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Nonfundamental Speculation Revisited

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ABSTRACT

We show that a linear pure strategy equilibrium may not exist in the model of Madrigal (1996), contrary to the claim of the original paper. This is because Madrigal's characterization of a pure strategy equilibrium omits a second-order condition. If the nonfundamental speculator's information about noise trading is sufficiently precise, a linear pure strategy equilibrium fails to exist. In parameter regions where a pure strategy equilibrium does exist, we identify a few calculation errors in Madrigal (1996) that result in misleading implications.

MADRIGAL (1996) PRESENTS AND SOLVES A MODEL IN WHICH a "nonfundamental speculator" observes superior order flow information that allows him to partly infer the insider's fundamental information. This type of behavior remains highly relevant today. For example, many investors and regulators suspect that high-frequency traders and other proprietary trading firms obtain valuable information about investors' order flows and profit from it.

However, the equilibrium solution by Madrigal is only partially correct. This note reports and corrects the errors in his original paper, using his original notation. Contrary to the claim of the original paper, a pure strategy equilibrium may fail to exist in Madrigal's model. Moreover, a few calculation errors in Madrigal (1996) result in misleading implications for some market outcomes, such as market liquidity and price discovery in the early period. Section I of this note reproduces Madrigal's (1996) model and states the correct characterization of a linear pure strategy equilibrium. Section II.A proves the nonexistence of a pure strategy equilibrium if the speculator's information is sufficiently precise. Section II.B corrects the calculation errors when a pure strategy equilibrium exists and discusses the implications for market outcomes.

I. The Setup of Madrigal (1996) and Characterization of a Pure Strategy Equilibrium

Madrigal (1996) considers a two-period Kyle (1985) model with one risky asset. The risky asset has a liquidation value given by a random variable

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$\tilde{v} \sim N(p_0, \Sigma_0)$ with $p_0 \in \mathbb{R}$ and $\Sigma_0 > 0$. The market is populated by four types of players: an insider, a (nonfundamental) speculator, a market maker, and noise traders. The insider places market orders x_1 and x_2 in periods 1 and 2, respectively. The nonfundamental speculator places an order y only in period 2. Noise traders trade in both periods, and they place an aggregate order of $\tilde{u}_t + \tilde{w}_t$ in period $t = 1, 2$, where $\tilde{u}_t \sim N(0, \sigma_u^2)$ (with $\sigma_u \geq 0$) and $\tilde{w}_t \sim N(0, \sigma_w^2)$ (with $\sigma_w \geq 0$). The random variables $(\tilde{v}, \tilde{u}_1, \tilde{w}_1, \tilde{u}_2, \tilde{w}_2)$ are mutually independent.

At the end of period t, after observing the total order flow ω_t for that period, the market maker sets price p_t according to the weak-efficiency rule, that is, $p_t = \mathrm{E}(\tilde{v}|\omega_s : s \leq t)$, for $t = 1, 2$.

Both the nonfundamental speculator and the insider trade to maximize their expected profits conditional on their private information. Prior to trading in period 2, the nonfundamental speculator observes the first-period order flow of noise trades $\tilde{w}_1$ (but not $\tilde{u}_1$). That is, the speculator's period-2 information set is $U = \{w_1, p_1\}$. Knowing $\tilde{w}_1$ gives the speculator an information advantage over the second-period market maker, and as a result he can make profits based on this information. Since this information is not about $\tilde{v}$ directly, Madrigal (1996) labels it nonfundamental information. Its quality is controlled by the parameter $k \equiv \frac{\sigma_w^2}{\sigma_u^2}$. The larger is k, the more variations in total noise trading comes from $\tilde{w}$ other than $\tilde{u}$, so that the order flow information $\tilde{w}$ is more useful.

The insider knows $\tilde{v}$ at the beginning of the economy. Madrigal (1996) assumes that the insider also observes w_1 in period 2. The insider's information sets at dates 1 and 2 are, therefore, $I_1 = \{v\}$ and $I_2 = \{v, w_1, p_1\}$, respectively.

In Madrigal's (1996) notation, the insider's trading strategies in periods 1 and 2 are[1]

$$x_1 = \alpha(v - p_0), \tag{1}$$

$$x_2 = a_2(v - p_1) - \frac{A}{2}(m - p_1), \tag{2}$$

where α, a_2, and A are endogenous constants, and

$$m = \mathrm{E}(\tilde{v}|w_1, p_1) \tag{3}$$

is the speculator's expectation on $\tilde{v}$ given his information set $U = \{w_1, p_1\}$.

