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Bilateral trading in divisible double auctions ☆

Songzi Du [a,*], Haoxiang Zhu [b]

[a] *Simon Fraser University, Department of Economics, 8888 University Drive, Burnaby, BC V5A 1S6, Canada*
[b] *MIT Sloan School of Management, 100 Main Street E62-623, Cambridge, MA 02142, United States*

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Abstract

Existing models of divisible double auctions typically require three or more traders—when there are two traders, the usual *linear* equilibria imply market breakdowns unless the traders' values are negatively correlated. This paper characterizes a family of *nonlinear ex post equilibria* in a divisible double auction with only two traders, who have interdependent values and submit demand schedules. The equilibrium trading volume is positive but less than the first best. Closed-form solutions are obtained in special cases. Moreover, no nonlinear ex post equilibria exist if: (i) there are $n \geq 4$ symmetric traders or (ii) there are 3 symmetric traders with pure private values. Overall, our nonlinear equilibria fill the "$n = 2$" gap in the divisible-auction literature and could be a building block for analyzing strategic bilateral trading in decentralized markets.

JEL classification: D44; D82; G14

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1. Introduction

Trading with demand schedules, in the form of double auctions, is common in many financial and commodity markets. In a typical model of divisible double auctions, traders simultaneously

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* Corresponding author.
E-mail addresses: songzid@sfu.ca (S. Du), zhuh@mit.edu (H. Zhu).

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submit linear demand schedules (i.e., a set of limit orders, or price-quantity pairs), and trading occurs at the market-clearing price. A large literature is devoted to characterizing the trading behavior in this mechanism as well as the associated price discovery and allocative efficiency (see, for example, Kyle, 1989; Vayanos, 1999; Vives, 2011; Rostek and Weretka, 2012, and Du and Zhu, 2016, among others). These models of double auctions typically require at least three traders for the existence of linear equilibria.

When there are exactly two traders, the existing theory predicts a market breakdown (no trade) unless traders' values are negatively correlated. While the $n \geq 3$ assumption is relatively innocuous for centralized markets, it is restrictive for decentralized, over-the-counter (OTC) markets, where trades are conducted bilaterally. Active OTC markets for divisible assets include those for corporate bonds, municipal bonds, structured products, interbank loans, repurchase agreements, and security lending arrangements, as well as spot and forward transactions in commodities and foreign currencies.

In this paper, we fill this gap by studying bilateral trading in divisible double auctions, which is largely unexplored in the previous literature. In our model, each trader receives a one-dimensional private signal about the asset and values the asset at a weighted average of his and the other trader's signals. That is, values are interdependent. In addition, the trader's marginal value for owning the asset declines linearly in quantity. Moreover, the traders can be asymmetric, in the sense that their values can have different weights on each other's signal, and that their marginal values can decline at different rates.

We characterize a family of nonlinear equilibria in this model. These equilibria can be ranked by their realized allocative efficiency, suggesting that efficiency is a natural equilibrium selection criterion. In an equilibrium, each trader's demand schedule is implicitly given by a solution to a nonlinear algebraic equation. We show that each equilibrium leads to a trading quantity that is positive and strictly lower (in absolute values) than the first best (efficient quantity). This behavior is consistent with the "demand reduction" property commonly seen in multi-unit auctions (see, for example, Ausubel et al., 2014). Moreover, the equilibria that we characterize are ex post equilibria; that is, the equilibrium strategies remain optimal even if each trader would observe the private information of the other trader. In the special case of constant marginal values, we obtain a trader's equilibrium demand schedule in closed form: it is simply a constant multiple of a power function of the difference between the trader's signal and the price, where the exponent is decreasing in the weight a trader assigns on his own signal.

Do these nonlinear ex post equilibria also exist in markets with at least three traders? We show that no nonlinear ex post equilibria exist if: (i) there are at least four symmetric traders or (ii) there are three symmetric traders who have pure private values. Thus, under fairly general conditions the only ex post equilibrium is the linear one identified in previous models. Not only does this result provide a justification for the widespread use of the linear equilibrium in the existing literature, it also suggests that bilateral double auctions behave qualitatively differently from multilateral ones, and hence merit further investigation.

An interesting and useful direction of further exploration is to use our bilateral double auction result as a strategic building block for analyzing dynamic trading in large OTC markets. So far, in the most widely used class of OTC market models that start from Duffie et al. (2005), the two agents in a pairwise meeting observe, by assumption, each other's valuation of the asset or continuation value, and trading happens by Nash bargaining (a split of total surplus by fixed portions). In contrast, the bilateral double auction in our model endogenously reveals asymmetric information to both counterparties through their fully strategic interactions. Thus, our model provides a strategic microfoundation for bilateral information transmission.

More recently, Duffie et al. (DMM, 2014) use *indivisible* bilateral double auctions, adapted from the literature pioneered by Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989), to model bilateral trading in large OTC markets. Our model of divisible double auctions allows arbitrary quantities and is hence better suited for modeling trade size and trading volume in OTC markets. Moreover, relative to DMM, our model allows more general information structures such as interdependent values. Finally, our model of bilateral trade is more tractable than that of DMM, partly because the divisible double auction allows optimization price by price, greatly simplifying the problem. Of course, our model has only two agents and is static. Extending it to a fully dynamic market with many (perhaps a continuum of) agents is an intriguing and important challenge that we leave for future research.

Our paper is also broadly related to the mechanism-design approach to bilateral trading. For example, in a bilateral trading setting with interdependent values, finite signals, and constant marginal values, Shimer and Werning (2015) show that mechanisms satisfying ex post participation constraints also present a tension between achieving efficiency and having fully revealing prices. Applying a similar mechanism design approach to our setting seems an interesting exercise and is left for future research.

2. Model

There are $n = 2$ players, whom we call "traders," trading a divisible asset. The total supply of the asset is normalized to zero. Each trader i observes a private signal, $s_i \in [\underline{s}, \overline{s}] \subset \mathbb{R}$, about the value of the asset. The distribution of (s_1, s_2) is arbitrary. We use j to denote the trader other than i. Trader i's value for owning the asset is:

$$v_i = \alpha_i s_i + (1 - \alpha_i) s_j, \tag{1}$$

where $\alpha_1 \in (0, 1]$ and $\alpha_2 \in (0, 1]$ are commonly known constants that capture the level of interdependence in traders' valuations. We assume that $\alpha_1 + \alpha_2 > 1$. We do not need the assumption of $\alpha_i > 1/2$ (placing more weight on one's own signal).

Remark. Du and Zhu (2016) derive the symmetric case ($\alpha_1 = \alpha_2 \in (0, 1]$) of valuation (1) in a setting where traders have common and private values and observe private, noisy signals about the common value. We emphasize that the contribution of this paper is to analyze the case of $\alpha_i \in (0, 1]$, so each trader i places a nonnegative weight $1 - \alpha_i$ on the other's information. In all previous models of divisible double auctions, as long as values have a nonnegative correlation, the existence of linear equilibrium requires $n \geq 3$. If, however, $\alpha_i > 1$ for each $i \in \{1, 2\}$, the two traders' values become negatively correlated. In and only in this case of negative value correlation, the linear equilibrium in Vives (2011, p. 1941–2) and Rostek and Weretka (2012) generates a positive trading volume with two traders.

We further assume that trader i's marginal value for owning the asset decreases linearly in quantity at a commonly known rate $\lambda_i \geq 0$. Thus, if trader i acquires quantity q_i at the price p, trader i has the ex post utility:

$$U_i(q_i, p; v_i) = v_i q_i - \frac{\lambda_i}{2}(q_i)^2 - p q_i. \tag{2}$$

By construction, if $q_i = 0$, then $U_i = 0$. This linear-quadratic utility function is also used in Vives (2011), Rostek and Weretka (2012), Du and Zhu (2016), among others.

The trading mechanism is a one-shot divisible double auction. We use $x_i(\,\cdot\,; s_i)$, where $x_i(\,\cdot\,; s_i) : [\underline{s}, \overline{s}] \to \mathbb{R}$, to denote the demand schedule that trader i submits conditional on his signal s_i. The demand schedule $x_i(\cdot\,; s_i)$ specifies that trader i wishes to buy a quantity $x_i(p; s_i)$ of the asset at the price p when $x_i(p; s_i)$ is positive, and that trader i wishes to sell a quantity $-x_i(p; s_i)$ of the asset at the price p when $x_i(p; s_i)$ is negative.

