommends that the SEC consider "its rule proposal requiring that internalized or preferenced orders only be executed at a price materially superior (e.g. 50 mils [0.5 cent] for most securities) to the quoted best bid or offer."

I now describe and solve a model of a limit-order dark pool that operates under a trade-at rule. The dark pool executes orders by price priority, and I model its trading mechanism as a double auction. The dark-pool execution price, p^*, is determined such that the aggregate limit buy orders (i.e. demand) at p^* is equal to the aggregate sell limit orders (i.e. supply) at p^*. Moreover, I model the effect of a trade-at rule by assuming that transaction prices in the dark pool must be within the interval $[-xS, xS]$, where $S > 0$ is the exchange spread and $x \in [0, 1]$ captures the strictness of the trade-at rule. Traders can only submit orders with a limit price within the interval $[-xS, xS]$; this restriction is almost without loss of generality.[8] The trade-through rule currently applied in the United States corresponds to $x = 1$, indicating a mandatory price improvement of zero. A midpoint-matching mechanism corresponds to $x = 0$, indicating a price improvement of the entire effective spread S. With the exception of this trade-at rule, the model of this section is identical to

[7] In Europe, MiFID uses a decentralized best-execution rule, by which investment firms decide whether an execution works for the best interest of investors.

[8] If there were no restriction on the limit price, then buyers who are willing to trade at xS would submit higher limit prices to gain priority. By the trade-at rule, a limit buy order with a price above xS will not be executed above xS, so it is risk-free to submit ever higher limit prices. Of course, if all buy orders above xS have the highest limit price, say \$999,999, then they will all get rationed at this highest price. Rationing them at a price higher than xS is equivalent to rationing them at xS. That is, as far as the prices and execution probabilities are concerned, it is almost without loss of generality that traders can only submit limit orders within $[-xS, xS]$.

the main model of Zhu (2013). Proposition 4 below characterizes an equilibrium that is analogous to Case 1 of Proposition 1 of Zhu (2013).[9] This result sheds light on how the trade-at rule affects the dark pool participation of informed traders.

Proposition 4. *In a market with an exchange and a dark pool that implements a double auction, there exists a unique threshold volatility $\bar{\sigma}(x) > 0$ with the property that, for any $\sigma \leq \bar{\sigma}(x)$, there exists an equilibrium $(\beta = 0, \alpha_d = \alpha_d^*, \alpha_e = 1 - \alpha_d^*)$, where $\alpha_d^* \in (0, G(1)]$ and μ_I^* solve*

$$\left[G^{-1}(\alpha_d) - \frac{xS}{\sigma}\right] \cdot (1 - \bar{r}_x) = \frac{\mu_I}{\mu_I + (1-\alpha_d)\mu_z}, \tag{34}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-\alpha_d)\mu_z}{\mu_I + (1-\alpha_d)\mu_z}\,\sigma\right), \tag{35}$$

where

$$\bar{r}_x = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^-}{(\alpha_d - G(xS/\sigma))Z^+}\right)\right]. \tag{36}$$

In this equilibrium with a fixed x:

1. *If $c \in [0, xS)$, a liquidity buyer (resp. seller) with a delay cost of c quotes a limit price of c (resp. $-c$) in the dark pool. If $c \in [xS, G^{-1}(\alpha_d^*)\sigma]$, then a liquidity buyer (resp. seller) with a delay cost of c quotes a limit price of xS (resp. $-xS$) in the dark pool. Liquidity traders with delay costs higher than $G^{-1}(\alpha_d^*)\sigma$ trade on the exchange.*

2. *The dark pool execution price is given by (43) in Section 4.1.*

3. *The dark pool participation rate α_d of liquidity traders, the mass μ_I of informed traders, and the scaled exchange spread S/σ are all strictly increasing in the value σ of information.*

4. *The exchange spread S and the signal-to-noise ratio $\mu_I/(\alpha_e\sigma_z)$ are higher than their counterparts in a market with only the lit exchange.*

Moreover, for $x \in (0, 1)$, the volatility threshold $\bar{\sigma}(x)$ is strictly decreasing in x.

