On Interest-Bearing Central Bank Digital Currency with Heterogeneous Banks

Rodney Garratt† Haoxiang Zhu‡

April 28, 2021

Abstract

We explore the implications of adding an interest-bearing central bank digital currency (CBDC) through commercial banks that differ in size. The large bank gives depositors a convenience value and hence possesses market power. As the interest on reserves (IOR) rate increases, the small bank gains market share in the deposit market and may increase lending. The interest rate on CBDC puts a lower bound on banks’ deposit interest rates, which is particularly binding on large banks. For low CBDC interest rates, the large bank deposit interest rate does not depend on the CBDC interest rate. However, as the CBDC interest rate continues to increase, the large bank’s deposit interest rate eventually rises and the small bank loses deposit market share. Other outcomes in deposit and lending markets can be ambiguous. We also explore a “direct” CBDC that is offered by the central bank or a government agency. In that scenario, which fits the recent “Banking for All Act,” the CBDC interest rate needs to be sufficiently high to have any impact at all.

Keywords: central bank digital currency, interest on (excess) reserves, deposit interest rates, bank lending

JEL Classification Numbers: E42, G21, G28, L11, L15

---

*We thank Todd Keister for multiple discussions in the early stages of this work. For additional helpful discussions and comments, we also thank David Andolfatto, Darrell Duffie, Zhiguo He, Yu Zhu, and Feng Zhu, as well as seminar participants at the Bank of Canada and Luohan Academy. Finally, we thank Jiaheng Yu for excellent research assistance.

†University of California Santa Barbara. Email: garratt@ucsb.edu
‡MIT Sloan School of Management and NBER. Email: zhuh@mit.edu.
“If all a CBDC did was to substitute for cash – if it bore no interest and came without any of the extra services we get with bank accounts – people would probably still want to keep most of their money in commercial banks.”
—Ben Broadbent, Deputy Governor of the Bank of England, in a 2016 speech

1 Introduction

A central bank digital currency (CBDC) “is a digital payment instrument, denominated in the national unit of account, that is a direct liability of the central bank” (BIS 2020). Since Ben Broadbent’s remarks (see above), policymakers have expressed increasing interest in the possibility that a CBDC may pay a non-zero interest. In fact, the Committee on Payments and Market Infrastructures defines interest bearing as one of five key design features of CBDCs.¹

In this paper, we propose a model of heterogeneous banks that we use to explore the implications of an interest-bearing CBDC. In our initial specification of the model, the CBDC is offered through the infrastructure of commercial banks and hence carries the same functionalities as regular bank deposits. Under this design, the interest paid on the CBDC puts a lower bound on the deposit interest rates of commercial banks. The model reveals how deposit interest rates and levels of deposits in the banking system depend on two key policy rates set by the central bank: the interest rate on reserves (IOR), which is paid to banks, and the interest on CBDC, which is paid to agents holding the CBDC.²

The commercial banking sector in the economy consists of a large bank and a small bank, both of which are strategic. The main distinction between them is that the large bank’s deposit offers a higher convenience value than the small bank’s deposit. For example, a large bank has a more expansive network of branches and ATMs. The difference in convenience value has a probability distribution across a continuum of agents who deposit in either bank. The higher convenience value of its deposit gives the large bank market power in the deposit market. Hence, the large bank offers a lower deposit interest rate than the small bank in equilibrium and yet has a larger market share.

The two commercial banks also lend to entrepreneurs who take on risky projects that differ in their quality (i.e., expected returns). When a loan is made by either bank, the bank creates a new deposit in the name of the entrepreneur as a new liability, which is exactly balanced by the new loan as a new asset. The entrepreneur immediately pays a randomly selected worker, and the deposit may flow out of the original bank to the other. For example, if the entrepreneur originally takes a loan from the small bank but the worker she hired has a high enough value for the convenience of the large bank, the deposit flows from the small bank to the large one, resulting in a flow of central bank reserves in the same direction.

¹The other four features are 24/7 availability, anonymity, transfer mechanism, and limits or caps.
²Since the financial crisis of 2008-09, interest on excess reserves has become the Federal Reserve’s main policy tool to adjust interest rate. We do not model reserve requirements (there currently are none in the United States) and hence we are effectively talking about excess reserves throughout the analysis. For simplicity, however, we will just refer to excess reserves as reserves.
Because the large bank has an inherent advantage in the deposit market, in equilibrium, the large bank’s newly created deposit in the lending process has a high probability of staying at the large bank. Thus, by lending out each dollar, not only does the large bank make a profit on the loan, it also earns the interest on reserves with a high probability on the newly created dollar of deposit. By contrast, the small bank is likely to lose the newly created deposit and the associated interest on the accompanying reserves. Therefore, holding fixed the quality of the entrepreneur’s project, the large bank earns a higher total profit by lending. Put differently, the large bank is willing to lend to lower-quality projects due to the higher expected profit of earning interest on reserves. This is a mechanism in which a deposit market advantage translates into a lending market advantage.

To see the impact of the CBDC, it is useful to start with the case without the CBDC as a benchmark. We vary the IOR rate and examine its impact on the deposit and loan markets. There are two cases. In the first case, the zero lower bound on the deposit interest rate binds. When the IOR rate is sufficiently low, the equilibrium deposit interest rates of the two banks are necessarily close to each other because both rates are between zero and the IOR rate. Due to the convenience value of the large bank’s deposit, the large bank pays depositors a zero interest rate and still retains the lion’s share of the deposit market. When the lower bound is binding, an increase in the IOR rate leads to a higher deposit interest rate at the small bank and increases its market share. But because the large bank’s deposit interest rate remains at zero and it still dominates the deposit market, the average deposit interest rate only rises by a small fraction of the IOR rate. A higher IOR rate raises the opportunity cost of capital for making loans at both banks, and the loan quality thresholds increase for both banks. The total loan volume drops, and so does the loan volume of the large bank. Interestingly, the loan volume of the small bank can increase or decrease in response to an increase in the IOR rate, where the potential increase comes from the small bank’s increasing market share in the deposit market.

Figure 1 shows the relationship between the interest rate paid on non-jumbo (i.e., less than $100K) deposits by U.S. commercial banks and the IOR rate since May 2009. As predicted by the model, the deposit interest rate does not respond to the increases in the IOR rate that began in December of 2015. When questioned about this fact, Federal Reserve Chair Jerome Powell said that deposit interest rates respond to changes in the IOR rate with a lag. This seems to hold true in Figure 1, in the sense that deposit interest rates fell slowly after the crisis in 2009, and rose slowly after lift-off began. The delayed response, however, does not explain why the deposit interest rate remained much lower than the IOR rate after it had time to catch up. Our model explains this fact without resorting to common frictions such as search costs in the deposit market. Instead, it is the depositors’ value for convenience that breaks the price competition and gives the large bank market power.

The second case of the equilibrium is when the zero lower bound on deposit interest rates does not bind. This happens when the IOR rate is sufficiently high. Both banks’ deposit interest rates are “interior” and have a constant spread between them, which is determined by the distribution of convenience value among depositors. In this case, a change in the IOR rate translates into the same change in deposit interest rates, but does not affect anything
Figure 1: Actual U.S. deposit interest rates from May 18, 2009 to February 1, 2021. Weekly deposit interest rates for amount less than $100,000 are obtained from FDIC through FRED. During this period, interest on reserves is taken to be the interest on excess reserves (IOER).

Figure 2: Actual U.S. deposit interest rates from 1986Q1 to 2008Q2. Domestic deposit interest rates are quarterly, calculated from call reports, as total interest expense on domestic deposit divided by total domestic deposit, multiplied by 4. During this period, interest on reserves is taken to be the actual Federal Funds rate.

else, such as the banks' market shares in the deposit market, the loan quality thresholds, or the loan volume. As an illustration of the second case, Figure 2 shows that before the 2008-09 crisis, consumer deposit interest rates tended to move proportionally with the Fed Funds rate (the relevant benchmark before IOR was introduced in 2009) when the Fed Funds rate was above approximately 3 percent. As in the post-crisis era, changes in the deposit interest rate appear to lag behind changes in the fed funds rate.

We contend that the heterogeneous-bank model is useful for evaluating the impact of the CBDC. Supposing that the CBDC is offered through commercial banks, and hence CBDC
inherits the convenience properties of the host institution, the lower bound on deposit interest rates at each bank becomes the interest rate on CBDC. The two cases of equilibrium again apply here analogously, and only the case where the lower bound is binding is relevant for our discussion (in the other case, the CBDC has no impact because its interest rate is too low). We consider the impact of increasing the CBDC interest rate while holding the IOR rate fixed. When the lower bound is binding, i.e., the large bank’s deposit interest rate is equal to the CBDC interest rate, an increase in the CBDC interest rate increases the large bank’s market share in the deposit market. The impact of the CBDC interest rate on the small bank’s deposit interest rate, however, is theoretically ambiguous. Because the large bank’s deposit interest rate increases with the CBDC interest rate one-for-one when the lower bound is binding, the small bank could either increase its own rate to compete on market share or reduce its own rate to maximize profit margin on its limited set of depositors. Likewise, the loan quality thresholds and loan volume have ambiguous responses to the increase in the CBDC interest rate. That said, if the small bank’s deposit interest rate increases in the CBDC interest rate, then the small bank’s loan quality threshold increases and its loan volume decreases. The fact that the loan market is affected at all by the CBDC interest rate, while holding the IOR rate fixed, is not mechanical. While bank lending in our model is not restricted by the amount of deposits or reserves, it is affected by the opportunity cost of capital, which, in turn, depends on the probability that a lent dollar returns to the lending bank and earns interest on reserves. It is through the CBDC’s impact on deposit market shares that the CBDC interest rate affects lending.