The nonfundamental speculator's trading strategy is

$$y = A(m - p_1), \tag{4}$$

where A is the same endogenous constant as in (2). (We show in Appendix A that (4) is the optimal response given (2).) Prices follow

$$p_t = p_{t-1} + \lambda_t \omega_t, \quad \text{for} \quad t = 1, 2, \tag{5}$$

[1] Equation (1) differs from the original equation (8) in Madrigal's (1996) Proposition 1 because we have adopted his simplified version, namely, equation (A29) on p. 571 of Madrigal (1996). Madrigal's original equation (8) has the form $x_1 = a_1(v - p_0) + b_1(E(m|v) - v)$.

where λ_t is the endogenous Kyle's lambda and the aggregate order flows in periods 1 and 2 are

$$\omega_1 = x_1 + u_1 + w_1 \quad \text{and } \omega_2 = x_2 + y + u_2 + w_2, \tag{6}$$

respectively.

The following proposition characterizes the pure strategy equilibrium if it exists. Lemma A1 in Appendix A provides additional details on the computation of the linear pure strategy equilibrium. The Internet Appendix provides a sample Matlab code for this computation.[2]

PROPOSITION 1 (Corrected Characterization from Madrigal (1996)): *A linear pure strategy equilibrium, if it exists, is defined by five unknowns: α, a_2, A, λ_1, and λ_2. These five unknowns are characterized by*

- *three first-order conditions, (A1), (A4), and (A13),*
- *two market maker's updating rules, (A15) and (A16), and*
- *two second-order conditions, (A2) and (A14).*

These equations are given in Appendix A.

II. Key Errors in Madrigal's (1996) Characterization

This section corrects the errors in Madrigal's (1996) characterization of a pure strategy equilibrium. First, a linear pure strategy equilibrium may not exist. Second, when a pure strategy equilibrium does exist, a few calculation errors lead to incorrect interpretations of some endogenous market outcomes.

A. Nonexistence of a Pure Strategy Equilibrium

We first show that a linear pure strategy equilibrium can fail to exist, contrary to proposition 1 of Madrigal (1996, p. 558).

Specifically, in period 1, the insider chooses x_1 to maximize

$$\mathrm{E}[(v - p_1)x_1 + (v - p_2)x_2 | v],$$

where x_2 is taken at its optimal strategy (2). Computation shows that the objective function is a quadratic function of x_1, and hence the first-order and second-order conditions are

$$c(v - p_0) - dx_1 = 0 \quad \text{and} - d \leq 0,$$

respectively, where c and d are endogenous parameters given by equations (A23) and (A24) in Appendix A.

Madrigal's (1996) computation does not check the second-order condition $-d \leq 0$. Condition (A14) in our Proposition 1 corrects this error.

[2] The Internet Appendix may be found in the online version of this article.

The other second-order condition, $\lambda_2 \geq 0$, is always satisfied in Proposition 1.

We find that, when parameter k is sufficiently large, that is, when the order-flow information of the nonfundamental speculator is sufficiently precise, a linear pure strategy equilibrium fails to exist. This result is formally stated in the next proposition and proven in Appendix B.

PROPOSITION 2: *If the nonfundamental speculator's signal is sufficiently precise (i.e., k is sufficiently large), then there is no linear pure strategy equilibrium.*

The intuition for the nonexistence of a linear pure strategy equilibrium is simple. If k is sufficiently large, the speculator observes period-1 noise trading almost perfectly and infers the insider's trade almost perfectly. If the insider were to use a pure strategy, the speculator would invert the insider's trade and learn his information almost perfectly. By inadvertently leaking the fundamental information through his order flow to the speculator, the insider's profit in period 2 is substantially reduced because of competition (see Holden and Subrahmanyam (1992)).

One could imagine a strategy under which the insider trades only in period 1 and hence the information leakage to the speculator would not affect the insider's total profit. But this strategy is not optimal and hence not an equilibrium. As in other Kyle-type models, the insider smoothes price impact across the two periods. As long as $p_1 \neq v$, the insider has an information advantage over the market maker and makes a positive profit in period 2. The only condition under which the insider never trades in period 2 is when $p_1 = v$, which cannot be guaranteed because of the noise traders in period 1. In addition, $p_1 = v$ implies zero profit for the insider in period 1. Therefore, period 2 is relevant and the speculator's presence imposes the risk of information leakage for the insider.