Proof. See Section 4.1. □

The equilibrium of Proposition 4 with a limit-order dark pool is qualitatively similar to the equilibrium characterized in Case 1 of Proposition 1 of Zhu (2013). The equilibrium is determined

[9] For tractability reasons, I have not characterized an equilibrium in which some informed traders send orders to the limit-order dark pool. A modeling challenge with informed participation in the limit-order dark pool is to calculate the expected loss of liquidity traders, conditional on order execution at each possible price in the interval $[-xS, xS]$, not only the midpoint. Boulatov and George (2012) model a nondisplayed market in which informed traders submit demand schedules (i.e. limit orders). Their model is tractable partly because their uninformed traders are noise traders and hence do not internalize the costs of trading against informed traders. By contrast, endogenous venue selection of liquidity traders is a key modeling objective of this paper.

by the marginal liquidity trader who is indifferent between the two venues, shown in (34), and the marginal for-profit trader who is indifferent about whether to acquire the information, shown in (35). If multiple equilibria exist, I select the equilibrium with the lowest α_d^* among those with the property that, as α_d varies in a neighborhood of α_d^*, the left-hand side of (34) crosses the right-hand side from below. The expressions of $\bar{\sigma}(x)$ and p^* in equilibrium are provided in Section 4.1.

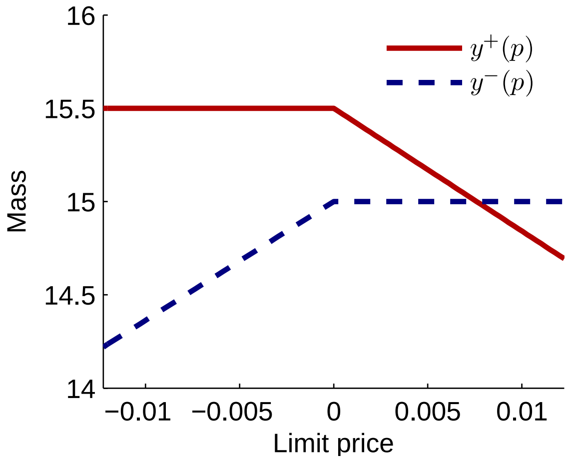
Naturally, in equilibrium liquidity traders who have higher delay costs submit more aggressive orders (i.e. buy orders with higher limit prices and sell orders with lower limit prices). Moreover, because a liquidity buyer's order is infinitesimal and has zero impact on the execution price p^*, she wishes to use a "truth-telling" strategy, that is, to submit a buy order whose limit price is equal to her delay cost.[10] If her decay cost $c < xS$, the trade-at rule is not binding, so she submit a dark pool buy order with the limit price c. If $c \geq xS$, the trade-at rule becomes binding at the price xS, so the liquidity buyer selects the highest limit price allowed, xS. In equilibrium, a strictly positive mass of liquidity buyers set the limit price xS and are rationed; their execution probability, $\bar{r}_x$, is given by (36). If the delay cost c is sufficiently high, the liquidity buyer trades on the exchange in order to avoid the risk of being rationed at the price xS. A liquidity seller's strategy is symmetric. The comparative statics (i.e. Part 3 and Part 4) of Proposition 4 have the same intuition as those in Proposition 2 and Proposition 3 of Zhu (2013). Because the limit-order dark pool in this case only attracts liquidity traders, adding it increases the exchange spread and the signal-to-noise ratio.