Our final theoretical exploration is an alternative CBDC design that is offered directly via the central bank or a government agency, but not commercial banks. For example, the Banking for All Act put forward in the U.S. Congress proposes using postal service locations for access to central bank accounts. In China, the central bank has experimented with its own CBDC mobile App in collaboration with state-owned banks. Clearly, these “direct CBDC” designs are unlikely to inherit all of the convenience aspects of commercial bank deposits. Therefore, we assume that a direct CBDC has the same convenience value of a small bank. This seemingly small change in design makes a substantial difference in the equilibrium outcome. Without being a credible substitute for a large bank’s deposit, the CBDC interest rate is no longer a lower bound of the large bank’s deposit interest rate. Depositors may accept a zero interest rate from the large bank and forgo a positive CBDC interest rate because the latter is not as convenient to use. Instead, the CBDC interest rate becomes the lower bound on the small bank’s deposit interest rate, but for this constraint to be binding and have any impact on the market, the CBDC interest rate needs to be sufficiently high. An increase in the CBDC interest rate still affects the deposit market and the loan market, but primarily through its impact on the small bank.

Overall, our analysis fleshes out the implications of introducing an interest-bearing CBDC to U.S. financial markets. In addition to the widely discussed benefit that a CBDC enhances the pass-through of monetary policy, the CBDC interest rate also affects the competitive positions of large and small banks. The CBDC interest rate has a distinct effect from the effect of the IOR rate, which further highlights the richness of monetary policy that is
conducted through two interest rates. We believe our results are useful in evaluating specific proposals for CBDC. For example, the Banking for All Act requires the CBDC interest rate be equal to the IOR rate, and our model with heterogeneous banks predicts that such a requirement would dramatically shrink the deposit base of small banks. Our model can also be easily adapted to evaluate policies such as the current practice of maintaining a constant gap between administered interest rates, such as the gap between the IOR and Overnight Reverse Repurchase Agreement rates set by the Fed.

2 Literature

Our work builds on previous literature that has modelled deposit and lending markets in the current regime of large excess reserves. In Martin, McAndrews and Skeie (2013), a loan is made if its return exceeds the marginal opportunity cost of reserves, which can be either the federal funds rate or the IOR rate, depending on the regime. Our model differs in that we have multiple banks and hence lent money may return to the same bank as new deposits. Hence, our opportunity cost of lending is lower. Nevertheless, we share the conclusion that the aggregate level of bank reserves does not determine the level of bank lending.

Our results on how changes in the federal funds rate affect deposit markets are connected to the literature on monetary policy transmission. Drechsler et al. (2017) provide a model in which increases in the federal funds rate give banks more market power in setting deposit rates. As a result, increases in the federal funds rate lead to an increase in the spread between the federal funds rate and deposit interest rates. Our model provides the same conclusion at lower levels of the federal funds rate. In our case, the increase in spread is amplified by the fact that in the constrained solution of our model with heterogeneous banks, the deposit interest rates at the large bank are completely non-responsive to changes in the federal funds rate.\(^3\) Drechsler et al. (2017) predict that a contraction in deposit supply induced by an increase in the federal funds rate will cause a reduction in lending. This does not occur in our model, since, in our model with large excess reserves, loans are not tied to deposit levels. Nevertheless, we also find that loan volumes decrease in response to an increase in the federal funds rate, since this change increases the opportunity cost of lending.

There is now a growing literature that seeks to examine the impact of CBDC on deposit and lending markets. The conclusions vary and depend upon the level of competition, the interest rate on the CBDC, and other features (e.g., liquidity properties of CBDC and reserve requirements). Keister and Sanches (2020) consider a competitive banking environment in which deposit interest rates are determined jointly by the transactions demand for deposits and the supply of investment projects. If the CBDC serves as a substitute for bank deposits, then its introduction causes deposit interest rates to rise, and levels of deposits and bank lending to fall.

In contrast, if banks have market power in the deposit market, the introduction of a

\(^3\)The “stickiness” of deposit rates in the United State is also documented in Driscoll and Judson (2013), who provide evidence that deposit interest rates respond less to increases in the federal funds rate than they do to decreases.
CBDC does not disintermediate banks, as banks can prevent consumers from holding the CBDC by matching its interest rate. This lowers their profit margin, but does not lower the level of deposits, and may even increase it. This is true in the model proposed by Andolfatto (2020), where the bank is a monopolist. In that work, an interest bearing CBDC causes deposit interest rates to raise and the level of deposits to increase. Likewise, in that work, banks have monopoly power in the lending market, and, as in Martin, McAndrews and Skeie (2013), lending is not tied directly to the level of deposits. Hence, a CBDC does not impact the interest rate on bank lending or the level of investment.

Chiu et al. (2019) also consider banks with market power and show that an interest-bearing CBDC can lead to more, fewer or no change in deposits, depending on the level of the CBDC interest rate. In an intermediate range of rates, the CBDC impacts the deposit market in a manner similar to Andolfatto (2020) in that banks offer higher deposit interest rates and increase deposits. Since, similar to Keister and Sanches (2020), lending is tied to the level of deposits, adding the CBDC results in increased lending.

Our work is closest to Andolfatto (2020). We do not specify the overlapping generations framework that he uses to make money essential. However, like Andolfatto, in our model, reserves are abundant, lending is determined by a performance threshold, and banks have monopoly power in the lending market. Hence, lending is determined not by deposit levels, but instead by the opportunity cost of funds. In our model, this opportunity cost is lower than the IOR rate, since we allow for the realistic feature that reserves come back to the lending bank with a probability that depends on the deposit market share.

The main difference between our model and those of the above CBDC papers is that we have heterogeneous banks. This allows us to look at the differential impact of the CBDC interest rate on the competitive landscape of banks in the deposit market and the loan market. The predictions regarding the large and small banks’ responses in deposit interest rates and market shares can be tested in the data. Moreover, our heterogeneous bank setting also implies that the design details of the CBDC matters for its impact. In particular, depositors in our model put an extra convenience value on the large bank’s deposit, so the CBDC cannot be a perfect substitute for large bank deposit and small bank deposit simultaneously. The degree to which the CBDC intends to compete with the large bank, the small bank, or a mix of the two is an active design choice.

The impact of adding a CBDC can be richer in the presence of other frictions. For example, in a model with purely real goods (no money) and competitive banks, Piazzesi and Schneider (2020) find that the introduction of CBDC is beneficial if all payments are made through deposits and the central bank has a lower cost in offering deposits. However, they also find that the CBDC can be harmful if the payer prefers to use a commercial bank credit line, but the receiver prefers central bank money. In the latter case, a transaction leads to a flow of funds from commercial banks to the central bank; if the liquid assets required to back up this flow are sufficiently costly for the paying bank, the CBDC increases overall costs.

Fernández-Villaverde et al. (2020) extend the analysis of CBDC to a Diamond and Dybvig (1983) environment in which banks are prone to bank runs. In this setting, the fact that the central bank may offer more rigid deposit contracts allows it to prevent runs. Since
commercial banks cannot commit to the same contract, the central bank becomes a deposit monopolist. Provided the central bank does not exploit this monopoly power, the first-best amount of maturity transformation in the economy is still achieved.

Brunnermeier and Niepelt (2019) and Fernández-Villaverde et al. (2020) derive conditions under which the addition of a CBDC does not affect equilibrium outcomes. Key to their result is the central bank’s active role in providing funding to commercial banks in order to neutralize the CBDC’s impact on their deposits.

3 Model and Equilibrium

3.1 Setup

The economy has a large bank (L) and a small bank (S). There are $X = X_S + X_L$ reserves in the banking system, where $X_S$ denotes the reserve holding of the small bank and $X_L$ denotes the reserve holding of the large bank. We normalize the size of an individual loan to be $1$, so reserves are in units of the standard loan size. For example, if a loan size is $1$ million and the actual reserve is $1$ trillion, then in our model, $X$ is interpreted as $\frac{1 \text{ trillion}}{1 \text{ million}} = 10^6$. Following Martin, McAndrews and Skeie (2013), we assume that the level of reserves $X$ is exogenously determined by the central bank and is assumed to be sufficiently high that the demand curve for reserves is flat at the exogenously determined interest rate $f$, which denotes the interest rate the central bank pays on reserves (IOR).

There is a unit mass of agents. Each agent potentially plays three roles in the model:

- **Entrepreneur:** agent $i \in [0, 1]$ is endowed with a project of quality (i.e., success probability) $q(i)$ that has a distribution $Q$. Project $i$ requires $1$ of investment and pays $A > 1$ with probability $q(i)$, and zero with probability of $1 - q(i)$, where $A$ is a commonly known constant. The expected payoff per dollar invested is thus $q(i)A$. Agent $i$ can only borrow from the bank where she keeps her deposit (the “relationship” bank).

- **Worker:** with some probability, an agent is randomly matched with an entrepreneur who receives financing from a bank. The first agent plays the role of worker (contractor) and gets paid the full $1$ to work on the project. An entrepreneur who receives financing can be a worker for another entrepreneur.

- **Depositor:** If an agent receives wage ($1$), she deposits it in a bank, chosen endogenously. The worker’s value for “convenience” of depositing at the large bank is $\delta$, which is a random variable with the cumulative distribution function $G$.