As an illustration for Proposition 2, for parameters $\sigma_u^2 = \Sigma_0 = 1$, numerical calculations show that, when $k \geq 30.88$, the second-order condition for the insider's date-1 problem, $d \geq 0$, is violated, so a linear pure strategy equilibrium does not exist. Although the threshold of 30.88 for k may appear large, it should not be taken literally given the highly stylized two-period model.

The economic significance of Proposition 2 is that the nature of the equilibrium is qualitatively different if the speculator receives precise order flow information. In fact, when the speculator's information is precise, there exists an equilibrium in which the insider uses a mixed strategy and adds random "noise" to the order flow. Although the added noise increases the transaction cost of the insider in period 1, it prevents the speculator from learning the insider's information with sufficient accuracy. The speculator, of course, anticipates this added noise and uses the correct inference. In a model that is related to but distinct from Madrigal's, Yang and Zhu (2016) solve a mixed strategy equilibrium and discuss its empirical relevance for the optimal execution of institutional investors when high-frequency traders may act as a nonfundamental speculator (or "back-runner").

B. Errors in Derivations When a Pure Strategy Equilibrium Exists

Table CI in Appendix C lists a few key errors in the derivation of Madrigal (1996) in parameter regions where a pure strategy equilibrium does exist.

Table CII in Appendix C reproduces Madrigal's Table I with the correct numbers, using the parameters $\sigma_u^2 = \Sigma_0 = 1$. Numerical analysis shows that the patterns are robust to the choice of parameter values. Madrigal's original results for λ_2, Σ_2, and the insider's profits remain qualitatively (but not numerically) correct. That is, the presence of the speculator reduces period-2 liquidity, improves period-2 price discovery, and reduces the insider's total profits. These effects are stronger if the speculator's order flow information is more precise.

Madrigal's original results for λ_1 and Σ_1, however, are misleading. In particular, for the k's used in Madrigal's Table I,

(i) the presence of the speculator may increase λ_1 if k is sufficiently large, and
(ii) as k increases, the conditional price variance at the end of period 1, Σ_1, is first increasing and then decreasing in k—in particular, the presence of the speculator increases Σ_1 if k is small and decreases Σ_1 if k is large.

From the corrected Table CII, we observe that

(i′) the presence of the speculator decreases λ_1, and
(ii′) as k increases, Σ_1 is increasing in k—in particular, the presence of the speculator increases Σ_1.

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Appendix A: Characterization of a Pure Strategy Equilibrium

This appendix provides details on the solution to the linear pure strategy equilibrium.

Nonfundamental speculator's date-2 problem: The speculator trades quantity y in the second period to maximize $\mathrm{E}[(v - p_2)y|w_1, p_1]$. Given the insider's trading strategy (2), the price function (5), and the expression of ω_2 in (6), we can compute the first-order condition as

$$y = \left(\frac{1 - \lambda_2 a_2}{2\lambda_2} + \frac{A}{4}\right)(m - p_1),$$

which, after matching with the conjectured trading strategy (4), implies

$$A = \frac{1 - \lambda_2 a_2}{2\lambda_2} + \frac{A}{4} \quad \Rightarrow \quad A = \frac{2(1 - \lambda_2 a_2)}{3\lambda_2}. \tag{A1}$$

The second-order condition is

$$\lambda_2 \geq 0. \tag{A2}$$

Insider's date-2 problem: In the second period, the insider chooses a trade x_2 to maximize $\mathrm{E}[(v - p_2)x_2 | p_1, w_1, v]$. Given the speculator's trading strategy (4), the price function (5), and the expression of ω_2 in (6), we can compute the first-order condition as

$$x_2 = \frac{v - p_1}{2\lambda_2} - \frac{A(m - p_1)}{2}, \tag{A3}$$

which, after matching with (2), implies

$$a_2 = \frac{1}{2\lambda_2}. \tag{A4}$$

The second-order condition is still given by $\lambda_2 \geq 0$ in (A2).

Insider's date-1 problem: In period 1, the insider chooses x_1 to maximize

$$\Pi(x_1) \equiv \mathrm{E}[(v - p_1)x_1 + (v - p_2)x_2 | v],$$

where x_2 is taken at its optimal quantity given by equation (A3).