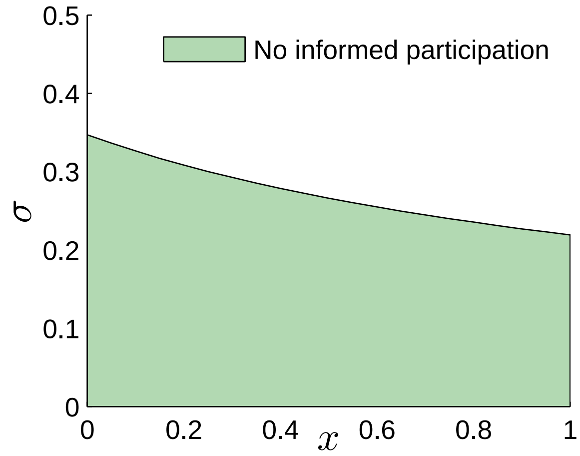
Proposition 4 further reveals that the trade-at rule has a material effect for the participation of informed traders in the dark pool. Because $\bar{\sigma}(x)$ is decreasing in x, the stricter is the trade-at rule, the less attractive is the dark pool to informed traders. The intuition is as follows. If an informed buyer were to deviate to the dark pool, she would select the most aggressive permissible limit price, xS, in order to maximize her execution probability. Although she would be rationed at the price xS, she would only compete with those liquidity traders who have a delay cost of xS or higher. The lower is x, the less scope there is for the informed trader to "step ahead of the queue" and gain execution priority. In particular, a midpoint dark pool with $x = 0$ has the greatest effectiveness in discouraging informed traders to participate.

The left-hand plot of Figure 2 shows the dark pool orders in the equilibrium of Proposition 4. In this example, $x = 0.8$, so the dark pool provides a price improvement equal to 20% of the exchange spread S. In this example, about 95% of liquidity traders in the dark pool set the most aggressive limit price, $\pm xS$. The dark pool transaction price in this case is about 0.007. The right-hand plot of Figure 2 shows that the volatility threshold $\bar{\sigma}(x)$ is strictly decreasing in x. With midpoint crossing ($x = 0$), informed traders avoid the dark pool if the value σ of information

[10] This strategy is reminiscent of the truth-telling strategy of MacAfee (1992), who considers a double auction with finitely many buyers and sellers. The double auction here has the institutional restriction that transaction prices are bounded by the trade-at rule.

Figure 2: A dark pool as a nondisplayed limit order book. The left-hand plot shows the aggregate limit orders in the dark pool, where $y^+(p)$ and $y^-(p)$ denotes the demand schedule and supply schedule, respectively. The right-hand plot shows the range of σ for which the equilibrium of Proposition 4 exists, that is, informed traders avoid the dark pool. Model parameters: $\mu_z = 60$, $\sigma_z = \sqrt{60}$, $\bar{\mu} = 20$, Z^+ and Z^- have $Gamma(30, 1)$ distributions, $G(s) = s/2$ for $s \in [0, 2]$, and $F(s) = 1 - e^{-s/2}$ for $s \in [0, \infty)$. The left-hand plot also uses $x = 0.8$, $\sigma = \bar{\sigma}(0.8) = 0.236$, and realizations $Z^+ = 31$ and $Z^- = 30$.





is lower than about 0.35. Under the current trade-through rule ($x = 1$), this volatility threshold is reduced to about 0.22.

The effect of the trade-at rule on informed participation in dark pools complements prior fairness-motivated arguments, which suggest that displayed orders should have strictly higher priority than nondisplayed orders at the same price (Joint CFTC-SEC Advisory Committee, 2011). Proposition 4 predicts that implementing a trade-at rule is likely to reduce informed participation in dark pools. It also predicts that dark pools operating as limit order books are more likely to attract informed traders and impatient liquidity traders than dark pools crossing at the midpoint.

4.1 Proof of Proposition 4

I prove this proposition in three steps. First, I calculate the execution price in the dark pool and the optimal limit prices chosen by liquidity traders (i.e. Part 1 and Part 2 of Proposition 4). Second, I derive incentive-compatibility conditions under which informed traders choose not to participate in the dark pool (i.e. equations (34) and (35)). Finally, I prove the comparative statics and conditions under which the equilibrium exists (i.e. Part 3 and Part 4 of Proposition 4, the cutoff volatility $\bar{\sigma}(x)$, and that $\bar{\sigma}'(x) < 0$).