Given the submitted demand schedules $(x_1(\cdot\,; s_1), x_2(\cdot\,; s_2))$, the auctioneer (a human or a computer algorithm) determines the transaction price $p^* \equiv p^*(s_1, s_2)$ from the market-clearing condition

$$x_1(p^*; s_1) + x_2(p^*; s_2) = 0. \tag{3}$$

After p^* is determined, trader i is allocated the quantity $x_i(p^*; s_i)$ of the asset and pays $x_i(p^*; s_i)p^*$. If no market-clearing price exists, there is no trade, and each trader gets a utility of zero. If multiple market-clearing prices exist, we can pick one arbitrarily.

We make no assumption about the distribution of (s_1, s_2). Therefore, the solution concept that we use is *ex post equilibrium*. In an ex post equilibrium, each trader has no regret—he would not deviate from his strategy even if he would learn the signal of the other trader.

Definition 1. An **ex post equilibrium** is a profile of strategies (x_1, x_2) such that for every profile of signals $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$, every trader i has no incentive to deviate from x_i, given the strategy x_j, $j \neq i$. That is, for any alternative strategy $\tilde{x}_i$ of trader i,

$$U_i(x_i(p^*; s_i), p^*; v_i) \geq U_i(\tilde{x}_i(\tilde{p}; s_i), \tilde{p}; v_i),$$

where v_i is given by Equation (1), p^* is the market-clearing price given x_i and x_j, and $\tilde{p}$ is the market-clearing price given $\tilde{x}_i$ and x_j.

In an ex post equilibrium, a trader can guarantee a non-negative ex post utility, since he can earn zero utility by submitting a demand schedule that does not clear the market (and hence trading zero quantity).

The ex post nature of the equilibrium implies that the equilibrium outcome is also robust to the way it is implemented. For instance, traders do not have to submit their demands at all prices simultaneously: they may go back and forth proposing prices and quantities that are subset of their demand schedules, until a market-clearing price emerges. Even if one trader learns something about the other's signal and demand schedule, the trader has no incentive to deviate from his ex post equilibrium demand schedule, as required by the ex post optimality condition. This robustness-to-implementation property is particularly desirable in bilateral trading in practice, where the bargaining protocol is usually not specified in rule books and is subject to high degrees of discretion and variation.

3. Characterize a family of ex post equilibria

We first define the sign function:

$$\operatorname{sign}(w) = \begin{cases} 1 & w > 0 \\ 0 & w = 0 \\ -1 & w < 0 \end{cases}. \tag{4}$$

Proposition 1. *Suppose that* $1 < \alpha_1 + \alpha_2 < 2$. *Let* C *be any positive constant such that*

$$C \geq \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_2}\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}, \text{ and} \tag{5}$$

$$C \geq \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1}\left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}. \tag{6}$$

Then, there exists a family (parameterized by C) of ex post equilibria in which:

$$x_i(p;s_i)=y_i(|s_i-p|)\cdot \operatorname{sign}(s_i-p),\ i\in\{1,2\}, \tag{7}$$

where, for $w_1, w_2 \in [0, \bar{s}-\underline{s}]$, $y_1(w_1)$ and $y_2(w_2)$ are the smaller solutions to

$$(2-\alpha_1-\alpha_2)w_1 = C\alpha_2 y_1(w_1)^{\alpha_1+\alpha_2-1}-\left(\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}\right)y_1(w_1), \tag{8}$$

$$(2-\alpha_1-\alpha_2)w_2 = C\alpha_1 y_2(w_2)^{\alpha_1+\alpha_2-1}-\left(\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}\right)y_2(w_2). \tag{9}$$

There is a unique equilibrium price $p^*=p^*(s_1,s_2)$, which is in between s_1 and s_2 and is given implicitly by[1]

$$p^*=\frac{\alpha_1 s_1+\alpha_2 s_2}{\alpha_1+\alpha_2}+\frac{\alpha_1\lambda_2-\alpha_2\lambda_1}{2(\alpha_1+\alpha_2)}x_1(p^*;s_1). \tag{10}$$

Moreover, among the family of equilibria, the one corresponding to the smallest C, subject to Conditions (5)–(6), maximizes trading volume and is the most efficient.

Fig. 1 demonstrates two equilibria of Proposition 1. The primitive parameters are $\alpha_1=0.7$, $\alpha_2=0.8$, $\lambda_1=0.1$, $\lambda_2=0.2$, $\underline{s}=0$, and $\bar{s}=1$. The realized signals are $s_1=0.3$ and $s_2=0.7$. On the left-hand plot, we show the equilibrium with $C=0.729$, which is the equilibrium that maximizes trading volume. The equilibrium price is $p^*=0.5126$, trader 1 gets $x_1(p^*;s_1)=-0.037$, and trader 2 gets $x_2(p^*;s_2)=0.037$. On the right-hand plot, we show the equilibrium with $C=1.458$. The equilibrium price is $p^*=0.513$, trader 1 gets $x_1(p^*;s_1)=-0.009$, and trader 2 gets $x_2(p^*;s_2)=0.009$.

While the full proof of Proposition 1 is provided in Section A.1, we briefly discuss its intuition here. The conditions (5)–(6) guarantee that the algebraic equations (8)–(9) have solutions. For example, the right-hand side of Equation (8), rewritten as

$$f_1(y_1)\equiv C\alpha_2 y_1^{\alpha_1+\alpha_2-1}-\left(\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}\right)y_1, \tag{11}$$

is clearly a concave function of y_1. Condition (5) ensures that the maximum of $f_1(y_1)$ is above $(2-\alpha_1-\alpha_2)(\bar{s}-\underline{s})$. Hence, by the Intermediate Value Theorem, a solution exists.

Moreover, whenever the inequality (5) is strict, there always exist two solutions $y_1(w_1)$, one of which is smaller than y_1^* and one larger than y_1^*, where y_1^* maximizes $f_1(y_1)$. Between the two, we select the former, for the following reason. It is easy to see that the smaller solution $y_1(w_1)$

[1] The uniqueness of solution p^* to Equation (10) can be seen as follows: suppose $\alpha_1\lambda_2>\alpha_2\lambda_1$, then the left-hand side of Equation (10) is increasing in p^*, while the right-hand side is decreasing in p^*.

If $\alpha_1\lambda_2<\alpha_2\lambda_1$, rewrite Equation (10) as:

$$p^*=\frac{\alpha_1 s_1+\alpha_2 s_2}{\alpha_1+\alpha_2}+\frac{\alpha_2\lambda_1-\alpha_1\lambda_2}{2(\alpha_1+\alpha_2)}x_2(p^*;s_2),$$

and then we can apply the above argument.

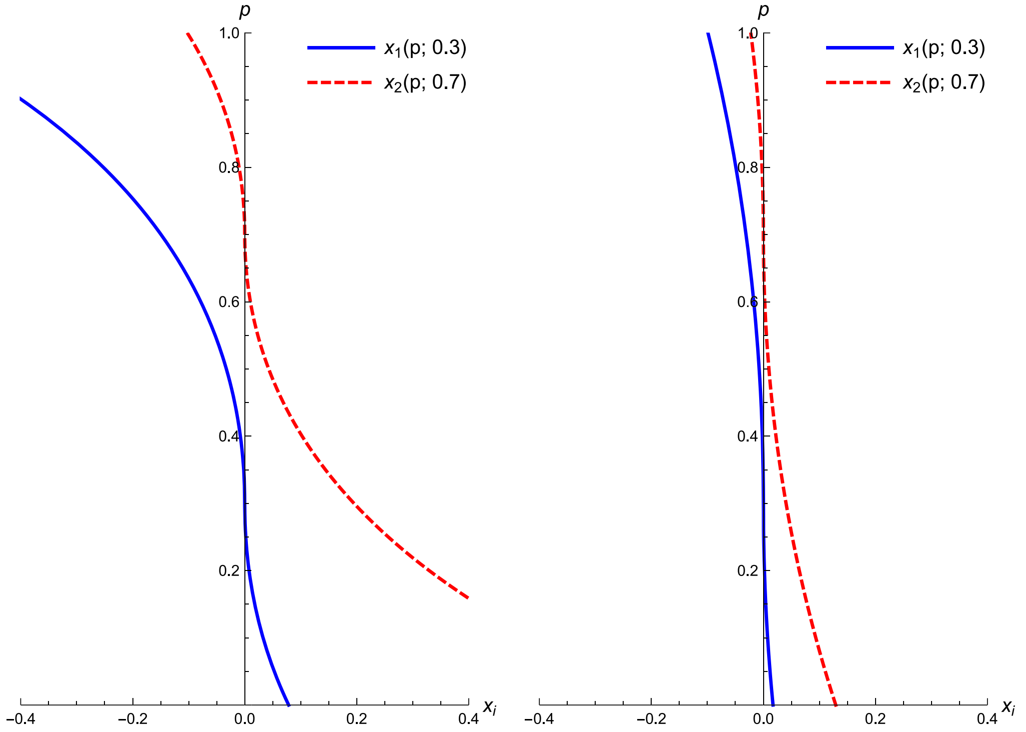

Fig. 1. Equilibria from Proposition 1.

is increasing in w_1 because $f_1(y_1)$ is increasing in y_1 before it obtains its maximum. This means that trader 1's demand $x_1(p; s_1)$ is decreasing in price p and is increasing in signal s_1, by (7). The other solution implies an upward-sloping demand schedule and should be discarded. Likewise for Condition (6) and Equation (9).