Using equations (5), (6), and (A3), we can compute

$$\Pi(x_1) = (v - p_0)x_1 - \lambda_1 x_1^2 + \mathrm{E}\left[\left.\begin{array}{c} (v - p_0 - \lambda_1(x_1 + u_1 + w_1))^2 \left(\frac{1}{4\lambda_2} - \frac{A}{2} + \frac{\lambda_2 A^2}{4}\right) \\ + (m - v)(v - p_0 - \lambda_1(x_1 + u_1 + w_1)) \left(-\frac{A}{2} + \frac{\lambda_2 A^2}{2}\right) \\ + \frac{\lambda_2 A^2}{4}(m - v)^2 \end{array}\right| v\right]. \tag{A5}$$

Note that $\{p_1, w_1\} = \{x_1 + u_1\}$ in terms of predicting $\tilde{v}$, and thus

$$m = \mathrm{E}(v | p_1, w_1) = p_0 + \beta(x_1 + u_1), \tag{A6}$$

where

$$\beta \equiv \frac{\mathrm{cov}(\alpha v, v)}{\mathrm{var}(\alpha v + u_1)} = \frac{\alpha \Sigma_0}{\alpha^2 \Sigma_0 + \sigma_u^2}. \tag{A7}$$

By (A6), we have

$$\mathrm{E}(m - v | v) = \beta x_1 - (v - p_0). \tag{A8}$$

Taking first-order and second-order derivatives of $\Pi(x_1)$ in (A5) and using the above expression of $\mathrm{E}(m - v|v)$ in (A8), we can get

$$\Pi'(x_1) = c(v - p_0) - dx_1, \tag{A9}$$

$$\Pi''(x_1) = -d, \tag{A10}$$

where

$$c = 1 + \frac{\lambda_1}{2}\left(A - \frac{1}{\lambda_2}\right) - \frac{\beta A}{2}, \tag{A11}$$

$$d = \lambda_1\left(2 - \frac{\lambda_1}{2\lambda_2} - A\lambda_1\left(\frac{\lambda_2 A}{2} - 1\right)\right) - \beta A\left(\lambda_1(1 - \lambda_2 A) + \frac{\lambda_2 A\beta}{2}\right). \tag{A12}$$

So, the first-order condition of the insider's date-1 problem implies

$$x_1 = \frac{c}{d}(v - p_0),$$

which, after matching with the initial conjecture (1), implies

$$\alpha = \frac{c}{d}. \tag{A13}$$

The second-order condition of the insider's date-1 problem is

$$d \geq 0. \tag{A14}$$

Market maker's date-1 problem: By equations (1), (5), and (6), we have

$$\lambda_1 = \frac{\mathrm{cov}(v, \omega_1)}{\mathrm{var}(\omega_1)} = \frac{\alpha\Sigma_0}{\alpha^2\Sigma_0 + (1+k)\sigma_u^2}. \tag{A15}$$

Market maker's date-2 problem: By equations (2) to (6), we have

$$\lambda_2 = \frac{\mathrm{cov}(v, \omega_2|\omega_1)}{\mathrm{var}(\omega_2|\omega_1)} = \frac{a_2\Sigma_1 + \frac{A}{2}\Sigma_{vm}}{a_2^2\Sigma_1 + \frac{A^2}{4}\Sigma_{mm} + a_2 A\Sigma_{vm} + (1+k)\sigma_u^2}, \tag{A16}$$

where the Σ's are the elements of $\mathrm{var}([v, m]|\omega_1)$, which are defined as

$$\Sigma_1 \equiv \mathrm{var}(v|\omega_1) = \Sigma_0 - \frac{S_{v\omega}^2}{S_{\omega\omega}}, \qquad \Sigma_{vm} \equiv \mathrm{cov}(v, m|\omega_1) = S_{vm} - \frac{S_{m\omega}S_{v\omega}}{S_{\omega\omega}},$$

$$\Sigma_{mm} \equiv \mathrm{var}(m|\omega_1) = S_{mm} - \frac{S_{m\omega}^2}{S_{\omega\omega}}, \tag{A17}$$

where the S's are elements of the unconditional variance matrix $\mathrm{var}([v, m, \omega_1])$, that is,

$$S_{vm} \equiv \mathrm{cov}(v, m) = \beta\alpha\Sigma_0, \qquad S_{v\omega} \equiv \mathrm{cov}(v, \omega_1) = \alpha\Sigma_0,$$

$$S_{m\omega} \equiv \mathrm{cov}(m, \omega_1) = \beta\left(\alpha^2\Sigma_0 + \sigma_u^2\right), \qquad S_{mm} \equiv \mathrm{var}(m) = \beta^2\left(\alpha^2\Sigma_0 + \sigma_u^2\right),$$

$$S_{\omega\omega} \equiv \mathrm{var}(\omega_1) = \alpha^2\Sigma_0 + \sigma_u^2(1+k).$$