We will look for an equilibrium in which the market shares, in terms of deposits, of the large bank and the small bank are $\alpha_L > 0.5$ and $\alpha_S = 1 - \alpha_L < 0.5$, both endogenously determined.

The timeline of the model is as follows.
$t = 0$: The banks set the deposit interest rates $r_L$ and $r_S$. There is a zero lower bound on the two deposit interest rates; that is, if the deposit interest rate is negative, then agents keep physical cash.

$t = 1$: Each agent is endowed with a project and goes to the relationship bank to borrow $1. Therefore, the relationship bank prices the loan as a monopolist.

$t = 2$: If a loan is granted, a funded entrepreneur pays a randomly matched worker $1 as wage. The worker chooses the bank to deposit and receives the bank’s deposit interest rate. The project succeeds or fails. For each dollar of defaulted loan, the bank suffers an additional cost of $\lambda > 0$, which may be viewed as a regulatory cost for a higher delinquency rate.

### 3.2 Bank deposit creation

For the purpose of illustration it is convenient to illustrate the deposit creation process by considering a discrete set-up, in which we characterize the bank’s decision to make a single loan. The condition on bank lending that we derive will be applicable to the continuum model in which borrowers (i.e., the entrepreneurs) are infinitesimal.

The tables below show the sequence of changes in the large bank’s balance sheet in the loan process. The changes in the small bank’s balance sheet in the loan process are entirely analogous.

1. Before lending, the large bank starts with $X_L$ reserves. Its balance sheet looks like:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves $X_L$</td>
<td>Deposits $X_L$</td>
</tr>
</tbody>
</table>

2. If the large bank makes a loan of $1, it immediately creates deposit of $1 in the name of the entrepreneur. The balance sheet of the bank becomes:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves $X_L$</td>
<td>Deposits $X_L$</td>
</tr>
<tr>
<td>Loans 1</td>
<td>New Deposits 1</td>
</tr>
</tbody>
</table>

3. Eventually, the entrepreneur will spend her money to pay a worker. The large bank anticipates that, in expectation, a fraction $\alpha_S$ of the $1 new deposit will be transferred to the small bank, leading to a reduction of reserves by the same amount. The fraction $\alpha_L$ remains in the bank because the worker has an account with the same bank. The bank’s balance sheet becomes:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reserves $X_L - \alpha_S$</td>
<td>Deposits $X_L$</td>
</tr>
<tr>
<td>Loans 1</td>
<td>New Deposits $\alpha_L$</td>
</tr>
</tbody>
</table>
If the large bank makes the $1 loan to entrepreneur $i$, and charges interest rate $R_i$, its total expected profit, by counting all items in the balance sheet, will be

$$\left(X_L - \alpha_S\right)f + \left[q(i)(1 + R_i) - \lambda(1 - q(i)) - 1\right] - \left(X_L + \alpha_L\right)r_L. \tag{1}$$

If the large bank does not make the loan, then its total profit will be

$$X_L(f - r_L). \tag{2}$$

The large bank’s marginal profit from making the loan, compared to not making it, is

$$\pi_i = q(i)(1 + R_i) - \lambda(1 - q(i)) - (1 + f) + \alpha_L(f - r_L). \tag{3}$$

In the expression of $\pi_i$, the net profit on the loan reflects the true opportunity cost of capital.

There are two cases of equilibrium, depending on whether the zero lower bound on the deposit interest rates is binding.

### 3.3 When the zero lower bound does not bind

In this case, $r_L$ and $r_S$ are both positive. We can solve the model backward in time.

**Deposit market at $t = 2$.** Faced with the two deposit interest rates $r_L$ and $r_S$, an agent with convenience value $\delta$ chooses the large bank if and only if

$$r_L + \delta > r_S \Rightarrow \delta > r_S - r_L. \tag{4}$$

Therefore, the eventual market shares of the banks are

$$\alpha_L = 1 - G(r_S - r_L), \tag{5}$$

$$\alpha_S = G(r_S - r_L). \tag{6}$$

**Loan market at $t = 1$.** We have derived in the previous section the marginal profit of a bank in making a loan. While the entrepreneur is infinitesimal here, the same expression (3) still applies.

To derive the lending criterion, the monopolist position of each bank in the lending market implies that a bank can make a take-it-or-leave-it offer to the entrepreneur. The bank’s optimal interest rate quote would be $R_i = A - 1$ (or just tiny amount below), and the entrepreneur, who has no alternative source of funds, would accept. The lending bank takes the full surplus.

Hence, the large bank makes the loan if and only if

$$q(i)A - \lambda(1 - q(i)) - (1 + f) + \alpha_L(f - r_L) > 0, \tag{7}$$

10
or
\[ q(i) > q^*_L = \frac{1 + f + \lambda - \alpha_L(f - r_L)}{A + \lambda}. \] (8)

Exactly the same calculation for the small bank yields the comparable investment threshold
\[ q^*_S = \frac{1 + f + \lambda - \alpha_S(f - r_S)}{A + \lambda}. \] (9)

**Choice of deposit interest rates at \( t = 0 \).** After a bank makes a loan of $1, the bank creates deposit of $1 in the name of the entrepreneur and also an asset (the loan itself) of $1. The entrepreneur immediately spends the $1 on a randomly chosen worker, whose convenience value is distributed according to \( G \). Therefore, the large bank anticipates that the newly created deposit will be deposited in the large bank with probability \( \alpha_L \). Likewise, the small bank anticipates that the newly created deposit will end up in the small bank with probability \( \alpha_S \).

Again, we start with the large bank. The large bank’s profit can be written as
\[ \Pi_L = \int_{q = q^*_L}^{1} [qA - \lambda(1 - q) - (1 + f) + \alpha_L(f - r_L)]dQ(q). \] (10)

Assuming that \( \Pi_L \) is strictly quasi-concave in \( r_L \) (for any given \( \alpha_L \)), the sufficient condition for a unique maximum of this function with respect to \( r_L \) is
\[
\frac{d\Pi_L}{dr_L} = \frac{d[\alpha_L(f - r_L)]}{dr_L} \int_{q = q^*_L}^{1} dQ(q) - \frac{d\alpha_L^*}{dr_L} \left[ \frac{A - \lambda(1 - q^*_L) - (1 + f) + \alpha_L(f - r_L)}{Q'(q^*_L)} \right] = 0, \text{ by the optimality of } q^*_L.
\]
(11)

As \( Q(q^*_L) < 1 \), the FOC expressed in (11) reduces to
\[ G'(r_S - r_L)(f - r_L) = 1 - G(r_S - r_L). \] (12)

An entirely analogous calculation for the small bank yields
\[ G'(r_S - r_L)(f - r_S) = G(r_S - r_L). \] (13)

Taking the difference of the two FOCs, we have \( G'(r_S - r_L)(r_S - r_L) = \alpha_L - \alpha_S > 0 \), so \( r_S > r_L \). Hence, the two FOCs imply that \( f > r_S > r_L \).

**Proposition 1** Suppose the environment is such that each bank’s profit function is strictly quasi-concave in its deposit interest rate. Define \( r_L \) and \( r_S \) to be unique the solutions to the following two FOCs:
\[ G'(r_S - r_L)(f - r_L) = 1 - G(r_S - r_L) \] (14)
\[ G'(r_S - r_L)(f - r_S) = G(r_S - r_L). \] (15)

If the solutions \( r_L \) and \( r_S \) are both strictly positive, then they are the deposit interest rates quoted by the large and small banks, respectively. In this equilibrium, \( f > r_S > r_L \). Moreover, the small bank’s investment criterion is more stringent than the large bank’s, i.e., \( q^*_L < q^*_S \).
In Appendix A, we show that the uniform distribution $G$ meets the requirement of Proposition 1. The last part of the proposition about the investment criterion can be seen as follows. Observe that $q_L^* < q_S^*$ if and only if $\alpha_L(f - r_L) > \alpha_S(f - r_S)$, or $\alpha_L/\alpha_S > (f - r_S)/(f - r_L)$. Then, observe that the two FOCs imply

$$\frac{\alpha_L}{\alpha_S} = 1 - \frac{G(r_S - r_L)}{G(r_S - r_L)} = \frac{f - r_L}{f - r_S} > \frac{f - r_S}{f - r_L},$$

where the last inequality follows from $r_L < r_S < f$.

We can characterize the equilibrium explicitly for the uniform distribution. Suppose that $G(\delta) = \delta/\Delta$, where $\delta \in [0, \Delta]$ for a sufficiently large $\Delta$. Then $G'(\cdot) = 1/\Delta$. The two first-order conditions reduce to

$$\frac{f - r_L}{\Delta} = 1 - \frac{r_S - r_L}{\Delta},$$

$$\frac{f - r_S}{\Delta} = \frac{r_S - r_L}{\Delta}.$$  

The unique solution is

$$r_S = f - \frac{\Delta}{3}, \quad r_L = f - \frac{2\Delta}{3}.$$  

It is easily verified that $r_S - r_L = \Delta/3 < \Delta$ (so it was fine to write $G(r_S - r_L) = (r_S - r_L)/\Delta$).