In these equilibria, each trader i buys $y_i(s_i - p)$ units of asset if the price p is below his signal s_i; he sells $y_i(p - s_i)$ units if p is above s_i. The constant C represents the aggressiveness of the bidding strategy; the smaller is C, the larger is $y_i(|s_i - p|)$, and hence the more aggressive the traders bid at each price.[2] The most aggressive equilibrium is also the most efficient one, because it maximizes the trading volume. In Section 3.2 we show that in all equilibria of Proposition 1 and for every realization of signals, the trading volume is less than that in the ex post efficient allocation, so a higher volume is closer to the ex post efficient allocation. On the other hand, as C tends to infinity, $y_i(w_i)$ tends to zero, and hence the amount of trading in equilibrium tends to zero. Among this family of ex post equilibria, the most efficient one is a natural candidate for equilibrium selection. It is also worth mentioning that we have not ruled out the existence of equilibria with other functional forms for demand.

As illustrated in Fig. 1, the equilibrium demand $x_i(p; s_i)$ is a concave function of p for $p \le s_i$ and a convex function of p for $p \ge s_i$. This is a consequence of $y_i(w_i)$ being convex in w_i (see Lemma 1). Put differently, each trader's price impact of buying or selling an additional marginal

[2] See Equations (8) and (9): as C gets larger, $y_i(w_i)$ must become smaller since the left-hand sides of (8) and (9) are not changing.

unit is decreasing in the overall (unsigned) traded quantity. To see this, note that the price impact of trader $j \neq i$ in the bilateral double auction can be measured by $1/|\frac{\partial x_i}{\partial p}| = 1/y_i'(w_i)$, where $w_i = |s_i - p|$. So the price impact is infinite if $s_i = p$ (since $y'(0) = 0$ from Lemma 1) and is smaller if s_i is further away from p (since $y_i'(w_i)$ is increasing in w_i). In general, the price impact is larger if $|s_i - p|/(\overline{s} - \underline{s})$ is smaller. If $\underline{s} \to -\infty$ or $\overline{s} \to \infty$, for any fixed $s_i - p$, $|s_i - p|/(\overline{s} - \underline{s}) \to 0$, and the price impact at such p becomes infinitely large, so its trading volume vanishes. That is, if the support of signals and prices is literally infinite, then the price impact faced by each trader is infinite almost everywhere, and the only equilibrium is the no trade equilibrium. However, as long as $\underline{s}$ and $\overline{s}$ are finite, trading volume is positive unless $s_i = s_j$, and the trading volume at a price far away from s_i and s_j is still large.

An economic interpretation of the nonlinear demand schedule and the decreasing price impact in quantity is the following. Without loss of generality, consider the buyer. In the nonlinear equilibrium, the equilibrium price is increasing and concave in the buyer's purchase quantity (see Fig. 1 for illustration). The strategic buyer wishes to engage in demand reduction to lower the price, but for each unit of demand reduction, the corresponding price reduction will be proportionally smaller than what it would be if the seller's demand schedule were linear. This nonlinear effect discourages demand reduction, especially if the traded quantity is already large. That the price impact is smaller for larger quantities resembles the familiar idea in mechanism design that "low types" are penalized so that high types would not imitate them (the incentive compatibility condition). From the perspective of the seller who posts the schedule, a "high type" is a buyer whose signal is far above the seller's signal, and a "low type" is a buyer whose signal is just above the seller's signal. Again, it is as if the seller sets a highly punitive price impact at a price close to her signal so as to prevent a high type buyer from "imitating" a low type one. Now suppose that we push the support of signals to the entire real line. Then, any finite signal of the buyer, say s_1, appears very low relative to the best possible realization, $+\infty$. The seller then sets a punitive, in fact infinite, price impact at the price p that is generated by s_1 and the seller's signal s_2. As s_1 varies, this price p spans the entire real line, and no trade takes place.[3]

Our nonlinear equilibria are broadly related to those of Glebkin (2015), who analyzes strategic trading among at least three strategic traders with symmetric information, CARA utility, and a general asset payoff distribution (in particular, without assuming normality). His equilibria and ours both demonstrate nonlinear price impact in traded quantity, but the shapes of the reactions are sometimes different. In our model, the equilibrium price is a concave function of the quantity demanded and is symmetric for buys and sells. But in his model, depending on the asset payoff distribution, the equilibrium price may be concave or convex in quantity, and the magnitudes may differ between buys and sells. This difference is likely due to the difference between the CARA utility in his model and the linear-quadratic utility in ours. Glebkin's model offers richer shapes of price impact in a centralized market with at least three traders, whereas our model is applicable to bilateral trading with interdependent values.[4] The two papers are hence complementary.

[3] Of course, this mechanism design analogy is informal and meant to illustrate the intuition. Formally applying the mechanism design approach to our setting (for example, by following Shimer and Werning, 2015) is beyond the scope of the paper.

[4] Glebkin (2015) does not consider the case of exactly two traders. In his setting, if there are only two traders and if the asset payoff is normal, an equilibrium does not exist. It remains an open question of under what payoff distributions an equilibrium would exist in Glebkin's setting if there are only two traders.

3.1. Special cases of Proposition 1

3.1.1. Constant marginal values

In the special case that the marginal values do not decline with quantity ($\lambda_1 = \lambda_2 = 0$), we obtain explicit closed-form solutions.

Corollary 1. *Suppose that $\alpha_1 + \alpha_2 > 1$ and $\lambda_1 = \lambda_2 = 0$. There exists a family of ex post equilibria in which:*

$$x_i(p; s_i) = C|\alpha_i(s_i - p)|^{\frac{1}{\alpha_1+\alpha_2-1}} \cdot \mathrm{sign}(s_i - p),\ i \in \{1, 2\}, \tag{12}$$

where C is any positive constant, and the equilibrium price is independent of C and is given by

$$p^*(s_1, s_2) = \frac{\alpha_1}{\alpha_1 + \alpha_2} s_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} s_2. \tag{13}$$

Corollary 1 shows that if $\lambda_1 = \lambda_2 = 0$, the equilibrium price $p^*(s_1, s_2)$ tilts toward the signal of the trader who assigns a larger weight on his private signal.

Corollary 1 imposes no restriction on the positive constant C since the marginal value does not decline with quantity. While trading volume can be unbounded in theory, in practice it is often bounded by institutional constraints. For instance, a bank may have an internal risk management policy that mandates an explicit maximal loan amount, say $Q > 0$, to each firm it lends to. In this case, the marginal value of the bank or the borrower is constant if the quantity $q_i \in [-Q, Q]$ and is zero if $q_i \notin [-Q, Q]$. The equilibria in Corollary 1 apply to this situation if the constant C satisfies $|x_i(p; s_i)| \le Q$ for every $(p, s_i) \in [\underline{s}, \overline{s}]^2$ and $i = 1, 2$, and the most aggressive equilibrium is one such that $\max(x_1(\underline{s}; \overline{s}), x_2(\underline{s}; \overline{s})) = Q$.

We now construct the equilibria for Corollary 1 to illustrate the general construction in Proposition 1.

Given a signal profile (s_1, s_2), trader i's ex post optimization problem is essentially selecting a market-clearing price p and getting the residual supply $-x_j(p; s_j)$, $j \ne i$. Thus, his ex post first order condition is that of a monopsonist facing a market supply of $-x_j(p; s_j)$:

$$\left.\frac{\partial U_i}{\partial p}\right|_{p=p^*} = x_j(p^*; s_j) + (\alpha_i s_i + (1 - \alpha_i)s_j - p^*)\left(-\frac{\partial x_j}{\partial p}(p^*; s_j)\right) = 0. \tag{14}$$

Let us conjecture that the market-clearing price p^* satisfies Equation (13). Consequently trader i can infer s_j from p^*. Substituting (13) into Equation (14), we get:

$$x_j(p^*; s_j) + (1 - \alpha_i - \alpha_j)(s_j - p^*)\left(-\frac{\partial x_j}{\partial p}(p^*; s_j)\right) = 0, \tag{15}$$

i.e., trader i's ex post first order condition becomes a differential equation on trader j's strategy. It is easy to solve the above differential equation:

$$x_j(p^*; s_j) = K_j(s_j - p^*)^{1/(\alpha_i+\alpha_j-1)}, \tag{16}$$

for any constant K_j. We can choose a constant K_j for $s_j > p^*$, and another one for $s_j < p^*$. Thus we let

$$x_j(p^*; s_j) = C_j|s_j - p^*|^{1/(\alpha_i+\alpha_j-1)}\,\mathrm{sign}(s_j - p^*) \tag{17}$$

for a positive constant C_j, to make $x_j(p^*; s_j)$ a decreasing function of p^* and thus a legitimate demand schedule. To satisfy our conjecture in Equation (13), we let $C_j = C \cdot (\alpha_j)^{1/(\alpha_i+\alpha_j-1)}$ for a positive constant C, $j = 1, 2$, which gives the equilibrium strategy in Corollary 1. Finally, we explicitly verify that our construction satisfies the ex post optimality condition in Section A.1.3.