LEMMA A1: *Let $k \equiv \frac{\sigma_w^2}{\sigma_u^2}$ denote the quality of order flow information. A linear pure strategy equilibrium in the form of equations (1) to (6) exists if and only if the following two conditions are satisfied:*

(1) There is a solution $q \in (0, 1)$ to the following seventh-order polynomial in terms of q^2:

$$A_7 q^{14} + A_6 q^{12} + A_5 q^{10} + A_4 q^8 + A_3 q^6 + A_2 q^4 + A_1 q^2 + A_0 = 0, \tag{A18}$$

where

$$\begin{aligned}
A_7 &= 9(8k+9)(k+1)^3,\\
A_6 &= -9(-16k+8k^2-27)(k+1)^2,\\
A_5 &= -9k(k+1)(74k+12k^2+65),\\
A_4 &= 3(-708k-570k^2-92k^3+16k^4-243),\\
A_3 &= -(1287k+135k^2-189k^3+4k^4+891),\\
A_2 &= -3(-66k-105k^2+4k^3+81),\\
A_1 &= -9(-29k+k^2-18),\\
A_0 &= 81.
\end{aligned}$$

(2) The following second-order condition is satisfied

$$d \equiv \lambda_1\left(2 - \frac{4}{9}\lambda_1 a_2\right) - \beta\frac{2a_2}{9}\left(2\lambda_1 + \frac{\beta}{2}\right) \geq 0, \tag{A19}$$

where

$$\begin{aligned}
\beta &= \frac{q\sqrt{1+k}}{q^2(1+k)+1}\frac{\Sigma_0^{1/2}}{\sigma_u},\\
\lambda_1 &= \frac{q}{(q^2+1)\sqrt{1+k}}\frac{\Sigma_0^{1/2}}{\sigma_u},\\
a_2 &= \sqrt{\frac{9(1+k)(q^2+1)(kq^2+q^2+1)\sigma_u^2}{(8kq^2+9q^2+9)\Sigma_0}}.
\end{aligned}$$

When a linear pure strategy equilibrium exists, the endogenous constants in (1) to (5) are given by

$$\alpha = q\frac{\sigma_u\sqrt{1+k}}{\Sigma_0^{1/2}}, \qquad a_2 = \sqrt{\frac{9(1+k)(q^2+1)(kq^2+q^2+1)\sigma_u^2}{(8kq^2+9q^2+9)\Sigma_0}},$$

$$A = \frac{2a_2}{3}, \qquad \lambda_1 = \frac{q}{(q^2+1)\sqrt{1+k}}\frac{\Sigma_0^{1/2}}{\sigma_u}, \quad \text{and} \quad \lambda_2 = \frac{1}{2a_2}.$$

We now prove Lemma A1.

Renormalize α as a linear transformation of q as follows:

$$\alpha = q\frac{\sigma_u\sqrt{1+k}}{\Sigma_0^{1/2}}. \tag{A20}$$

We can then express the system of equations in terms of this single unknown q. The idea is to substitute other equations into (A13) and then rearrange it as a seventh-order polynomial of q^2.

Simplify the expressions of c and d in (A11) and (A12): By (A4), we have

$$\lambda_2 = \frac{1}{2a_2}. \tag{A21}$$

By (A1) and (A21), we can compute

$$A = \frac{2a_2}{3}. \tag{A22}$$

Using (A4) and (A21), we can rewrite the expressions of c and d in (A11) and (A12) as

$$c = 1 - \frac{a_2}{3}(2\lambda_1 + \beta), \tag{A23}$$

$$d = \lambda_1\left(2 - \frac{4}{9}\lambda_1 a_2\right) - \beta\frac{2a_2}{9}\left(2\lambda_1 + \frac{\beta}{2}\right). \tag{A24}$$

Express λ_1, β, and a_2 as functions of q: To express (A13) as an equation of q, we need to calculate the expressions of λ_1, β, and a_2 in (A23) and (A24) as functions of q. By (A15) and (A20), we can express λ_1 as

$$\lambda_1 = \frac{q}{(q^2+1)\sqrt{1+k}}\frac{\Sigma_0^{1/2}}{\sigma_u}. \tag{A25}$$

Using (A7) and (A20), we can compute

$$\beta = \frac{q\sqrt{1+k}}{q^2(1+k)+1}\frac{\Sigma_0^{1/2}}{\sigma_u}. \tag{A26}$$