3.4 When the zero lower bound binds

The second case of the equilibrium is if the zero lower bound on the deposit interest rate is binding for at least one of the two banks. Intuitively, the lower bound is likely to bind if $f$ is very low. Denote the large bank’s constrained deposit interest rate, i.e., the rate after imposing a zero lower bound, by $r_L^+ \geq 0$. Likewise, let $r_S^+ \geq 0$ denote the small bank’s deposit interest rate under the zero lower bound constraint. There are two possibilities if the zero lower bound binds: $r_S^+ > 0$ and $r_L^+ = 0$; and $r_S^+ = r_L^+ = 0$.

We start by looking for an equilibrium in which only the large bank is constrained by the lower bound on deposit interest rates, i.e., $r_L^+ = 0$ but $r_S^+ > 0$. The large bank’s profit function $\Pi_L$ at this corner $r_L^+ = 0$ is no longer differentiable because the left derivative with respect to $r_L^+$ and the right derivative are different. The right derivative $d\Pi_L/dr_L^+$ is the same as before:

$$\left. \frac{d\Pi_L}{dr_L^+} \right|_{r_L^+ \to 0^+} = \left[ G'(r_S^+ - r_L^+)(f - r_L^+) - (1 - G(r_S^+ - r_L^+)) \right] \int_{q = q_L^*}^1 dQ(q),$$

where we have used the first-order condition of the optimal $q_L^*$. In contrast, as $r_L$ decreases from 0 to a mildly negative rate, $r_L^+$ stays at zero, and there is no change to $\Pi_L$. Thus, the left derivative $d\Pi_L/dr_L$ is zero. The only way that setting a zero deposit interest rate is optimal for the large bank is that the right derivative at $r_L^+ = 0$ is nonpositive, or

$$G'(r_S^+)f - (1 - G(r_S^*)) \leq 0.$$  

12
Now we turn to the small bank. Because we have conjectured that \( r_S^+ > 0 \), the small bank’s first-order condition is the same as before, or
\[
\frac{d\Pi_S}{dr_S} = [G'(r_S^+)(f - r_S^+) - G(r_S^+)] \int_{q=q_S^*}^1 dQ(q) = 0 \tag{22}
\]
This equation has only one unknown, \( r_S^+ \). Provided that \( \Pi_S \) is strictly quasi-concave in \( r_S \), this FOC has a unique solution \( r_S^+ \). If that solution also satisfies the large bank’s first-order condition, then it would be the large bank constrained equilibrium we are looking for.

Specifically, let \( \Phi(x) = G'(x)(f - x) - G(x) \), for \( f \geq 0 \) and \( x \geq 0 \). It is easy to see that \( \Phi(0) = G'(0)f - G(0) \geq 0 \) and \( \Phi(f) = -G(f) \leq 0 \). So a sufficient condition for \( \Phi(x) \) to have a unique root is if \( \Phi'(x) < 0 \), that is, if
\[
G''(x)(f - x) - 2G'(x) < 0, \text{ for all } f \geq 0, x \geq 0. \tag{23}
\]
We will maintain this assumption in the remainder of the paper.

**Proposition 2** Suppose that environment is such that the small bank’s profit function is strictly quasi-concave in its deposit interest rate and the large bank’s profit function is strictly decreasing in its deposit interest rate. For any given \( f \), denote the unique root of (22) by \( r_S^+ \). If \( r_S^+ > 0 \) and \( G'(r_S^+)(f - (1 - G(r_S^+))) \leq 0 \), then it is an equilibrium that the small bank quotes a deposit interest rate of \( r_S^+ \) and the large bank quotes a deposit interest rate of zero. In this equilibrium, \( q_L^* < q_S^* \), that is, the investment criterion is more stringent for the small bank than it is for the large bank.

See Appendix A for a description of environments that meet the requirements of Proposition 2. The last part of the proposition, about investment criterion, follows from an argument that is analogous to what we used following Proposition 1. To show \( q_L^* < q_S^* \), we need to show
\[
(1 - G(r_S^+))f \geq G(r_S^+)(f - r_S^+), \text{ or } \frac{1-G(r_S^+)}{G(r_S^+)} \geq \frac{f-r_S^+}{f}. \tag{24}
\]
The two FOCs, \( G(r_S^+) = G'(r_S^+)(f - r_S^+) \) and \( 1 - G(r_S^+) \geq G'(r_S^+)f \), together imply that \( \frac{1-G(r_S^+)}{G(r_S^+)} \geq \frac{f-r_S^+}{f} > \frac{f-r_S^+}{f}. \)

For an explicit example, we again use the uniform distribution: \( G(\delta) = \delta/\Delta \). Then, condition (23) is then trivially satisfied. Moreover, \( r_S^+ \) solves \( (f-r_S^+)/\Delta - r_S^+ / \Delta = 0 \), i.e., \( r_S^+ = f/2 > 0 \). At this rate, we also need the large bank’s first-order condition \( G'(f/2)f - (1 - G(f/2)) \leq 0 \), i.e., \( \Delta \geq \frac{3}{2}f \).

By now it becomes clear that there does not exist an equilibrium in which both banks are constrained to quote a zero deposit interest rate, unless \( f \) is zero itself. To see this, note that by the same logic as above, the right derivative of the small bank should also be nonpositive if it is constrained to quote a zero deposit interest rate. But this means that
\[
G'(0)f - G(0) \leq 0, \tag{24}
\]
which is impossible unless \( f = 0 \) (assuming \( G' > 0 \) over its entire domain).
4 Impact of IOR on Market Outcomes

Having characterized the two cases of equilibrium, we now compute the impact of a change in IOR on deposit interest rates, levels of deposits, and total lending.

4.1 When the zero lower bound is not binding

This case turns out to be quite straightforward. In the equilibrium of Proposition 1, we have \( G'(r_S - r_L)(r_S - r_L) = 1 - 2G(r_S - r_L). \) Thus, the spread \( r_S - r_L \) is invariant to \( f \). The two first-order conditions in Proposition 1 imply that \( r_L \) and \( r_S \) increase one-for-one with \( f \). However, because the deposit interest rate spread \( r_S - r_L \) is invariant to \( f \), so are the deposit market shares. Finally, both \( q^*_L \) and \( q^*_S \) increase in \( f \) as both \( \alpha_L(f - r_L) \) and \( \alpha_S(f - r_S) \) are invariant to \( f \). Hence, loan volume decreases in \( f \). These implications are summarized in the following proposition.

**Proposition 3** In the equilibrium of Proposition 1, an increase in IOR leads to higher deposit interest rates offered by both banks, but it does not change the two banks’ market shares in the deposit market. The loan quality thresholds increase and the loan volume decreases in IOR.

4.2 When the zero lower bound binds

The case with the zero lower bound is more involved and interesting. Consider an increase in \( f \) that leaves (21) satisfied. Differentiating (22), we get

\[
[G''(r_S^+)(f - r_S^+) - G'(r_S^+)]dr_S + G'(r_S^+)df = 0,
\]

which yields

\[
\frac{dr_S^+}{df} = -\frac{G'(r_S^+)}{G''(r_S^+)(f - r_S^+) - 2G'(r_S^+)}. \quad (26)
\]

The condition that guarantees a unique solution for \( r_S^+ \) in Proposition 2 implies that \( G''(r_S^+)(f - r_S^+) - 2G'(r_S^+) < 0 \) and hence \( dr_S^+/df > 0 \).

What happens to deposit shares as \( f \) increases? Differentiating (5) gives,

\[
\frac{\partial \alpha_L}{\partial f} = -G'(r_S^+)\frac{dr_S^+}{df} = \frac{(G'(r_S^+))^2}{G''(r_S^+)(f - r_S^+) - 2G'(r_S^+)} < 0, \quad (27)
\]

and then we have \( \frac{\partial \alpha_S}{\partial f} = -\frac{\partial \alpha_L}{\partial f} > 0 \). So the share of deposits in the large bank shrinks and the share in the small bank increases in the range of \( f \) such that (21) holds.

What happens to the credit market as \( f \) increases? Let us first calculate the loan quality thresholds \( q^*_L = \frac{1 + f + \lambda - G(r_S^+)}{A + \lambda} \) and \( q^*_S = \frac{1 + f + \lambda - G(r_S^+)(f - r_S^+)}{A + \lambda} \). For \( q^*_L \), we have

\[
\frac{dq^*_L}{df} = \frac{1 - (1 - G(r_S^+)) + G'(r_S^+)(f - r_S^+)}{A + \lambda} \frac{dr_S^+}{df} > 0, \quad (28)
\]

14
that is, the loan quality of the large bank improves. For $q^*_S$, we have
\[
\frac{dq^*_S}{df} = \frac{1 - G(r^+_S) + \left[-G'(r^+_S) f + G'(r^+_S) r^+_S + G(r^+_S)\right] dr^+_S / df}{A + \lambda} > 0,
\]
(29)
where we have used the first-order condition $G'(r^+_S)(f - r^+_S) = G(r^+_S)$. So the small bank’s loan quality also improves.