3.1.2. Pure private values

Pure private values correspond to $\alpha_1 = \alpha_2 = 1$. Strictly speaking, private values are not covered by Proposition 1, but one can easily obtain ex post equilibria using the same line of arguments as in Proposition 1.

Corollary 2. *Suppose that $\alpha_1 = \alpha_2 = 1$. Let C_1 and C_2 be positive constants satisfying*

$$C_1 - C_2 = \frac{\lambda_1 - \lambda_2}{2}, \tag{18}$$

and

$$C_i \geq \frac{\lambda_1 + \lambda_2}{2}\left(\log\frac{2(\bar{s} - \underline{s})}{\lambda_1 + \lambda_2} + 1\right), i \in \{1, 2\}. \tag{19}$$

There exists a family (parameterized by C_1 and C_2) of ex post equilibria in which:

$$x_i(p; s_i) = y_i(|s_i - p|) \cdot \text{sign}(s_i - p),\ i \in \{1, 2\}, \tag{20}$$

where, for $w_i \in [0, \bar{s} - \underline{s}]$, $y_i(w_i)$ is the smaller solution to

$$w_i = C_i y_i(w_i) - \frac{\lambda_1 + \lambda_2}{2} y_i(w_i) \log(y_i(w_i)). \tag{21}$$

There is a unique equilibrium price $p^* = p^*(s_1, s_2)$, which is in between s_1 and s_2 and is given implicitly by

$$p^* = \frac{s_1 + s_2}{2} + \frac{\lambda_2 - \lambda_1}{4} x_1(p^*; s_1). \tag{22}$$

Moreover, among the family of equilibria, the one corresponding to the smallest C_1 and C_2, subject to Conditions (18)–(19), maximizes trading volume and is the most efficient.

As in Proposition 1, the right-hand side of Equation (21) is a concave function of $y_i(w_i)$. Condition (19) ensures that Equation (21) has two solutions in $y_i(w_i)$, and the smaller of the two solutions increases with w_i. Moreover, the smaller is the constant C_i, the more aggressive is trader i's equilibrium strategy in Equation (21). Hence, the most efficient equilibrium of this family corresponds to the smallest possible C_1 and C_2.

3.1.3. Pure common value

Pure common value corresponds to the case of $\alpha_1 + \alpha_2 = 1$. While pure common value is not covered by our model, we cover cases arbitrarily close to pure common value, i.e., $\alpha_1 + \alpha_2 > 1$ can be arbitrarily close to 1. We show here that as we approach a common value setting the equilibrium trade disappears, which is consistent with the intuitions from the no trade theorem of Milgrom and Stokey (1982).

Corollary 3. *Suppose that $\lambda_1 + \lambda_2 > 0$. As $\alpha_1 + \alpha_2$ tends to 1, trading vanishes in every equilibrium (x_1, x_2) from Proposition 1:*

$$\lim_{\alpha_1 + \alpha_2 \to 1} \sup_{(p, s_i) \in [\underline{s}, \bar{s}]^2} |x_i(p; s_i)| = 0, \tag{23}$$

for every $i \in \{1, 2\}$.

We do not require a trader placing more weight on his own signal. We allow, for example, that $\alpha_1 = 1$ and $\alpha_2 = \epsilon > 0$, where ϵ is small. When the signals s_1 and s_2 are independent, this information structure corresponds to a "lemon" setting in which player 1 is perfectly informed about the almost-common value, while player 2 is very uninformed. The above result implies that the equilibrium trade must vanish as $\epsilon \to 0$.

3.2. Demand reduction and efficiency

Now, we turn to the efficiency properties of the bilateral ex post equilibria of Proposition 1. To guarantee the existence of the efficient allocation, we will assume that $\lambda_1 + \lambda_2 > 0$. Otherwise (i.e., if $\lambda_1 = \lambda_2 = 0$), the allocative efficiency can always be improved by moving a marginal unit of asset from the trader with a lower value to the trader with a higher value.

For welfare comparison, let us first sketch the competitive equilibrium. A competitive equilibrium demand schedule $x_i^c(p^c; s_i)$ takes the price as given (each trader has no effect on the price) and solves:

$$x_i^c(p^c; s_i) \in \operatorname*{argmax}_{q_i \in \mathbb{R}} v_i(s_i, p^c)q_i - \frac{\lambda_i (q_i)^2}{2} - p^c q_i, \tag{24}$$

where $v_i(s_i, p^c)$ is trader i's value given signal s_i and competitive equilibrium price p^c. With only two traders, the competitive equilibrium is meant to be a theoretical benchmark and not a realistic description of market reality.

Let us conjecture that the competitive equilibrium price satisfies:

$$p^c = a_1 s_1 + a_2 s_2. \tag{25}$$

Trader i infers from the competitive equilibrium price p^c:

$$v_i(s_i, p^c) = \alpha_i s_i + (1 - \alpha_i)s_j = \alpha_i s_i + (1 - \alpha_i)\frac{p^c - a_i s_i}{a_j}, \tag{26}$$

and as a result,

$$x_i^c(p^c; s_i) = \frac{1}{\lambda_i}\left(\alpha_i s_i + (1 - \alpha_i)\frac{p^c - a_i s_i}{a_j} - p^c\right). \tag{27}$$

The competitive equilibrium price satisfies $x_1^c(p^c; s_1) + x_2^c(p^c; s_2) = 0$, thus

$$p^c = \frac{\left(\frac{\alpha_1}{\lambda_1} - \frac{a_1(1-\alpha_1)}{a_2\lambda_1}\right)s_1 + \left(\frac{\alpha_2}{\lambda_2} - \frac{a_2(1-\alpha_2)}{a_1\lambda_2}\right)s_2}{\frac{1}{\lambda_1} - \frac{1-\alpha_1}{a_2\lambda_1} + \frac{1}{\lambda_2} - \frac{1-\alpha_2}{a_1\lambda_2}}. \tag{28}$$

Matching the above coefficients with those in (25) gives the following unique non-trivial solution:

$$a_1 = \frac{(1 - \alpha_2)\lambda_1 + \alpha_1\lambda_2}{\lambda_1 + \lambda_2}, \qquad a_2 = \frac{(1 - \alpha_1)\lambda_2 + \alpha_2\lambda_1}{\lambda_1 + \lambda_2}. \tag{29}$$

Substitute the above solution into Equations (25) and (27) gives the competitive equilibrium:

$$x_i^c(p; s_i) = \frac{\alpha_1 + \alpha_2 - 1}{\alpha_j\lambda_i + (1 - \alpha_i)\lambda_j}(s_i - p), \qquad j \neq i, \tag{30}$$

$$p^c = \frac{(1 - \alpha_2)\lambda_1 + \alpha_1\lambda_2}{\lambda_1 + \lambda_2}s_1 + \frac{(1 - \alpha_1)\lambda_2 + \alpha_2\lambda_1}{\lambda_1 + \lambda_2}s_2. \tag{31}$$

It is easy to see that the competitive equilibrium always obtains the ex post efficient allocation: for every $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$,

$$x_1(p^c; s_1) \in \operatorname*{argmax}_{q_1 \in \mathbb{R}} \; v_1 q_1 - \frac{\lambda_1}{2}(q_1)^2 + v_2(-q_1) - \frac{\lambda_2}{2}(-q_1)^2. \tag{32}$$

Proposition 2. *Suppose that* $\lambda_1 + \lambda_2 > 0$.

1. *For every ex post equilibrium* (x_1, x_2) *of Proposition 1 and for every signal profile* $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$*, the ex post equilibrium trades strictly less than the ex post efficient allocation:* $|x_1(p^*; s_1)| < |x_1^c(p^c; s_1)|$.
2. *The ex post equilibrium prices from Proposition 1 and the competitive equilibrium price are the same if and only if* $\lambda_1\alpha_2 = \lambda_2\alpha_1$.