Substituting (A4) and (A21) into (A16), we have

$$a_2^2 = \frac{(1+k)\sigma_u^2}{\Sigma_1 - \frac{\Sigma_{mm}}{9}}. \tag{A27}$$

By (A17), (A20), and (A26), we can compute

$$\Sigma_1 = \frac{\Sigma_0}{q^2+1} \quad \text{and} \quad \Sigma_{mm} = \frac{q^2 k \Sigma_0}{(q^2(1+k)+1)(q^2+1)},$$

which we substitute into (A27) to obtain

$$a_2 = \sqrt{\frac{9(1+k)(q^2+1)(kq^2+q^2+1)\sigma_u^2}{(8kq^2+9q^2+9)\Sigma_0}}. \tag{A28}$$

Rewrite (A13) as a function of q: Substituting (A23) and (A24) into (A13) and rearranging, we have

$$\frac{1-\frac{a_2}{3}(2\lambda_1+\beta)}{\lambda_1\left(2-\frac{4}{9}\lambda_1 a_2\right)-\beta\frac{2a_2}{9}\left(2\lambda_1+\frac{\beta}{2}\right)} = \alpha, \tag{A29}$$

which implies

$$1-2\alpha\lambda_1 = \frac{a_2}{3}\left[(2\lambda_1+\beta)-\alpha\frac{2}{3}\left(2\lambda_1^2+\beta\left(2\lambda_1+\frac{\beta}{2}\right)\right)\right]. \tag{A30}$$

Using (A20) and (A25), we can compute

$$1-2\alpha\lambda_1 = \frac{1-q^2}{q^2+1}.$$

Using (A20), (A25), (A26), and (A28), we can compute

$$\frac{a_2}{3}\left[(2\lambda_1+\beta)-\alpha\frac{2}{3}\left(2\lambda_1^2+\beta\left(2\lambda_1+\frac{\beta}{2}\right)\right)\right]$$
$$= \sqrt{\frac{(q^2+1)(kq^2+q^2+1)}{8kq^2+9q^2+9}}\,\frac{q(k+3kq^2+3q^2+3)(2kq^2+3q^2+3)}{3(q^2+1)^2(kq^2+q^2+1)^2}.$$

Thus, equation (A30) becomes

$$1-q^2 = q\sqrt{\frac{kq^2+q^2+1}{(q^2+1)(8kq^2+9q^2+9)}}\,\frac{(k+3kq^2+3q^2+3)(2kq^2+3q^2+3)}{3(kq^2+q^2+1)^2}. \tag{A31}$$

Conditions in terms of q: Now the system is characterized by equation (A31) and two second-order conditions, $\lambda_2 \geq 0$ and $d \geq 0$. We next recharacterize these conditions as conditions given in Lemma A1.

First, equation (A31) implies that $q \in (0,1)$. To see this, note that if $q < 0$, then by (A25) and (A26), we have $\beta < 0$ and $\lambda_1 < 0$. By the second-order condition $\lambda_2 \geq 0$ and equation (A4), we have $a_2 \geq 0$. So, by equation (A24), we have $d = \lambda_1(2-\frac{4}{9}\lambda_1 a_2) - \beta\frac{2a_2}{9}(2\lambda_1+\frac{\beta}{2}) < 0$, which violates the second-order condition of the insider's date-1 problem. If $q = 0$, then, by (A31), we have $1 - q^2 = 0 \Rightarrow q^2 = 1$, a contradiction. Thus, we must have $q > 0$. Again, when $q > 0$, by equation (A31), we have $1 - q^2 > 0 \Rightarrow q^2 < 1 \Rightarrow q < 1$. Thus, we have $q \in (0, 1)$.

Second, note that, when $q \in (0, 1)$, by equation (A28), we have $a_2 \in (0, \infty)$. Then, by (A21), we have $\lambda_2 = \frac{1}{2a_2} > 0$, so that the second-order condition in period 2 is satisfied. Thus, we are only left with one second-order condition, $d \geq 0$. By noting that d is given by (A24), we obtain the second-order condition (A19) specified in the lemma.

Finally, squaring equation (A31) and rearranging, we can obtain the seventh-order polynomial (A18) given in the lemma. The expressions of α, a_2, A, λ_1, and λ_2 in the lemma are obtained by equations (A20), (A28), (A22), (A25), and (A21), respectively.