Recall that in equilibrium, the large bank has $\alpha^*_L = 1 - G(r^+_S - r^+_L)$ depositors and each depositor gets a draw $q(i)$ from the probability distribution function $Q$ defined on $[0, 1]$. Hence, the probability depositor $i$ has a suitable project at the large bank is $(1 - Q(q^*_L))$, and so expected lending by the large bank is $\alpha_L(1 - Q(q^*_L))$. Likewise, expected lending at the small bank is $\alpha_S(1 - Q(q^*_S))$. Total lending is therefore equal to
\[
T = \alpha_L(1 - Q(q^*_L)) + \alpha_S(1 - Q(q^*_S)) = 1 - (1 - G(r^+_S))Q(q^*_L) - G(r^+_S)Q(q^*_S)
\]
\[
= 1 - Q(q^*_L) + G(r^+_S)(Q(q^*_L) - Q(q^*_S)).
\]
(30)
We can thus compute
\[
\frac{dT}{df} = -Q'(q^*_L)\frac{dq^*_L}{df} + G'(r^+_S)(Q(q^*_L) - Q(q^*_S))\frac{dr^+_S}{df} + G(r^+_S)(Q(q^*_L)\frac{dq^*_L}{df} - Q(q^*_S)\frac{dq^*_S}{df})
\]
\[
= -(1 - G(r^+_S))Q'(q^*_L)\frac{dq^*_L}{df} + G'(r^+_S)(Q(q^*_L) - Q(q^*_S))\frac{dr^+_S}{df} - G(r^+_S)Q'(q^*_S)\frac{dq^*_S}{df} < 0,
\]
(31)
where the inequality follows because each of the three terms is negative (the middle term being negative follows from $q^*_L < q^*_S$).

Intuitively, an increase in $f$ causes both sizes of banks to raise their lending thresholds and hence lend less. Meanwhile, depositors (potential borrowers from each bank) move from large banks to small banks. Since small banks (will still) have a higher lending threshold than large banks, the reduction on lending at large banks is greater than the increase at small banks.

We can also separately characterize the loan volume made by the large bank, $T_L$, and the small bank, $T_S$, where
\[
T_L = \alpha_L(1 - Q(q^*_L)), T_S = \alpha_S(1 - Q(q^*_S)).
\]
(32)
We have
\[
\frac{dT_L}{df} = -(1 - G(r^+_S))Q'(q^*_L)\frac{dq^*_L}{df} - G'(r^+_S)\frac{dr^+_S}{df}(1 - Q(q^*_L)) < 0,
\]
(33)
and
\[
\frac{dT_S}{df} = G'(r^+_S)\frac{dr^+_S}{df}(1 - Q(q^*_S)) - G(r^+_S)Q'(q^*_S)\frac{dq^*_S}{df}.
\]
(34)
Clearly, the volume of loans made by the large bank declines in $f$ unambiguously, but because $dr^+_S / df > 0$ and $dq^*_S / df > 0$, the volume of loans made by the small bank may increase or decrease in $f$.  

15
Proposition 4  In the equilibrium of Proposition 2, an increase in the IOR rate leads to:

- A higher interest rate of the small bank $r_S^+$;
- A higher market share of the small bank $\alpha_S$;
- Higher quality thresholds of loans by both banks, $q_L^*$ and $q_S^*$;
- A lower volume of loans issued by the large bank and both banks combined, but an ambiguous change in loan amount issued by the small bank.

Figure 3 shows the deposit interest rates and market share of the two banks as functions of the interest on reserves $f$ for a uniform distribution $G(\delta) = \delta/0.035$. We observe that the deposit interest rate offered by the large bank remains at zero until the IOR rises to a high level (left panel), which in this case is $f = 2\Delta/3 = 2.33\%$. In addition, the market share of the large bank is larger for IOR closer to zero (right panel).

These two properties combined produce predicted average deposit interest rates that are lower and less responsive to changes in the IOR rate $f$ when it is near zero. When $f$ is near zero, the share of deposits at the large bank is near 1 and the large bank offers a deposit interest rate equal to 0. As $f$ increases, the share of deposits held by the small bank increases and the small bank offers a positive rate, so the the average deposit interest rate increases slowly. Eventually, as $f$ increases further, we move to the interior solution described in Proposition 1, where deposit shares are constant (with respect to $f$) and deposit interest rates rise proportionately with $f$.

We illustrate the predictive performance of our model using U.S. data on deposit interest rates from 1986 to 2020. The opportunity cost of funds for banks is determined in part by either the IOR rate or the federal funds rate, whichever is larger. In the period before
the 2008-09 crisis the relevant rate was the federal funds rate. Figure 4 shows actual and predicted U.S. deposit interest rates from 1986Q1 to 2008Q2 relative to the federal fund rate. The choice of the upper bound on the convenience value for deposits at the large bank is somewhat arbitrary, but 3.5 percent seems reasonable as a maximum value and the resulting predicted data series seems to fit the actual data reasonably well. The main deviation of the predicted series from the actual one is that our predicted rates respond too quickly (and hence to much) to changes in the federal funds rate. This is expected since we do not build in any "stickiness" into the deposit interest rate that would be necessary to match the actual data more closely. Given our assumed distribution for the convenience parameter for large bank deposits, our predicted equilibrium involves a binding zero lower bound on the rate at the large bank when the reference rate is less than $2 \frac{1}{3}$ percent. This is true in the data during the pre-crisis period from 2001Q3 to 2004Q3. During this period we predict deposit interest rates near zero, however actual deposit interest rates remained close to, and even slightly above, the federal funds rate.

Figure 4: Actual and predicted U.S. deposit interest rates from 1986Q1 to 2008Q2. Domestic deposit interest rates are quarterly, calculated from call reports, as total interest expense on domestic deposit divided by total domestic deposit, multiplied by 4. During this period, interest on reserves is taken to be the actual Federal Funds rate. The model’s prediction assumes a uniform distribution for the convenience parameter with $\Delta = 0.035$. The model-implied interest rate is the weighted average of the large bank’s and the small bank’s deposit interest rates, weighted by their market shares.

In the period after the crisis the relevant rate for computing the opportunity cost of loaned funds was (generally) the IOR rate. Figure 5 shows actual and predicted U.S. deposit interest rates from May 2009 to February 2021 relative to the IOR rate. This period is characterized by a long stretch of near zero rates in the Federal Funds market and an IOR rate of 25 basis points. Again consistent with the market-lag story, deposit interest rates fell slowly during this period toward zero until the Fed began to raise rates in December 2015. The Fed raised the IOR rate multiple time reaching a peak of 2.40% from December 2018.
to April 2019, but deposit interest rate reacted very slowly. Again, our model reacts too quickly to the changes in the IOR rate, but still our predicted rate stays quite low relative to the increases in the IOR rate. This is because at the low IOR rates that existed up until December 2018, the zero lower bound is binding in our model and hence average market rate are determined largely by large bank deposit interest rates which are at zero.

Figure 5: Actual and predicted U.S. deposit interest rates from May 18, 2009 to February 1, 2021. Weekly deposit interest rates for amount less than $100,000 are obtained from FDIC through FRED. During this period, interest on reserves is taken to be the interest on excess reserves (IOER). The model’s prediction assumes a uniform distribution for the convenience parameter with $\Delta = 0.035$. The model-implied interest rate is the weighted average of the large bank’s and the small bank’s deposit interest rates, weighted by their market shares.

Figure 6 shows the threshold loan quality and loan volume for the two banks as functions of the central bank rate $f$ for the uniform distribution $G(\delta) = \delta/0.035$. As we show in (28) and (29), both thresholds are increasing in $f$, which is intuitive since $f$ is a key determinant of the opportunity cost of lending. The small bank’s threshold stays above the large bank’s, because the opportunity cost for the large bank is smaller since they expect that a larger share of loaned money would return in the form of new deposits. As $f$ increases from 0, the small bank’s threshold increases faster than the large bank’s, until deposit shares stabilize at the point where the zero lower bound is no binding (left panel). A rising loan quality threshold and a decreasing share of deposits, which means lower demand for loans, both lead to a decrease in loan volume for the large bank as $f$ increases. However, a rising threshold and an increasing market share push loan volumes in opposite directions for the small bank. The net effect is positive in the case of the uniform distribution (right panel). Total loan volume always declines in $f$. While the zero lower bound is binding, depositors move from the large bank with a lower threshold to the small bank with a higher threshold as $f$ increases. This substitution leads to an overall decline in lending. Once market shares stabilize, the continued increase in loan quality thresholds for both banks as $f$ increases
causes total lending to decline.

Figure 6: Equilibrium loan market under a uniform distribution, $G(\delta) = \delta/0.035$.

In the case of the deposit market, our predictions depend on the convenience parameter for large banks, which, it is reasonable to assume, does not vary much over time. Hence, we were able to match our predicted deposit interest rates as $f$ varies, to actual time series. It is more difficult to illustrate the predictions of our model for loan quality and loan volume, as these predictions depend on factors that vary over time. For example, a key determinant of loan volume is the demand for loans, which will vary over time with the business cycle or other events such as the financial crisis or COVID-19. In our model, the demand for loans from bank $j$ is determined by $\alpha_j$, which is constant in a steady-state solution. As such, we do not attempt to predict how loan volume varies with respect to $f$ over time.

5 Interest-Bearing CBDC

Suppose the central bank offers a digital alternative to cash in the form of an interest-bearing and universally accessible CBDC. While the exact details of implementing the CBDC is not critical for our model, it is useful to think about the CBDC accounts as administered by commercial banks. That is, depositors earn interest paid by the central bank and the CBDC is the central bank’s liability, but various account services such as money transfer are provided by existing commercial banks’ infrastructure. For this reason, the CBDC administered by each bank is a close substitute for the commercial bank deposit of the same bank. In particular, the convenience value of the CBDC inherits the convenience value of the commercial bank deposit. The procedural for characterizing the equilibrium is similar to that for the baseline model.

Let the interest rate on CBDC equal $s \in (0, f]$. We follow Andolfatto (2020) and take the CBDC interest rate as exogenous. The immediate implication is that the CBDC interest
rate becomes a new lower bound on the deposit interest rate. The s-constrained deposit interest rate for the large bank is \( r^+_L \geq s \). Likewise, write the s-constrained deposit interest rate for the small bank as \( r^+_S \geq s \).