Thus, when traders are asymmetric, the price in a strategic equilibrium is generally different from that in the competitive equilibrium. The prices are equal if and only if $\lambda_1\alpha_2 = \lambda_2\alpha_1$, which is not an obvious result ex ante. Intuitively, if λ_1/λ_2 is larger, trader 1 is less aggressive than trader 2 due to a higher inventory cost; but if α_1/α_2 is larger, trader 1 is more aggressive than trader 2 because a larger fraction of trader 1's value comes from his private signal. It turns out that in a strategic equilibrium the effects of these two incentives on the price offset each other when the two ratios are equal, i.e., $\lambda_1\alpha_2 = \lambda_2\alpha_1$, restoring the competitive equilibrium price.

4. Non-existence of nonlinear ex post equilibria if $n > 2$

So far, we have characterized a class of nonlinear ex post equilibria for $n = 2$. A natural question is whether the equilibrium construction generalizes to a market with $n \geq 3$ traders. We answer this question in the negative by showing that, when all traders' preferences are symmetric,[5] a nonlinear ex post equilibrium does not exist for $n \geq 4$. This non-existence result also holds if $n = 3$ and the three symmetric traders have pure private values. For these non-existence results we restrict attention to demand schedules that are twice continuously differentiable, globally downward sloping in price, and globally upward sloping in signals, which are plausible assumptions for practical applications.

In this section, there are $n \geq 3$ symmetric traders, with $\alpha_i = \alpha \in (0, 1]$ and $\lambda_i = \lambda > 0$, $1 \leq i \leq n$, where

$$v_i = \alpha s_i + \frac{1-\alpha}{n-1}\sum_{j \neq i} s_j, \tag{33}$$

and the utility $U_i(q_i, p; v_i)$ is still given by Equation (2). As before, the total supply of the asset is normalized to zero. As in Definition 1, in an ex post equilibrium every trader i would not deviate from his equilibrium strategy even if he would observe the realization of others' signals $s_{-i} = (s_1, \ldots s_{i-1}, s_{i+1}, \ldots, s_n)$, if others are following their equilibrium strategies.

[5] We focus on symmetric traders for the following reason. Rostek and Weretka (2012) show that when traders are asymmetric, the linear equilibrium is not an ex post equilibrium because the market-clearing price does not reveal all payoff-relevant information. For this reason, it would be too difficult a task to look for ex post equilibria with asymmetric traders. But with symmetric traders, ex post equilibrium is a suitable solution concept as it exactly selects the canonical linear equilibrium in the literature mentioned before (see Equation (34)).

It is known from the divisible double auction literature (see Vives, 2011; Rostek and Weretka, 2012, and Du and Zhu, 2016, among others) that if (and only if) $n\alpha > 2$, the following strategy constitutes an ex post equilibrium:

$$x_i(p; s_i) = \frac{n\alpha - 2}{\lambda(n-1)}(s_i - p)\,. \tag{34}$$

And the equilibrium price is

$$p^*(s) = \frac{1}{n}\sum_{i=1}^{n} s_i\,. \tag{35}$$

Proposition 3. *Suppose that either (i) $n \geq 4$ or (ii) $n = 3$ and $\alpha = 1$. Let $(x_1, \ldots, x_n)$ be an ex post equilibrium such that a market-clearing price $p^*(s)$ exists at every $s \in [\underline{s}, \overline{s}]^n$, and x_i is twice continuously differentiable, $\frac{\partial x_i}{\partial p}(p; s_i) < 0$, and $\frac{\partial x_i}{\partial s_i}(p; s_i) > 0$ for every s_i, p and i. Then, for every $s \in [\underline{s}, \overline{s}]^n$ and every i, at the market-clearing price $p = p^*(s)$, $x_i(p; s_i)$ is equal to the demand in Equation (34).*[6]

Proposition 3 states a strong and novel uniqueness result: the linear equilibrium in Equation (34) is the only ex post equilibrium, without a priori restricting to linear or symmetric strategies. (As usual, for any fixed s_i, the uniqueness of $x_i(p; s_i)$ in Proposition 3 applies only to market-clearing prices, i.e., $p = p^*(s_i, s_{-i})$ for some $s_{-i} \in [\underline{s}, \overline{s}]^{n-1}$, since the demands at non-market-clearing prices need not satisfy any optimality condition.) Proposition 3 thus provides a justification for the widespread use of the linear equilibrium in the literature. The parameter conditions (i) and (ii) in Proposition 3 cover almost the entire case of $n > 2$, with the only exception of $\{n = 3, \alpha < 1\}$. Our proof technique does not work for this rather specific case, and it remains an open question whether nonlinear equilibria exist for $\{n = 3, \alpha < 1\}$.

To convey the intuition of Proposition 3 and a flavor of the formal argument, let us sketch here a key step in the proof. For simplicity, let us assume private values ($\alpha = 1$) and $n \geq 3$. Fix an ex post equilibrium $(x_1, \ldots, x_n)$ that satisfies the regularity conditions in Proposition 3. We work with the inverse function of $x_i(p; \cdot)$, to which we refer as $\tilde{s}_i(p; \cdot)$. That is, for any realized allocation $y_i \in \mathbb{R}$, we have $x_i(p; \tilde{s}_i(p; y_i)) = y_i$. Because $x_i(p; s_i)$ is strictly increasing in s_i, $\tilde{s}_i(p; y_i)$ is strictly increasing in y_i. With an abuse of notation, we denote $\frac{\partial x_i}{\partial p}(p; y_i) \equiv \frac{\partial x_i}{\partial p}(p; \tilde{s}_i(p; y_i))$.

For a signal profile $s \in (\underline{s}, \overline{s})^n$, the market-clearing price must be interior (see footnote 10), so the ex post first order condition is satisfied by $(x_1, \ldots, x_n)$; we write it in terms of the inverse functions (cf. Equation (14)):

$$\sum_{j \neq i} \frac{\partial x_j}{\partial p}(p; y_j) = \frac{-y_i}{\tilde{s}_i(p; y_i) - p - \lambda y_i}, \qquad \text{where } y_i = -\sum_{j \neq i} y_j\,. \tag{36}$$

The above equation holds for *all* $(p, y_j)_{j \neq i} \in (\bar{p} - \epsilon, \bar{p} + \epsilon) \times \prod_{j \neq i}(\bar{y}_j - \epsilon, \bar{y}_j + \epsilon)$, where $(\bar{p}, \bar{y}_j)_{j \neq i}$ is the realized price and allocations of the signal profile s and ϵ is sufficiently small.[7]

[6] The assumption of a compact signal support $[\underline{s}, \overline{s}]^n$ is not necessary for Proposition 3, since if the ex post equilibrium condition holds over the signal space $[0, \infty)^n$, it also holds over any compact subset $[\underline{s}, \overline{s}]^n$, and Proposition 3 for compact signal support then applies.

[7] If $\tilde{s}_i(p; y_i) - p - \lambda y_i = 0$, then $y_i = 0$ by the ex post first order condition; but $y_i = -\sum_{j \neq i} y_j \neq 0$ for a generic $(y_j)_{j \neq i}$.

We pick any $j_1 \neq i$ and $j_2 \neq i$. In the neighborhood $(y_{j_1}, y_{j_2}) \in (\bar{y}_{j_1} - \epsilon, \bar{y}_{j_1} + \epsilon) \times (\bar{y}_{j_2} - \epsilon, \bar{y}_{j_2} + \epsilon)$, we differentiate Equation (36) by y_{j_1} (the first equality below) and by y_{j_2} (the last equality below) to get:

$$\frac{\partial}{\partial y_{j_1}}\left(\frac{\partial x_{j_1}}{\partial p}(p; y_{j_1})\right) = \frac{\partial}{\partial y_{j_1}}\left(\frac{-y_i}{\tilde{s}_i(p; y_i) - p - \lambda y_i}\right) = \frac{\partial}{\partial y_{j_2}}\left(\frac{-y_i}{\tilde{s}_i(p; y_i) - p - \lambda y_i}\right) = \frac{\partial}{\partial y_{j_2}}\left(\frac{\partial x_{j_2}}{\partial p}(p; y_{j_2})\right), \tag{37}$$

where the second equality follows because y_i has a coefficient of -1 on both y_{j_1} and y_{j_2}. Thus, $\frac{\partial}{\partial y_{j_1}}\left(\frac{\partial x_{j_1}}{\partial p}(p; y_{j_1})\right)$ and $\frac{\partial}{\partial y_{j_2}}\left(\frac{\partial x_{j_2}}{\partial p}(p; y_{j_2})\right)$ can depend only on p. Therefore, we have

$$\frac{\partial x_j}{\partial p}(p; y_j) = G(p) y_j + H_j(p) \tag{38}$$

for every $j \neq i$. We then substitute Equation (38) back to the ex post first order condition (36), and this imposes a strong restriction on the functional form of $\tilde{s}_i(p; y_i)$, which can be satisfied only by a linear function of y_i and p. The details can be found in the appendix.