Appendix B: Proof of Proposition 2

Note that, in a pure strategy equilibrium, we have $\alpha > 0$ (because $q > 0$ and $\alpha = q\frac{\sigma_u\sqrt{1+k}}{\Sigma_0^{1/2}}$) and $d > 0$ (the second-order condition). So, $c > 0$ by equation (A13). Using the expression of c in (A23), we have

$$1 - \frac{a_2}{3}(2\lambda_1 + \beta) > 0 \Rightarrow \frac{a_2^2}{9}(2\lambda_1 + \beta)^2 < 1. \tag{B1}$$

Plugging the expressions of λ_1, β, and a_2 in (A25), (A26), and (A28) into the left-hand side (LHS) of (B1), we find that (B1) is equivalent to

$$-q^2(1 - q^2)^2k^2 + q^2(11 - q^2)(q^2 + 1)k + 9(q^2 + 1)^2 > 0. \tag{B2}$$

By Lemma A1, in a pure strategy equilibrium, we have $q^2 \in (0, 1)$. First, if as $k \to \infty$ q^2 does not go to zero or one, then the highest order of the LHS of (B2), $-q^2(1 - q^2)^2k^2$, is strictly negative, which means that (B2) is violated.

Second, suppose that for any sequence of $k \to \infty$, we have $q^2 \to 1$ in a pure strategy equilibrium. Then the highest order of the LHS of the polynomial in Lemma A1 is $-64k^4$, which is strictly negative. This contradicts Lemma A1, which says that the polynomial is equal to zero in a pure strategy equilibrium. (That is, the condition of $\frac{d}{c} = \alpha$ is violated.)

Finally, suppose $q^2 \to 0$ in a pure strategy equilibrium for some sequence of $k \to \infty$. By (A29), we have

$$1 - \frac{a_2}{3}(2\lambda_1 + \beta) = \alpha\left[2\lambda_1 - \frac{4}{9}\lambda_1^2 a_2 - \beta\frac{2a_2}{9}\left(2\lambda_1 + \frac{\beta}{2}\right)\right] < 2\lambda_1\alpha,$$

because $a_2 > 0$, $\beta > 0$, and $\lambda_1 > 0$ in a pure strategy equilibrium. Substituting (A20) and (A25) yields

$$1 - \frac{a_2}{3}(2\lambda_1 + \beta) < \frac{2q^2}{q^2 + 1} \Rightarrow \frac{a_2^2}{9}(2\lambda_1 + \beta)^2 > \left(1 - \frac{2q^2}{q^2 + 1}\right)^2. \tag{B3}$$

Combining (B1) and (B3), we have $\frac{a_2^2}{9}(2\lambda_1+\beta)^2 \to 1$ as $k \to \infty$. Plugging the expressions of λ_1, β, and a_2 into (A25), (A26), and (A28) and matching orders, we can show that $q = \frac{1}{3k} + o(\frac{1}{k})$. Substituting this expression of q into equations (A25), (A26), and (A28), which in turn are substituted into the expression of d in (A24), we can show that

$$d = -\frac{1}{81\sqrt{k}}\frac{\Sigma_0^{1/2}}{\sigma_u} + o\left(\frac{1}{\sqrt{k}}\right).$$

So, for large k, this second-order condition is violated.

Appendix C: Correction of Calculation Errors in the Pure Strategy Characterization

Table CI
Correction of a Few Key Equations in the Derivation of Madrigal (1996)