Once again, there are three possibilities for equilibrium deposit interest rates: \( r^+_S > r^+_L > s \); \( r^+_S > s \) and \( r^+_L = s \); and \( r^+_S = r^+_L = s \). The equilibrium with \( r^+_S > s \) and \( r^+_L = s \) requires
\[
G'(r^+_S - s)(f - s) - (1 - G(r^+_S - s)) \leq 0  \tag{35}
\]
and
\[
\frac{d\Pi_s}{dr_s} = [G'(r^+_S - s)(f - r^+_S) - G(r^+_S - s)] \int_{q = q_s}^1 dQ(q) = 0. \tag{36}
\]

By analogous calculations as before, we have the following characterizations of the two cases of the equilibrium. Details are omitted due to their similarity to the baseline model. The same logic as before also shows that it is not an equilibrium to set \( r^+_L = r^+_S = s \) unless \( s = \delta \), which is a knife-edge case.

**Proposition 5** Suppose the environment is such that each bank’s profit function is strictly quasi-concave in its deposit interest rate. Let \( r^+_S \) and \( r^+_L \) solve the following two equations:
\[
G'(r^+_S - r^+_L)(f - r^+_L) = 1 - G(r^+_S - r^+_L), \tag{37}
\]
\[
G'(r^+_S - r^+_L)(f - r^+_S) = G(r^+_S - r^+_L). \tag{38}
\]
If \( r^+_S \) and \( r^+_L \) are both above \( s \), then it is an unconstrained equilibrium that the large bank and the small bank set deposit interest rate \( r^+_L \) and \( r^+_S \), respectively. In this equilibrium, the large bank has a larger market share in the deposit market and a less stringent lending standard compared to the small bank.

**Proposition 6** Suppose the environment is such that the small bank’s profit function is strictly quasi-concave in its deposit interest rate and the large bank’s profit function is strictly decreasing in its deposit interest rate.\(^4\) Let \( r^+_S \in [s, f] \) be the unique solution to
\[
G'(r^+_S - s)(f - r^+_S) = G(r^+_S - s). \tag{39}
\]
If \( G'(r^+_S - s)(f - s) < 1 - G(r^+_S - s) \), then it is an equilibrium that the large bank and the small bank set the deposit interest rates \( s \) and \( r^+_S \), respectively. In this equilibrium, the large bank has a larger market share in the deposit market and a less stringent lending standard compared to the small bank.

Figure 7 illustrates the behavior of banks’ deposit interest rates and market shares as functions of the central bank’s interest rate (IOR), with and without the CBDC. We use the uniform distribution \( G(\delta) = \delta / 0.035 \). We assume that the CBDC pays an interest of

\(^4\)In particular, it means that \( G'(r^+_S - s)(f - s) - (1 - G(r^+_S - s)) < 0 \). Moreover, a sufficient condition for \( d\Pi / dr_L \) to remain strictly negative for \( r_L > s \) is that \( G'(r^+_S - r^+_L)(f - r^+_L) + G(r^+_S - r^+_L) \) is strictly decreasing in \( r^+_L \) for all \( r^+_L \in [s, r^+_S] \), i.e., \(-G''(r^+_S - r^+_L)(f - r^+_L) - 2G'(r^+_S - r^+_L) < 0\).
Figure 7: The effect of adding a CBDC on equilibrium behavior under a uniform distribution, \( G(\delta) = \delta/0.035 \). Only when \( f > 2\% \) is the CBDC added and it pays a rate \( s = 2\% \).

\( s = 2\% \). We do not expect that the central bank would ever set a CBDC interest rate above IOR, so the figure is only drawn for IOR rates above 2%. The presence of the CBDC increases the deposit interest rates paid by both types of banks, and the increase for the large bank is larger. After an initial jump in the large bank’s deposit interest rate at \( f = 2\% \), it becomes flat for a wider range of IOR. The addition of the CBDC also increases the large bank’s market share. Once the IOR reaches a sufficiently high level, which in this example is \( f = 4.33\% \), the lower bound of CBDC stops being binding and everything becomes identical to the world without the CBDC.

We are interested in how market outcomes change as \( s \) changes, while holding \( f \) fixed. Differentiating (36), holding \( f \) fixed, we get

\[
\begin{align*}
&[G''(r^+_S - s)(f-r^+_S) - G'(r^+_S - s) - G'(r^+_S - s)]dr_s - [G''(r^+_S - s)(f-r^+_S) - G'(r^+_S - s)]ds = 0, \\
&\text{(40)}
\end{align*}
\]

which yields

\[
\frac{dr^+_S}{ds} = \frac{G''(r^+_S - s)(f-r^+_S) - G'(r^+_S - s)}{G''(r^+_S - s)(f-r^+_S) - 2G'(r^+_S - s)}. \\
\text{(41)}
\]

The sufficient condition that guarantees the uniqueness of the constrained solution implies that the denominator is negative. Hence, as \( s \) increases, the deposit interest rate on small banks increases if \( G''(r^+_S - s)(f-r^+_S) < G'(r^+_S - s) \). In the uniform case, where \( G''(r^+_S) = 0 \), \( \frac{dr^+_S}{ds} = \frac{1}{2} \). But if \( G'(r^+_S - s) < G''(r^+_S - s)(f-r^+_S) < 2G'(r^+_S - s) \), we have the more surprising outcome that the small bank’s deposit interest rate decreases in \( s \).
What about the market shares in the deposit market? The small bank’s market share is $\alpha_S = G(r_S^{++} - s)$. We have

$$\frac{d\alpha_S}{ds} = G'(r_S^{++} - s) \left( \frac{dr_S^{++}}{ds} - 1 \right).$$  \hspace{1cm} (42)

From the expression of $\frac{dr_S^{++}}{ds}$, we have

$$\frac{dr_S^{++}}{ds} - 1 = \frac{G'(r_S^{++} - s)}{G''(r_S^{++} - s)(f - r_S^{++}) - 2G'(r_S^{++} - s)} < 0.$$  \hspace{1cm} (43)

Thus, $\frac{d\alpha_S}{ds} < 0$, i.e., the small bank’s market share decreases in $s$.

Returning, yet again, to the uniform distribution, where $G(\delta) = \delta/\Delta$ for a sufficiently large $\Delta$. The lower bound on the deposit interest rates is the CBDC interest rate $s$. When this lower bound is not binding, we can solve, as before, $r_S^{++} = f - \Delta/3$ and $r_L^{++} = f - 2\Delta/3$. The condition for the lower bound not binding is $r_L^{++} > s$, or $\Delta < \frac{3}{2}(f - s)$. When the lower bound of $s$ is binding for the large bank, $r_S^{++}$ solves $(f - r_S^{++})/\Delta - (r_S^{++} - s)/\Delta = 0$, i.e., $r_S^{++} = (f + s)/2 \in (s, f)$. We also need the large bank’s first-order condition $G'((f - s)/2)(f - s) - (1 - G((f - s)/2)) \leq 0$, i.e., $\Delta \geq \frac{3}{2}(f - s)$. In this example, a higher $s$ raises the deposit interest rate at the small bank and weakens the requirement for (35) to hold. The implication of the latter effect is that the large bank deposit interest rate remains flat longer than in the cash case.

Finally, we turn to lending. In the constrained equilibrium with the CBDC, the quality thresholds of the loans are

$$q_L^* = \frac{1 + f + \lambda - \alpha_L(f - s)}{A + \lambda},$$ \hspace{1cm} (44)

$$q_S^* = \frac{1 + f + \lambda - \alpha_S(f - r_S^{++})}{A + \lambda}.$$ \hspace{1cm} (45)

Both thresholds can be ambiguous in $s$. Although a higher $s$ reduces $\alpha_S$ and increases $\alpha_L$, it reduces the large bank’s deposit interest rate spread $f - s$, so $\alpha_L(f - s)$ may be increasing or decreasing. Likewise, while a higher $s$ reduces $\alpha_S$, its impact on $r_S^{++}$ is unclear. If $r_S^{++}$ increases in $s$, then $\alpha_S(f - r_S^{++})$ decreases in $s$ and $q_S^*$ increases in $s$. Finally, the impact of $s$ on loan volumes at the two banks, $\alpha_L(1 - Q(q_L^*))$ and $\alpha_S(1 - Q(q_S^*))$, is also ambiguous.

**Proposition 7** In the constrained equilibrium with the CBDC, an increase in the CBDC interest rate $s$ leads to:

- A higher deposit interest rate at the small bank if $G''(r_S^{++} - s)(f - r_S^{++}) < G'(r_S^{++} - s)$ and a lower deposit interest rate at the small bank if $G'(r_S^{++} - s) < G''(r_S^{++} - s)(f - r_S^{++}) < 2G'(r_S^{++} - s)$.

- A smaller market share of the small bank in the deposit market.
Ambiguous changes in loan quality thresholds and loan volume in general. But if \( dr_{S}^{++}/ds > 0 \), then the small bank’s loan quality threshold increases and loan volume decreases in s.