To obtain the functional form in Equation (38), which is an important step in the proof, it is crucial that there are two independent variables y_{j_1} and y_{j_2}. Hence, Equation (38) does not apply if $n = 2$. Intuitively, in Equation (37) we are varying s_{j_1} and s_{j_2} in a way that holds the market-clearing price fixed, and ex post optimality subject to such variation imposes so strong a restriction on the shape of the equilibrium that only linear strategies satisfy it.

The above argument does not work when $\alpha < 1$, since then the right-hand side of Equation (36) would contain $\tilde{s}_j(p; y_j)$. In the appendix we uses an alternative argument for $\alpha < 1$ which relies on $n \geq 4$.

5. Conclusion

Existing models of divisible double auctions prove to be important tools for analyzing trading in centralized markets. The existence of their linear equilibria, however, typically require three or more traders. This requirement limits the applicability of those models in decentralized markets, where each transaction occurs between exactly two traders.

This paper fills this "$n = 2$" gap. We construct a family of non-linear, ex post equilibria in divisible double auction with two traders, who have interdependent values and submit demand schedules. The equilibria are characterized by solutions to algebraic equations. The equilibrium trading volume is positive but less than the first best. Our results open the possibility of using bilateral double auctions as a building block for analyzing strategic trading in decentralized markets of divisible assets.

Appendix A. Proofs for Section 3

A.1. *Proof of Proposition 1*

A.1.1. *Step 1: writing the first order conditions as differential equations*

Given a signal profile (s_1, s_2), trader i's ex post optimization problem is essentially selecting a market-clearing price p and getting the residual supply $-x_j(p; s_j)$ of trader $j \neq i$, since whatever the demand schedule trader i uses, his final allocation must clear the market, i.e., equal to $-x_j(p; s_j)$ at some price p. Let

$$\Pi_i(p) = (v_i - p)(-x_j(p; s_j)) - \frac{\lambda_i}{2}(-x_j(p; s_j))^2, \tag{39}$$

which is trader i's payoff given residual supply $-x_j(p; s_j)$. We construct (x_1, x_2) such that the following ex post first order conditions are always satisfied: for every $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$, $i \in \{1, 2\}$ and $j \neq i$,

$$\begin{aligned} \Pi_i'(p^*) &= x_j(p^*; s_j) + \left(\alpha_i s_i + (1-\alpha_i)s_j - p^* + \lambda_i x_j(p^*; s_j)\right)\left(-\frac{\partial x_j}{\partial p}(p^*; s_j)\right) = 0, \\ & x_1(p^*; s_1) + x_2(p^*; s_2) = 0, \end{aligned} \tag{40}$$

where $p^* \equiv p^*(s_1, s_2)$ is the market-clearing price. Note that trader i's demand x_i cannot depend on s_j; this distinguishes an ex post equilibrium from a full-sharing equilibrium in which traders truthfully share their signals s_1 and s_2 before trading. Trader i must infer s_j from the market-clearing price p^*. The price inference is accomplished by the following conjecture on the market-clearing price:

$$p^* = as_1 + (1-a)s_2 + \Lambda x_1(p^*; s_1) \tag{41}$$

where a and Λ are constants to be uniquely determined in Step 2. We also verify Conjecture (41) in Step 2. Intuitively, the equilibrium price should be a linear function of the signals and the equilibrium demand, since each trader's marginal value depends on a weighted average of signals and decreases linearly with quantity, as can be seen in Equation (40). Thus Conjecture (41) is a natural starting point.

Given Conjecture (41), we have

$$\begin{aligned} & v_1 - p^* + \lambda_1 x_2(p^*; s_2) \\ &= \alpha_1(s_1 - p^*) + (1-\alpha_1)(s_2 - p^*) + \lambda_1 x_2(p^*; s_2) \\ &= -\frac{\alpha_1}{a}((1-a)(s_2 - p^*) + \Lambda x_1(p^*; s_1)) + (1-\alpha_1)(s_2 - p^*) + \lambda_1 x_2(p^*; s_2) \\ &= \left(\alpha_1 \frac{1-a}{a} - (1-\alpha_1)\right)(p^* - s_2) + \left(\frac{\alpha_1}{a}\Lambda + \lambda_1\right) x_2(p^*; s_2), \end{aligned} \tag{42}$$

and

$$\begin{aligned} v_2 - p^* + \lambda_2 x_1(p^*; s_1) &= \left(\alpha_2 \frac{a}{1-a} - (1-\alpha_2)\right)(p^* - s_1) \\ &\quad + \left(-\frac{\alpha_2}{1-a}\Lambda + \lambda_2\right) x_1(p^*; s_2). \end{aligned} \tag{43}$$

Intuitively, in Equations (42) and (43) the value v_i is inferred from the market-clearing price using Conjecture (41). A subtlety here is that the inference on v_i is made with the variables of trader j. This is because we want to rewrite the first order condition of trader i in (40) as a differential equation that involves only trader j, $j \neq i$:

$$\begin{aligned} x_1(p^*; s_1) &= \Bigg(\left(\alpha_2 \frac{a}{1-a} - (1-\alpha_2)\right)(p^* - s_1) \\ &\quad + \left(\lambda_2 - \frac{\alpha_2}{1-a}\Lambda\right) x_1(p^*; s_1)\Bigg) \frac{\partial x_1}{\partial p}(p^*; s_1), \end{aligned} \tag{44}$$

$$x_2(p^*; s_2) = \left(\left(\alpha_1 \frac{1-a}{a} - (1-\alpha_1)\right)(p^* - s_2) + \left(\lambda_1 + \frac{\alpha_1}{a}\Lambda\right) x_2(p^*; s_2)\right) \frac{\partial x_2}{\partial p}(p^*; s_2). \tag{45}$$

Thus, we have two differential equations that can be solved separately. If Equation (44) holds for every $(s_1, p^*) \in [\underline{s}, \overline{s}]^2$ and Equation (45) for every $(s_2, p^*) \in [\underline{s}, \overline{s}]^2$, and if the market-clearing price satisfies Conjecture (41), then the first order conditions in Equation (40) must also hold for every $(s_1, s_2) \in [\underline{s}, \overline{s}]^2$.

To solve Equations (44) and (45), we first solve a simpler equation:

$$y(w) = (\eta w - \lambda y(w))y'(w), \quad y(0) = 0, \quad y'(w) > 0 \text{ for } w \in (0, \overline{s} - \underline{s}], \tag{46}$$

where η and λ are constants. Then, set

$$\begin{aligned} &\eta = \alpha_2 \frac{a}{1-a} - (1 - \alpha_2), && \lambda = \lambda_2 - \frac{\alpha_2}{1-a}\Lambda, \\ &w = |s_1 - p^*|, && x_1(p^*; s_1) = y(|s_1 - p^*|) \cdot \mathrm{sign}(s_1 - p), \end{aligned} \tag{47}$$

we see that for every (s_1, p^*) Equations (44) is satisfied since $\frac{\partial x_1}{\partial p}(p^*; s_1) = -y'(|s_1 - p^*|)$. Likewise for Equations (45).

Lemma 1. *Suppose that $0 < \eta < 1$ and $\lambda > 0$. The differential equation*

$$y(w) = (\eta w - \lambda y(w))y'(w) \tag{48}$$

is solved by the implicit solution to:

$$(1 - \eta)w = Cy(w)^\eta - \lambda y(w), \tag{49}$$

where C is a positive constant. If

$$C \geq \left(\frac{\lambda}{\eta}\right)^\eta (\overline{s} - \underline{s})^{1-\eta}, \tag{50}$$

we can select $y(w)$ that solves (49) such that $y(0) = y'(0) = 0$, $y(w) > 0$, $y'(w) > 0$ and $y''(w) > 0$ for every $w \in (0, \overline{s} - \underline{s}]$.