Equation	Expression
Madrigal's (A16)	$\Sigma_2 = \frac{A^2}{4}\frac{(\Sigma_1\Sigma_{mm}-\Sigma_{vm}^2)+\Sigma_1(\sigma_u^2(1+k)-3/2A^2\Sigma_{vm})}{a_2^2\Sigma_1+(A^2/4)\Sigma_{mm}+(1+k)\sigma_u^2}$
Correction	$\Sigma_2 = \frac{\frac{A^2}{4}(\Sigma_1\Sigma_{mm}-\Sigma_{vm}^2)+\Sigma_1(1+k)\sigma_u^2}{a_2^2\Sigma_1+\frac{A^2}{4}\Sigma_{mm}+a_2A\Sigma_{vm}+(1+k)\sigma_u^2}$
Madrigal's (A18)	$d = \lambda_1(2-\lambda_1 a_2 - \frac{2}{3}A\lambda_1) - \frac{\beta}{3}(2\lambda_1 A + \frac{A\beta}{2})$
Correction	$d = \lambda_1(2-\lambda_1 a_2 + \frac{5}{6}A\lambda_1) - \frac{\beta}{3}(2\lambda_1 A + \frac{A\beta}{2})$
Madrigal's (A32)	$d = \lambda_1(2 - \frac{\lambda_1}{2\lambda_2} + A\lambda_1(\frac{\lambda_2 A}{2}-1)) - \beta A(\lambda_1(1+\lambda_2 A) + \frac{\lambda_2 A\beta}{2})$
Correction	$d = \lambda_1(2 - \frac{\lambda_1}{2\lambda_2} - A\lambda_1(\frac{\lambda_2 A}{2}-1)) - \beta A(\lambda_1(1-\lambda_2 A) + \frac{\lambda_2 A\beta}{2})$
Madrigal's (A41)	$\lambda_2 = \frac{a_2\Sigma_1+\frac{A}{2}\Sigma_{vm}}{a_2^2\Sigma_1+\frac{A^2}{4}\Sigma_{mm}+(1+k)\sigma_u^2}$
Correction	$\lambda_2 = \frac{a_2\Sigma_1+\frac{A}{2}\Sigma_{vm}}{a_2^2\Sigma_1+\frac{A^2}{4}\Sigma_{mm}+a_2A\Sigma_{vm}+(1+k)\sigma_u^2}$
Madrigal's (A50)	$a_2^2 = (\sigma_u^2+\sigma_w^2)(\Sigma_1 + 2/3\Sigma_{vm} - 1/9\Sigma_{mm})^{-1/2}$
Correction	$a_2^2 = \frac{(1+k)\sigma_u^2}{\Sigma_1 - \frac{\Sigma_{mm}}{9}}$
Madrigal's (A51)	$\Sigma_2 = \frac{\Sigma_1(\frac{A^2}{4}\Sigma_{mm}+(1+k)\sigma_u^2) - A\Sigma_{vm}(a_2\Sigma_1+\frac{A}{4}\Sigma_{vm})}{a_2^2\Sigma_1+\frac{A^2}{4}\Sigma_{mm}+(1+k)\sigma_u^2}$
Correction	$\Sigma_2 = \frac{\Sigma_1(\frac{A^2}{4}\Sigma_{mm}+(1+k)\sigma_u^2) - \frac{A^2}{4}\Sigma_{vm}^2}{a_2^2\Sigma_1+\frac{A^2}{4}\Sigma_{mm}+a_2A\Sigma_{vm}+(1+k)\sigma_u^2}$
Madrigal's (A53)	$a_1 = d^{-1}(1-\lambda_1 a_2 + (\beta - 2\lambda_1)(\frac{\lambda_2 A^2}{4} - \frac{A}{2}) + \frac{\beta\lambda_2 A^2}{4})$
Madrigal's (A54)	$b_1 = d^{-1}\frac{A}{2}(\lambda_1 - \lambda_1\lambda_2 A + \lambda_2 A\beta)$
Corrections	$a_1 = h^{-1}(1-\lambda_1 a_2 + (\beta - 2\lambda_1)(\frac{\lambda_2 A^2}{4} - \frac{A}{2}) + \frac{\beta\lambda_2 A^2}{4})$
	$b_1 = h^{-1}\frac{A}{2}(\lambda_1 - \lambda_1\lambda_2 A + \lambda_2 A\beta)$
	where $h = [2 - \frac{\lambda_1}{2\lambda_2} + (\beta - 2\lambda_1)(\frac{\lambda_2 A^2}{4} - \frac{A}{2}) + \frac{\beta\lambda_2 A^2}{4}]\lambda_1$

Table CII
Reproduced and Corrected Table I of Madrigal (1996), with $\sigma_u^2 = \Sigma_0 = 1$

$(\sigma_w^2/\sigma_u^2) = k$	0.01	0.25	1	5.6808	20	No Speculator
			Panel A: First-Period Variables			
a_1	0.84099	0.92509	1.13512	1.84912	2.86275	0.66712
b_1	0.24773	0.29271	0.42647	1.12242	2.57563	–
λ_1/λ_1^N	0.99976	0.99432	0.98084	0.94207	0.91262	1
			Panel B: Second-Period Variables			
$a_2/\sqrt{1+k}$	1.20207	1.20146	1.20008	1.19717	1.19609	1.20210
$A/\sqrt{1+k}$	0.80138	0.80097	0.80005	0.79812	0.79739	–
λ_2/λ_2^N	1.00002	1.00054	1.00168	1.00411	1.00502	1
			Panel C: Conditional Variances			
Σ_1	0.69229	0.69820	0.71204	0.74668	0.76924	0.69202
Σ_2	0.34579	0.34094	0.32949	0.29991	0.27925	0.34601
			Panel D: Profits			
π^i/π^N	0.99924	0.98237	0.94234	0.83793	0.76493	1
$\pi^s/\sqrt{1+k}$	0.00057	0.01308	0.04246	0.11720	0.16805	–

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix S1: Internet Appendix.