The following example illustrates the ambiguous comparative statics with respect to the CBDC interest rate. Suppose that the convenience value \( \delta \) of the large bank has the following distribution function

\[
G(\delta) = \frac{(d + \epsilon - \delta)^{-1} - (d + \epsilon)^{-1}}{\epsilon^{-1} - (d + \epsilon)^{-1}}, \quad \text{for } \delta \in [0, d],
\]

(46)

where \( d > 0 \) and \( \epsilon > 0 \) are constants. The density,

\[
G'(\delta) = \frac{1}{\epsilon^{-1} - (d + \epsilon)^{-1}} \frac{1}{(d + \epsilon - \delta)^2},
\]

(47)

is increasing in \( \delta \in [0, d] \). Appendix B characterizes conditions under which the equilibrium characterization in Proposition 6 is valid for the distribution specified in (46).

As illustrated in Figure 8 below, the distribution specified in (46) leads to a U-shaped deposit interest rate of the small bank as the CBDC interest rate increases. The large bank’s deposit interest rate is equal to the CBDC interest rate. The market share of the small bank in the deposit market declines in the CBDC interest rate, which is theoretically unambiguous. Figure 9 shows that the large bank’s loan quality threshold can also be non-monotone in the CBDC interest rate. But, because the large bank’s market share increases in the CBDC interest rate, loan volume increases for the large bank and decreases for the small bank.

Some theoretical ambiguities in Proposition 7 can be eliminated for a class of distribution functions \( G \), as shown in the following corollary.

**Corollary 1** If the distribution \( G \) is such that \( G'/G \) is decreasing, then an increase in the CBDC interest rate \( s \) in the constraint equilibrium with CBDC leads to a higher deposit interest rate, a higher loan quality threshold, and a lower loan volume at the small bank.

To see why an decreasing \( G'/G \) eliminates the theoretical ambiguity about the small bank’s deposit interest rate, note that the combination of \( G'(r_{S}^{++} - s) < G''(r_{S}^{++} - s)(f - r_{S}^{++}) \) and the first-order condition \( G'(r_{S}^{++} - s)(f - r_{S}^{++}) = G(r_{S}^{++} - s) \) implies \( [G'(r_{S}^{++})]^2 < G''(r_{S}^{++} - s)G(r_{S}^{++} - s) \). But a decreasing \( G'/G \) means that for all \( \delta \), \( G''(\delta)G(\delta) < [G'(\delta)]^2 \), which is contrary to the required condition for an ambiguous change in \( r_{S}^{++} \). Once the small bank’s deposit interest rate is increasing in the CBDC interest rate, the implications for its loan quality threshold and loan volume would follow. Note, however, that Corollary 1 does not eliminate the theoretical ambiguity about the large bank’s loan quality threshold.

We illustrate the results of Proposition 7 and Corollary 1 for the uniform distribution, which satisfies the condition that \( G'/G \) is decreasing. Figures 10 and 11 show the impact of changes in the CBDC interest rate \( s \) in the deposit and loan markets when the IOR rate is fixed at 2%. In the case of uniform \( G \) a higher CBDC interest rate leads to a higher deposit interest rate for the both banks (left panel of Figure 10). Moreover, at the point where
Figure 8: Equilibrium deposit market as CBDC interest rate varies from 0 to 2%. Convenience values of the large bank have the distribution (46) with $\epsilon = 0.002$ and $d = 0.02$.

Figure 9: Equilibrium loan market as CBDC interest rate varies from 0 to 2%. Convenience values of the large bank have the distribution (46) with $\epsilon = 0.002$ and $d = 0.02$.

$s = f$ the rates are equal at $s$ (this convergence occurs for any distribution $G$). Because the relative increase in the large bank’s (constrained) deposit interest rate is greater than the small banks, we see a corresponding shift in market share away from the small bank and
toward the large bank as $s$ increases (right panel of Figure 10).

Figure 10: Equilibrium deposit market as CBDC interest rate varies from 0 to 2%. Convenience values of the large bank have a uniform distribution, $G(\delta) = \delta/0.035$.

Turning to Figure 11, loan quality thresholds for both banks are increasing in $s$, which follows immediately from the fact that, in this example, small bank deposit interest rates are increasing in $s$ (left panel of Figure 11). These thresholds converge to the same value at the point where $s = f$, since at this point the profit on deposits from a loan (see equation (3)) is the same (zero) for each bank. The corresponding changes in loan volume for an increase in $s$ are the opposite of what we saw for an increase in $f$, reflecting the aforementioned fact that an increase in $s$ has a greater impact on the loan quality threshold of the large bank than the small bank (right panel of Figure 11).

6 Direct Central Bank Account and Narrow Bank

So far, we considered a CBDC that is offered through commercial banks and that inherits the convenience value of deposits from the host bank. In this section, we consider a different design: direct central bank accounts that do not go through commercial banks. For example, if the central bank offers a separate mobile App for users, we would consider such a design as a direct account. In this case, the separate CBDC mobile App does not inherit the functionalities that commercial bank Apps or websites already perform, such as money transfer and paperless check deposits. A narrow bank whose only objective is to collect deposits, put it in the central bank balance sheet, and collect interest may be viewed as a direct account because it does not inherently have the payment functionality of commercial bank deposits.
As a conservative modeling choice, we assume that the direct CBDC has the same convenience value of the small bank, which we normalize to 0. The convenience value of the large bank’s deposit is still $\delta$, distributed across agents according to $G$. Let $s$ be the CBDC interest rate, as before. In this design, it is the small bank’s deposit interest rate that may be bound by the CBDC interest rate $s$; if $r_S < s$, the CBDC account strictly dominates the small bank’s deposits. The large bank may offer $r_L < s$ because of the extra convenience value. The large bank’s market share is still $1 - G(r_S - r_L)$ and the small bank’s is $G(r_S - r_L)$. The small bank’s deposit interest rate has a lower bound of $s$, and the large bank’s deposit interest rate has a lower bound of 0.\footnote{We assume the $\Delta$ is sufficiently large so that the large bank is able to attract some customers at near zero rates, despite the existence of a CBDC that pays interest rate $s$.}

1. \textit{Unconstrained equilibrium}. Let $r_S$ and $r_L$ solve the following two equations:

$$G'(r_S - r_L)(f - r_L) = 1 - G(r_S - r_L),$$

$$G'(r_S - r_L)(f - r_S) = G(r_S - r_L).$$

If $r_S > s$ and $r_L > 0$, then it is an unconstrained equilibrium.

2. \textit{Large bank constrained equilibrium}. Let $r_S$ solve the following equation:

$$G'(r_S)(f - r_S) = G(r_S).$$

Figure 11: Equilibrium loan market as CBDC interest rate varies from 0 to 2%. Convenience values of the large bank have a uniform distribution, $G(\delta) = \delta/0.035$.\footnote{We assume the $\Delta$ is sufficiently large so that the large bank is able to attract some customers at near zero rates, despite the existence of a CBDC that pays interest rate $s$.}
If \( r_S > s \) and \( G'(r_S)f \leq 1 - G(r_S) \), then it is a large-bank constrained equilibrium.

3. **Small bank constrained equilibrium.** Let \( r_L \) solve the following equation:

\[
G'(s - r_L)(f - r_L) = 1 - G(s - r_L).
\]  

(51)

If \( r_L > 0 \) and \( G'(s - r_L)(f - s) \leq G(s - r_L) \), then it is a small-bank constrained equilibrium.

4. **Both bank constrained equilibrium.** If

\[
G'(s)(f - s) \leq G(s),
\]  

(52)

\[
G'(s)f \leq 1 - G(s),
\]  

(53)

then \( r_L = 0 \) and \( r_S = s \) is a both-bank constrained equilibrium.

Let’s look at the uniform example again, with \( G(\delta) = \delta/\Delta \). The following table summarizes the four equilibrium cases.

<table>
<thead>
<tr>
<th>Constrained</th>
<th>( r_L )</th>
<th>( r_S )</th>
<th>( r_L ) cond.</th>
<th>( r_S ) cond.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neither</td>
<td>( f - \frac{2}{3}\Delta )</td>
<td>( f - \frac{4}{3}\Delta )</td>
<td>( f &gt; \frac{5}{3}\Delta )</td>
<td>( f &gt; s + \frac{1}{3}\Delta )</td>
</tr>
<tr>
<td>Large</td>
<td>0</td>
<td>( \frac{f}{2} )</td>
<td>( f \leq \frac{5}{3}\Delta )</td>
<td>( f &gt; 2s )</td>
</tr>
<tr>
<td>Small</td>
<td>( \frac{f + s - \Delta}{2} )</td>
<td>( s )</td>
<td>( f + s &gt; \Delta )</td>
<td>( f \leq s + \frac{1}{3}\Delta )</td>
</tr>
<tr>
<td>Both</td>
<td>0</td>
<td>( s )</td>
<td>( f + s &lt; \Delta )</td>
<td>( f &lt; 2s )</td>
</tr>
</tbody>
</table>

Table 1: Four equilibrium cases when CBDC is offered independently. The solutions are obtained under a uniform distribution of convenience value, \( G(\delta) = \delta/\Delta \).

The following figure illustrates the four cases of equilibrium. The point that the four lines intersect is \( s = \Delta/3 \) and \( f = 2\Delta/3 \).