Proof of Lemma 1. We first show the solution of the differential equation (48) is implicitly defined by (49). For notional simplicity let us suppress the dependence of y on w and rewrite (48) as:

$$y\,dw + (\lambda y - \eta w)\,dy = 0. \tag{51}$$

We use the standard integrating factor technique to convert (51) into an exact differential equation; that is, multiplying both sides of (51) by the integrating factor $e^{\int(-\eta-1)/y\,dy} = y^{-1-\eta}$, we get:

$$y^{-\eta}\,dw + y^{-1-\eta}(\lambda y - \eta w)\,dy = 0, \tag{52}$$

which is an exact differential equation since $\frac{\partial}{\partial y}(y^{-\eta}) = \frac{\partial}{\partial w}(y^{-1-\eta}(\lambda y - \eta w))$. Thus, there exists a function $F(y, w)$ such that $\frac{\partial F}{\partial y} = y^{-1-\eta}(\lambda y - \eta w)$ and $\frac{\partial F}{\partial w} = y^{-\eta}$; it is easy to see that

$$F(y, w) = y^{-\eta}w + \lambda\frac{y^{1-\eta}}{1-\eta}. \tag{53}$$

Thus, the solution is implicitly defined by

$$y^{-\eta}w + \lambda\frac{y^{1-\eta}}{1-\eta} = K \tag{54}$$

for a constant K. Letting $C \equiv K(1 - \eta)$, it is easy to see that (54) is equivalent to (49).

For the second part, let

$$f(y) = Cy^{\eta} - \lambda y. \tag{55}$$

The function f is clearly strictly concave and obtains its maximum at

$$y^* = \left(\frac{C\eta}{\lambda}\right)^{\frac{1}{1-\eta}}. \tag{56}$$

We choose $C > 0$ so that

$$f(y^*) = C\left(\frac{C\eta}{\lambda}\right)^{\frac{\eta}{1-\eta}} - \lambda\left(\frac{C\eta}{\lambda}\right)^{\frac{1}{1-\eta}} = C^{\frac{1}{1-\eta}}\left(\frac{\eta}{\lambda}\right)^{\frac{\eta}{1-\eta}}(1-\eta) \geq (1-\eta)(\overline{s} - \underline{s}) \tag{57}$$

which is equivalent to (50). Given this choice of C, for every $w \in [0, \overline{s} - \underline{s}]$, by the Intermediate Value Theorem there is a unique $y(w) \in [0, y^*]$ that solves $f(y(w)) = (1-\eta)w$. Since $f'(y(w)) > 0$ for $y(w) \in (0, y^*)$, we have $y'(w) = \frac{1-\eta}{f'(y(w))} > 0$ for $w \in (0, \overline{s} - \underline{s}]$. And clearly, $y'(0) = 0$ if $\eta < 1$.

Finally, we differentiate both sides of (48) to obtain:

$$y'(w) = (\eta w - \lambda y(w))y''(w) + (\eta - \lambda y'(w))y'(w), \tag{58}$$

i.e.,

$$(\eta w - \lambda y(w))y''(w) = (1-\eta)y'(w) + \lambda y'(w)^2. \tag{59}$$

Since $y'(w) > 0$ and $\eta w - \lambda y(w) = \frac{y(w)}{y'(w)} > 0$, we conclude that $y''(w) > 0$ for $w > 0$. □

A.1.2. Step 2: deriving the equilibrium strategy

For $w_1, w_2 \in [0, \overline{s} - \underline{s}]$, let $y_1(w_1)$ and $y_2(w_2)$, be implicitly defined by (via Lemma 1):

$$\left(1 - \alpha_2\frac{a}{1-a} + (1-\alpha_2)\right)w_1 = C_1 y_1(w_1)^{\alpha_2\frac{a}{1-a} - (1-\alpha_2)} - \left(\lambda_2 - \frac{\alpha_2}{1-a}\Lambda\right)y_1(w_1), \tag{60}$$

$$\left(1 - \alpha_1\frac{1-a}{a} + (1-\alpha_1)\right)w_2 = C_2 y_2(w_2)^{\alpha_1\frac{1-a}{a} - (1-\alpha_1)} - \left(\lambda_1 + \frac{\alpha_1}{a}\Lambda\right)y_2(w_2), \tag{61}$$

and suppose the conditions in the second part of Lemma 1 are satisfied. Define the following strategies:

$$x_1(p; s_1) = y_1(|s_1 - p|)\cdot \operatorname{sign}(s_1 - p), \tag{62}$$

$$x_2(p; s_2) = y_2(|s_2 - p|)\cdot \operatorname{sign}(s_2 - p). \tag{63}$$

Let $w_1 = |s_1 - p^*|$ and $w_2 = |s_2 - p^*|$, we rewrite Conjecture (41) as

$$aw_1 - (1-a)w_2 + \Lambda y(w_1) = 0. \tag{64}$$

Clearly, Equation (60) is equivalent to

$$\left(1 - \alpha_2\frac{a}{1-a} + (1-\alpha_2)\right)\left(w_1 + \frac{\Lambda}{a}y_1(w_1)\right) \tag{65}$$

$$= C_1 y_1(w_1)^{\alpha_2\frac{a}{1-a} - (1-\alpha_2)} - \left(\lambda_2 - \frac{\alpha_2}{1-a}\Lambda - \left(1 - \alpha_2\frac{a}{1-a} + (1-\alpha_2)\right)\frac{\Lambda}{a}\right)y_1(w_1).$$

By the definition of the market-clearing price p^*, we have $y(w_1) = y(w_2)$. Thus, to satisfy both Equations (61) and (65), Conjecture (64), and $y(w_1) = y(w_2)$, we must have the same exponent in Equations (61) and (65):

$$\alpha_2 \frac{a}{1-a} - (1-\alpha_2) = \alpha_1 \frac{1-a}{a} - (1-\alpha_1), \tag{66}$$

i.e.,

$$a = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \tag{67}$$

To satisfy Conjecture (64), we make Equation (61) equal to $\frac{a}{1-a} = \frac{\alpha_1}{\alpha_2}$ times Equation (65):

$$C_1 = \alpha_2 C, \quad C_2 = \alpha_1 C, \tag{68}$$

for a constant $C > 0$, and

$$\lambda_1 + \frac{\alpha_1}{a}\Lambda = \frac{\alpha_1}{\alpha_2}\left(\lambda_2 - \frac{\alpha_2}{1-a}\Lambda - \left(1 - \alpha_2 \frac{a}{1-a} + (1-\alpha_2)\right)\frac{\Lambda}{a}\right), \tag{69}$$

i.e.,

$$\Lambda = \frac{\alpha_1\lambda_2 - \alpha_2\lambda_1}{2(\alpha_1+\alpha_2)}. \tag{70}$$

Substituting (67), (68) and (70) into (60) and (61) gives:

$$(2-\alpha_1-\alpha_2)w_1 = C\alpha_2 y_1(w_1)^{\alpha_1+\alpha_2-1} - \left(\lambda_2\left(1-\frac{\alpha_1}{2}\right) + \lambda_1\frac{\alpha_2}{2}\right)y_1(w_1), \tag{71}$$

$$(2-\alpha_1-\alpha_2)w_2 = C\alpha_1 y_2(w_2)^{\alpha_1+\alpha_2-1} - \left(\lambda_1\left(1-\frac{\alpha_2}{2}\right) + \lambda_2\frac{\alpha_1}{2}\right)y_2(w_2). \tag{72}$$

The following lemma gives conditions that guarantee the existence of market-clearing price.

Lemma 2. *Suppose that*

$$C \geq \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_2}\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}, \textit{ and} \tag{73}$$

$$C \geq \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1}\left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}. \tag{74}$$

Then for every profile $(s_1, s_2) \in [\underline{s}, \bar{s}]^2$, there exists a unique $p^* \in [\underline{s}, \bar{s}]$ that satisfies $x_1(p^*; s_1) + x_2(p^*; s_2) = 0$; that is, there exist unique $w_1 \geq 0$ and $w_2 \geq 0$ such that $w_1 + w_2 = |s_1 - s_2|$ and $y_1(w_1) = y_2(w_2)$.

Proof. By Lemma 1, conditions (73) and (74) give $y_1 : [0, \bar{s}-\underline{s}] \to [0, \infty)$ and $y_2 : [0, \bar{s}-\underline{s}] \to [0, \infty)$, respectively, that are strictly increasing and convex.

Without loss of generality, suppose that $s_2 < s_1$. There exists a minimum $\bar{y} > 0$ that solves[8]:

[8] By construction, we have

$$(2-\alpha_1-\alpha_2)(s_1-s_2) = C\alpha_2 y^{\alpha_1+\alpha_2-1} - \left(\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}\right)y$$

$$(2-\alpha_1-\alpha_2)(s_1-s_2)=C(\alpha_1+\alpha_2)\bar{y}^{\alpha_1+\alpha_2-1}-(\lambda_1+\lambda_2)\,\bar{y}. \tag{75}$$

Let w_1 satisfies

$$(2-\alpha_1-\alpha_2)w_1=C\alpha_2\bar{y}^{\alpha_1+\alpha_2-1}-\left(\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}\right)\bar{y}, \tag{76}$$

and let w_2 satisfies

$$(2-\alpha_1-\alpha_2)w_2=C\alpha_1\bar{y}^{\alpha_1+\alpha_2-1}-\left(\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}\right)\bar{y}. \tag{77}$$

Clearly, we have $w_1>0$, $w_2>0$ and $w_1+w_2=s_1-s_2$. Let $p^*=s_2+w_1$. Then we have $w_1=p^*-s_2$, $w_2=s_1-p^*$, and $y_1(w_1)=y_2(w_2)=\bar{y}$, i.e., $x_1(p^*;s_1)=-x_2(p^*;s_2)$.