Step 1: Price p^* of execution and optimal prices of limit orders

I let $y^+ : [-xS, xS] \to [0, \infty)$ be the aggregate *downward-sloping* demand schedule of liquidity buyers in the dark pool, and let $y^- : [-xS, xS] \to [0, \infty)$ be the aggregate *upward-sloping* supply schedule of liquidity sellers. For each p, $y^+(p)$ is the total mass of limit buy orders that have a limit price of p or higher, and $y^-(p)$ is the total mass of limit sell orders that have a limit price of p or lower. Because the dark pool crosses orders by price priority, its execution price p^* is

$$p^* = \begin{cases} xS, & \text{if } y^+(p) > y^-(p) \text{ for all } p \in [-xS, xS]. \\ -xS, & \text{if } y^+(p) < y^-(p) \text{ for all } p \in [-xS, xS]. \\ \{p : y^+(p) = y^-(p)\}, & \text{otherwise.} \end{cases} \tag{37}$$

I proceed under the conjecture that the set $\{p : y^+(p) = y^-(p)\}$ contains at most one element, in which case p^* of (37) is uniquely well-defined. I later verify this conjecture. Once p^* is determined, buy orders with limit prices above or equal to p^* are matched, at the price of p^*, with sell orders whose prices are at most p^*. If there is a positive mass of buy or sell orders at the price p^*, then traders setting the limit price p^* are rationed pro-rata, as before.

I now derive the optimal limit prices of liquidity traders in the dark pool, under the conjecture that the probability distribution of p^* has no atom in $(-xS, xS)$. This no-atom conjecture, verified later, implies that a liquidity trader quoting a price of $p \in (-xS, xS)$ has her order filled with certainty (i.e. is not rationed) if $p^* = p$. Thus, a liquidity buyer who has a delay cost of $c \in [0, xS)$ and quotes a price of p in the dark pool has the expected payoff (negative cost)

$$\begin{aligned} X_d(p; c) &= -\mathbb{E}\left[\mathbb{I}_{\{p \geq p^*\}} p^* + \mathbb{I}_{\{p < p^*\}} c\right] \\ &= -c - \int_{-xS}^{p} (p^* - c)\, dH(p^*), \end{aligned} \tag{38}$$

where $\mathbb{I}(\,\cdot\,)$ is the indicator function and $H(p^*)$ is the cumulative distribution function of p^*. Because there is no adverse selection in the dark pool, the execution cost for this liquidity buyer is either the payment p^* or the delay cost c. Conjecturing that $H(p^*)$ is differentiable with $H'(p^*) > 0$ for $p^* \in (-xS, xS)$, properties that are also verified later, we obtain

$$\frac{dX_d(p; c)}{dp} = -(p - c)H'(p). \tag{39}$$

Because (39) shows that the sensitivity of expected payoff to the limit price p is positive for $p < c$ and negative for $p > c$, the optimal limit price for the liquidity buyer is her delay cost c. Symmetrically, the optimal limit price for a liquidity seller with a delay cost of $c \in [0, xS)$ is $-c$.

This "truth-telling" strategy is also ex-post optimal, in that no one wishes to deviate even after observing the execution price. The first-order condition (39) also implies that xS is the highest limit price in the dark pool, and that $-xS$ is the lowest limit price.[11]

Let $y(p)$ be the downward-sloping demand schedule in the dark pool if $Z^+ = 1$. Because a limit price $p \in [0, xS)$ is submitted by the liquidity buyer with the delay cost p,

$$y(p) = \alpha_d - G\left(\frac{\max(0,p)}{\sigma}\right), \quad -xS < p < xS. \tag{40}$$

By symmetry, the liquidity buyers' demand schedule and the liquidity sellers' supply schedule in the dark pool are, respectively,

$$y^+(p) = Z^+ y(p), \tag{41}$$
$$y^-(p) = Z^- y(-p). \tag{42}$$

Because the equation $y^+(p) = y^-(p)$ has at most one root, we have verified our earlier conjecture that the dark pool execution price p^* is uniquely well-defined.