In the direct CBDC design, the small bank’s deposit interest rate is more sensitive to the CBDC interest rate \( s \). Moreover, it is clear from Figure (12) that a very small \( s \) does not have any impact on deposit interest rate because the equilibrium is in “large bank constrained” or “neither bank constrained,” in which cases neither bank’s deposit interest rate depends on \( s \). Only if \( s \) is sufficiently large is the equilibrium in “small bank constrained,” where \( dr_L/ds = 1/2 \) and \( dr_S/ds = 1 \), or in “both bank constrained,” where \( dr_L/ds = 0 \) and \( dr_S/ds = 1 \). In contrast, if the CBDC is offered through commercial banks and inherits the convenience value of bank deposits, deposit interest rates respond to changes in the CBDC interest rate even when the latter is small.
Figure 12: Four cases of equilibrium under an independent CBDC. The graph assumes an uniform distribution of convenience value, $G(\delta) = \delta/\Delta$, and shows the parameter range $s < \Delta$ and $s < f$.

7 Concluding Remarks

Our objective in this study is to understand how an interest-bearing CBDC might impact deposit and lending markets. Our institutional setup is most suited for the United States in the current environment, where there are large quantities of excess reserves and an administered rate, the interest on reserves, that determines banks’ opportunity cost of lending. We construct a model that describes how banks set deposit interest rates and decide on the project quality threshold for lending. In this regard we believe it is crucial to consider multiple, heterogeneous banks. Heterogeneous banks are required to capture differences in depositor preferences for banks of different sizes, where size is proxy for an array of preferred services and economies of scale provided by larger banks. It turns out that this modelling feature is key to understanding the impact of the zero lower bound, since differences in preferences for deposits at banks of various sizes lead to an array of equilibrium scenarios that differ in terms of whether or not the zero-lower bound is binding and how responsive deposit interest rates are to changes in the IOR rate.

The first test of our model is its ability to capture a key feature of deposit interest rates: the lack of responsiveness to changes in the IOR rate at low levels. This rigidity arises
in our model because banks have different opportunity costs of credit provision due to the differing likelihoods that loan reserves will return as new deposits. This introduces a channel through which changes in the opportunity cost of credit provision — federal funds rate or IOR, whichever is higher — leads to changes not only in deposit interest rates at the small and large banks, but also in the share of depositors at each bank. Interestingly, an increase in this opportunity cost reduces lending at the large bank, but leads to an ambiguous effect on lending volume at the small bank. Nevertheless, the overall effect is necessarily a reduction in total lending.

Our initial approach to modelling a CBDC is to assume it would be offered via the commercial banks and inherit the convenience of the host institution. In this setting, an interest-bearing CBDC leads to a lower market share for small banks and can lead to lower deposit interest rates for small banks. When all deposit interest rates increase in response to the introduction of an interest-bearing CBDC, the trade-off for loan market is the usual quality vs quantity one: A higher CBDC interest rate will increase the quality of loan portfolios, but reduce quantity.

An alternative approach is to assume that the CBDC is offered directly by the central bank, perhaps utilizing services and location of a government entity such as the post office. This case is interesting because it fits the proposal outlined in the Banking for All Act, and it also captures the scenario where a CBDC substitute is offered by a narrow bank. In this case, we assume the CBDC inherits the convenience value of the small bank. This approach opens up the possibility for more equilibrium scenarios in which a lower bound on deposit interest rates is binding on the banks. In particular, it is possible to have equilibria in which the small bank is constrained by the lower bound in equilibrium and the large bank is not. Under this design, the CBDC interest rate needs to be set sufficiently high in order to have any impact at all.

An interesting aspect of multiple previous studies on CBDC is that its provision impacts equilibrium outcomes despite not being held in equilibrium. This is true, for example, in Chiu et al. (2019) and Garratt and Lee (2020), where the option to use CBDC changes the equilibrium outcome even if it is not exercised. An exception is Keister and Sanches (2020), where the CBDC has specific liquidity benefits that leads to its use. In the environment of indirect CBDC provision by banks that we consider in this paper, small bank customers strictly prefer to hold deposits (because the small bank pays a higher interest rate than the CBDC interest rate), while the allocation between CBDC and deposits at the large bank is indeterminate (the two forms of money pay the same interest and are equally convenient). In our model, this indeterminacy does not matter much, since banks are flush with reserves and are not constrained in terms of lending by how much they have in deposits. In practice, the composition of balances at large banks might be determined by tie-breakers (e.g., social or political views) that are outside this model.
Appendix

A Verify the Equilibrium for the Uniform Distribution

In this appendix, we verify the equilibrium for the uniform distribution \( G \). We will do so in the economy that has the CBDC; the world without CBDC is equivalent to setting \( s = 0 \).

For the small bank, we have

\[
\frac{d\Pi_s}{ds} = [G'(r_s^+ - r_l^+)(f - r_s^+) - G(r_s^+ - r_l^+)](1 - Q(q_s^+)).
\] (54)

We want to show that the term in the squared bracket is positive if \( r_s^+ \) is below a cutoff and negative if \( r_s^+ \) is above that cutoff. Its derivative with respect to \( r_s^+ \) is \( G''(r_s^+ - r_l^+) - 2G'(r_s^+ - r_l^+) \), which is strictly negative for the uniform distribution \( G \). Hence, \( \Pi_s \) has a single peak in \( r_s^+ \in [r_l^+, f] \). The peak does not happen at \( r_s^+ = r_l^+ \) because at this point \( d\Pi_s/dr_s < 0 \). Nor does the peak happen at \( r_s^+ = f \) for the same reason. Thus, the unique maximum of \( \Pi_s \) is obtained from the first-order condition \( d\Pi_s/dr_s = 0 \).

For the large bank,

\[
\frac{d\Pi_L}{dr_L} = [G'(r_s^+ - r_L^+)(f - r_L^+) - (1 - G(r_s^+ - r_L^+))](1 - Q(q_L^+)).
\] (55)

For the uniform distribution, the derivative of the terms in the square bracket with respect to \( r_L^+ \) is \( -G''(r_s^+ - r_L^+) - 2G'(r_s^+ - r_L^+) \), which is strictly negative. If \( G'(r_s^+ - s)(f - s) - (1 - G(r_s^+ - s)) < 0 \), then \( d\Pi_L/dr_L < 0 \) for all \( r_L^+ \in [s, r_s^+] \), and the equilibrium is constrained. If, however, \( G'(r_s^+ - s)(f - s) - (1 - G(r_s^+ - s)) \geq 0 \), then the optimal \( r_L^+ \) is given by the first-order condition \( d\Pi_L/dr_L = 0 \), which is the unconstrained equilibrium.

B Verify the Constrained Equilibrium for Distribution

(46)

In this appendix, we verify that a sufficient condition for the constrained equilibrium under the distribution function (46) is that \( d \geq f \) and the positive constant \( \epsilon \) is sufficiently small.

As before,

\[
\frac{d\Pi_s}{ds} = [G'(r_s^+ - s)(f - r_s^+) - G(r_s^+ - s)](1 - Q(q_s^+)).
\] (56)

We want to find conditions under which \( G'(r_s^+ - s)(f - r_s^+) - G(r_s^+ - s) < 0 \) if and only if \( r_s^+ \) is smaller than a cutoff value. We have

\[
f - r_s^+ < \frac{G(r_s^+ - s)}{G'(r_s^+ - s)} = [d + \epsilon - (r_s^+ - s)] - \frac{(d + \epsilon - (r_s^+ - s))^2}{d + \epsilon}.
\] (57)
At \( r_S^{++} = s \), the left-hand side is \( f - s \geq 0 \) and the right-hand side is 0. At \( r_S^{++} = f \), the left-hand side is 0 and the right-hand side is

\[
(d + \epsilon - f + s) \left( 1 - \frac{d + \epsilon - f + s}{d + \epsilon} \right) > 0. \tag{58}
\]

Moreover, the derivative of the left-hand side with respect to \( r_S^{++} \) is \(-1\), whereas the derivative of the right-hand side is

\[
-1 + \frac{2(d + \epsilon - r_S^{++} + s)}{d + \epsilon} > -1. \tag{59}
\]

Therefore, the function \( G'(r_S^{++} - s)(f - r_S^{++}) - G(r_S^{++} - s) \) is positive if \( r_S^{++} \) is below a cutoff value and turns negative if \( r_S^{++} \) is above a cutoff value. Thus, the small bank’s optimal deposit interest rate \( r_S^{++} \) is given by the first-order condition \( G'(r_S^{++} - s)(f - r_S^{++}) - G(r_S^{++} - s) = 0 \).

For the large bank,

\[
\frac{d\Pi_L}{dr_L} = [G'(r_S^{++} - r_L^{++})(f - r_L^{++}) - 1 + G(r_S^{++} - r_L^{++})](1 - Q(q_L)). \tag{60}
\]

We want to show that this derivative is negative for all \( r_L^{++} \in [s, r_S^{++}] \). The distribution (46) has \( G'(\delta) > 0 \). So, for any \( r_L^{++} \in [s, r_S^{++}] \), \( G'(r_S^{++} - r_L^{++})(f - r_L^{++}) - 1 + G(r_S^{++} - r_L^{++}) \) is decreasing in \( r_L^{++} \). It is sufficient to show that

\[
G'(r_S^{++} - s)(f - s) - 1 + G(r_S^{++} - s) < 0. \tag{61}
\]

Given the small bank’s first-order condition \( G'(r_S^{++} - s)(f - r_S^{++}) - G(r_S^{++} - s) = 0 \), it is equivalent to show \( G'(r_S^{++} - s)(2f - s - r_S^{++}) < 1 \), or

\[
2f - s - r_S^{++} < (\epsilon^{-1} - (d + \epsilon)^{-1})(d + \epsilon - r_S^{++} + s)^2, \tag{62}
\]

which holds for a sufficiently small (but positive) \( \epsilon \).
References