Given $y(p)$, the execution price p^* in the dark pool is

$$p^* = \begin{cases} +xS, & \text{if } [\alpha_d - G\left(\frac{xS}{\sigma}\right)]Z^+ \geq \alpha_d Z^-, \\ +\sigma G^{-1}\left[\alpha_d\left(1 - \frac{Z^-}{Z^+}\right)\right], & \text{if } \left[\alpha_d - G\left(\frac{xS}{\sigma}\right)\right] Z^+ < \alpha_d Z^- \leq \alpha_d Z^+, \\ -\sigma G^{-1}\left[\alpha_d\left(1 - \frac{Z^+}{Z^-}\right)\right], & \text{if } \left[\alpha_d - G\left(\frac{xS}{\sigma}\right)\right] Z^- < \alpha_d Z^+ \leq \alpha_d Z^-, \\ -xS, & \text{if } [\alpha_d - G\left(\frac{xS}{\sigma}\right)]Z^- \geq \alpha_d Z^+. \end{cases} \tag{43}$$

Because the total trading interest Z^+ of liquidity buyer and the total trading interest Z^- of liquidity sellers are identically distributed, the dark pool execution price p^* has a mean of zero. By the differentiability of G and of the distribution function of Z^-/Z^+, $H(p^*)$ is continuous, differentiable, and strictly increasing on $(-xS, xS)$, as conjectured earlier.

Step 2: Incentive conditions for participation

What remains to be shown are the incentive-compatibility conditions of liquidity traders who set the limit price xS or $-xS$ in the dark pool, as well as the incentive-compatibility condition of informed traders, who avoid the dark pool. A liquidity buyer quoting the limit price xS in the

[11] If the maximum limit price were lower, say $p_0 < xS$, then a liquidity buyer with a delay cost of $p_0 + \epsilon$ for some small $\epsilon > 0$ would deviate to the dark pool and quote $p_0 + \epsilon$. This deviating buyer has an execution probability of 1 and pays at most $p_0 + \epsilon < xS \leq S$, which is better than execution on the exchange. The argument for the lowest limit price is symmetric.

dark pool has an execution probability of $\bar{r}_x$ and an expected payoff, given the delay cost c, of

$$X_d(xS; c) = -(1 - \bar{r}_x)(c - xS). \tag{44}$$

This expected payoff calculation follows from the fact that $\mathbb{E}(p^*) = 0$ and the fact that failing to cross in the dark pool incurs a delay cost of c but saves the payment xS.

Because informed traders avoid the dark pool with probability 1 in the conjectured equilibrium, an informed buyer who *deviates* to the dark pool also has the crossing probability $\bar{r}_x$. Moreover, in order to get the highest priority, this deviating informed trader sets the highest limit price xS. Her expected profit in the dark pool is thus

$$W_d = \sigma - (1 - \bar{r}_x)(\sigma - xS). \tag{45}$$

As before, for any delay cost $c \leq \sigma$,

$$W_d - X_d(xS; c) = \sigma \bar{r}_x + c(1 - \bar{r}_x) \leq \sigma = W_e - X_e. \tag{46}$$

That is, an informed buyer behaves in the same way as does a liquidity buyer who has a delay cost of σ. If informed traders do not participate in the dark pool, an equilibrium is determined by a marginal liquidity trader who is indifferent between the dark pool and the exchange. Given α_d, this liquidity trader has a delay cost of $G^{-1}(\alpha_d)\sigma$. So we must have $X_d(xS; G^{-1}(\alpha_d)) = -S$, or (34). Thus, (34) and (35) characterize an equilibrium. We now look for conditions under which, in equilibrium, $\alpha_d^* \leq G(1)$. In this equilibrium, $\beta = 0$.

Step 3: Comparative statics and conditions for the existence of equilibria

I now calculate the comparative statics, assuming the existence of an equilibrium, and then show conditions under which the stated equilibrium exists. Total differentiation of (34) and (35) with respect to σ yields

$$\underbrace{\left(\frac{\partial[lhs(34)]}{\partial \alpha_d} - \frac{\partial[rhs(34)]}{\partial \alpha_d} \right)}_{>0} \frac{d\alpha_d}{d\sigma} + \underbrace{\left(\frac{\partial[lhs(34)]}{\partial \mu_I} - \frac{\partial[rhs(34)]}{\partial \mu_I} \right)}_{<0} \frac{d\mu_I}{d\sigma} = 0, \tag{47}$$

$$\underbrace{\left(1 - \frac{\partial[rhs(35)]}{\partial \mu_I} \right)}_{>0} \frac{d\mu_I}{d\sigma} = \underbrace{\frac{\partial[rhs(35)]}{\partial \alpha_d}}_{<0} \frac{d\alpha_d}{d\sigma} + \underbrace{\bar{\mu} F'(\sigma - S) \left(1 - \frac{S}{\sigma} \right)}_{>0}. \tag{48}$$

As before, if $d\alpha_d/d\sigma \leq 0$ at some σ_0, then (47) implies that $d\mu_I/d\sigma \leq 0$ at σ_0 as well. But this contradicts (48). Thus, we have the comparative statics with respect to σ. And given the equilibrium, the dark pool execution price p^* and the optimal limit prices follow from calculations done in Step 1 of the proof. A market without a dark pool is characterized by setting $\bar{r}_x = 0$ in (34), and the effects of adding a dark pool for exchange spread and signal-to-noise ratio follow from the same argument in Case 1 of Proposition 3 of Zhu (2013).

Now I characterize the condition for the existence of an equilibrium and the threshold volatility $\bar{\sigma}(x)$. For $x \in [0,1]$, I define $\bar{K}(x)$ implicitly by

$$(1 - x\bar{K}(x))\left\{1 - \mathbb{E}\left[\min\left(1, \frac{G(1)Z^-}{[G(1) - G(x\bar{K}(x))]Z^+}\right)\right]\right\} = \bar{K}(x). \tag{49}$$

This $\bar{K}(x)$ is uniquely well-defined because the left-hand side of (49) is decreasing in $\bar{K}(x)$ and the right-hand side is strictly increasing in $\bar{K}(x)$. Moreover, total differentiation of (49) with respect to x yields

$$\underbrace{\left(\frac{\partial[lhs(49)]}{\partial \bar{K}(x)} - 1\right)}_{<0} \bar{K}'(x) + \underbrace{\frac{\partial[lhs(49)]}{\partial x}}_{<0} = 0.$$

So we have $\bar{K}'(x) < 0$.

On the other hand, given $\bar{K}(x)$, I define $\mu_I^*(x)$ by

$$\frac{\mu_I^*(x)}{\mu_I^*(x) + (1 - G(1))\mu_z} = \bar{K}(x),$$

and define $\bar{\sigma}(x)$ by

$$\mu_I^*(x) = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{\mu_I^*(x) + (1 - G(1))\mu_z}\,\bar{\sigma}(x)\right).$$

Because $\mu_I^*(x)$ is strictly increasing in $\bar{K}(x)$ and because $\bar{\sigma}(x)$ is strictly increasing in $\mu_I^*(x)$, $\bar{\sigma}(x)$ is strictly increasing in $\bar{K}(x)$. Because $\bar{K}'(x) < 0$, $\bar{\sigma}'(x) < 0$.

What remains to be shown is that, for $\sigma \leq \bar{\sigma}(x)$, an equilibrium characterized by Proposition 4 exists. Clearly, once α_d is determined, μ_I is uniquely determined by (35). For sufficiently small α_d, the left-hand side of (34) is negative, whereas the right-hand side is strictly positive. For $\alpha_d = G(1)$, (35) implies that

$$\mu_I = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{\mu_I + (1 - G(1))\mu_z}\,\sigma\right),$$

which is no larger than $\mu_I^*(x)$. Thus,

$$K \equiv \frac{\mu_I}{\mu_I + (1 - G(1))\mu_z} = \frac{S}{\sigma}\bigg|_{\alpha_d = G(1)} \leq \bar{K}(x),$$

and, by the definition of $\bar{K}(x)$,

$$(1-xK)\left\{1-\mathbb{E}\left[\min\left(1, \frac{G(1)Z^-}{[G(1)-G(xK)]Z^+}\right)\right]\right\} > K.$$

That is, at $\alpha_d = G(1)$, the left-hand side of (34) is weakly higher than the right-hand side. Therefore, there exists a solution $\alpha_d^* \in (0, G(1))$ to (34), and an equilibrium exists.

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