Finally, the uniqueness of p^* follows from the fact that both $x_1(p;s_1)$ and $x_2(p;s_2)$ are strictly decreasing in p. $\square$

A.1.3. Step 3: verifying ex post optimality

Finally, we directly verify the ex post optimality of the profile (x_1,x_2) constructed in Step 2. We will show that

$$\Pi_i(p^*)\ge\Pi_i(p), \tag{78}$$

for every $p\in[\underline{s},\overline{s}]$ and every $(s_1,s_2)\in[\underline{s},\overline{s}]^2$, where Π_i is defined in Equation (39).

Without loss of generality, fix $i=1$ and $s_2<s_1$. By construction, we have $s_2<p^*<s_1$, $x_1(p^*;s_1)=-x_2(p^*;s_2)>0$. Since $x_1(p^*;s_1)>0$, the first order condition (40) implies that

$$v_1-p^*-\lambda_1x_1(p^*;s_1)=v_1-p^*+\lambda_1x_2(p^*;s_2)>0. \tag{79}$$

For later reference let $\bar{p}\in(p^*,\overline{s})$ be such that

$$v_1-\bar{p}+\lambda_1x_2(\bar{p};s_2)=0. \tag{80}$$

We note that

$$\Pi_1'(p)=(v_1-p+\lambda_1x_2(p;s_2))\left(-\frac{\partial x_2}{\partial p}(p;s_2)\right)+x_2(p;s_2)<0 \tag{81}$$

for $p\ge\bar{p}$. Thus, $\Pi_1(p)$ cannot be maximized by $p\in[\bar{p},\overline{s}]$.

We have

$$\Pi_1(p^*)=\int_0^{x_1(p^*;s_1)}(v_1-p^*-\lambda_1q)\,dq>0. \tag{82}$$

On the other hand, when $p\le s_2$, we have $x_2(p;s_2)\ge0$, hence $\Pi_1(p)\le0$. Thus, $\Pi_1(p)$ cannot be maximized by $p\in[\underline{s},s_2]$.

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when $y=y_1(s_1-s_2)$, and

$$(2-\alpha_1-\alpha_2)(s_1-s_2)=C\alpha_1y^{\alpha_1+\alpha_2-1}-\left(\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}\right)y$$

when $y=y_2(s_1-s_2)$. Hence, by the Intermediate Value Theorem, there exists a $\bar{y}\le\min(y_1(s_1-s_2),y_2(s_1-s_2))$ that satisfies Equation (75).

For $p \in (s_2, \overline{s}]$, we have $x_2(p; s_2) = -y_2(p - s_2)$, and hence:

$$\begin{aligned}\Pi_1'(p) =&(v_1 - p - \lambda_1 y_2(p - s_2))y_2'(p - s_2) - y_2(p - s_2) \\ =&(v_1 - p - \lambda_1 y_2(p - s_2))y_2'(p - s_2) \\ &- ((\alpha_1 + \alpha_2 - 1)(p - s_2) - (\lambda_1 + (\alpha_1 + \alpha_2)\Lambda)y_2(p - s_2))y_2'(p - s_2) \\ =&(v_1 - p - (\alpha_1 + \alpha_2 - 1)(p - s_2) + (\alpha_1 + \alpha_2)\Lambda y_2(p - s_2))y_2'(p - s_2)\end{aligned} \tag{83}$$

where the second line follows by the differential equation in (46). Since $y'(p - s_2) > 0$ for $p > s_2$, $\Pi_1'(p) = 0$ for $p > s_2$ if and only if

$$v_1 - p - (\alpha_1 + \alpha_2 - 1)(p - s_2) + (\alpha_1 + \alpha_2)\Lambda y_2(p - s_2) = 0 \tag{84}$$

for $p > s_2$.

We distinguish between two cases:

1. When $\Lambda \leq 0$, the left-hand side of (84) is strictly decreasing in p, since by Lemma 1 $y(p - s_2)$ is strictly increasing in p. Thus, Equation (84) has only one solution: $p = p^*$ (by the construction in Step 1 and 2, we have $\Pi_1'(p^*) = 0$).
2. When $\Lambda > 0$, the left-hand side of (84) is strictly convex in p, since by Lemma 1 $y(p - s_2)$ is strictly convex in p. Thus, Equation (84) has at most two solutions (one of the solutions is $p = p^*$). However, we know that for any $p \geq \bar{p}$, the left-hand side of the (84) is negative, since $\Pi_1'(p) < 0$ (see Equation (81)). Therefore, $p = p^*$ is the only solution to (84).

Therefore, Equation (84) has only one solution on $(s_2, \overline{s}]$: $p = p^*$. This implies that $\Pi_1'(p) = 0$ has only one solution on $(s_2, \overline{s}]$: $p = p^*$. Since the maximum point of $\Pi_i(p)$ over $[\underline{s}, \overline{s}]$ cannot be in $[\underline{s}, s_2]$ or in $[\bar{p}, \overline{s}]$, it must be in $(s_2, \bar{p})$ and satisfies $\Pi_i'(p) = 0$. We thus conclude that $p = p^*$ maximizes $\Pi_i(p)$ over all $p \in [\underline{s}, \overline{s}]$.

A.2. *Proof of Corollary 2*

The proof of Corollary 2 follows the same steps as that of Proposition 1, with the following modifications:

- In Step 1, Lemma 1, we solve the differential equation:

  $$y(w) = (w - \lambda y(w))y'(w), \tag{85}$$

  whose solution is given by the implicit equation

  $$w = Cy(w) - \lambda y(w)\log(y(w)), \tag{86}$$

  where C is a constant. If

  $$C \geq \lambda \left(\log \frac{\overline{s} - \underline{s}}{\lambda} + 1\right), \tag{87}$$

  the implicit solution $y(w)$ can be selected to satisfy the second part of Lemma 1.
- In Step 2, we have $a = 1/2$, and we let $y_1(w_1)$ and $y_2(w_2)$, where $w_1, w_2 \geq 0$, be implicitly defined by

  $$w_1 = C_1 y_1(w_1) - (\lambda_2 - 2\Lambda)y_1(w_1)\log(y_1(w_1)), \tag{88}$$

  $$w_2 = C_2 y_2(w_2) - (\lambda_1 + 2\Lambda)y_2(w_2)\log(y_2(w_2)), \tag{89}$$

where Equation (88) is equivalent to

$$w_1 + 2\Lambda y_1(w_1) = (C_1 + 2\Lambda)y_1(w_1) - (\lambda_2 - 2\Lambda)y_1(w_1)\log(y_1(w_1)). \tag{90}$$

To satisfy Condition (64), we let

$$C_1 + 2\Lambda = C_2, \qquad \lambda_2 - 2\Lambda = \lambda_1 + 2\Lambda, \tag{91}$$

i.e.,

$$\Lambda = \frac{\lambda_2 - \lambda_1}{4}. \tag{92}$$

A.3. Proof of Corollary 3

Without loss let (x_1, x_2) be the most aggressive equilibrium in Proposition 1. By definition, $\sup_{(p,s_i)\in[\underline{s},\overline{s}]^2} |x_i(p; s_i)| = y_i(\overline{s} - \underline{s})$. Suppose $i = 1$. Let $f_1(y_1)$ be defined by Equation (11), i.e., the right-hand side of Equation (8). The maximum of $f_1(y_1)$ is at

$$y_1^* = \left(\frac{C\alpha_2(\alpha_1 + \alpha_2 - 1)}{\lambda_2\left(1 - \frac{\alpha_1}{2}\right) + \lambda_1\frac{\alpha_2}{2}}\right)^{1/(2-\alpha_1-\alpha_2)}. \tag{93}$$

By definition, $y_1(\overline{s}-\underline{s}) \le y_1^*$. As $\alpha_1+\alpha_2 \to 1$, the constant C for the most aggressive equilibrium is bounded above, so the corollary follows.

A.4. Proof of Proposition 2

The second part of the proposition follows by an easy comparison of prices and is omitted. Let us denote

$$q^c \equiv |x_1^c(p^c; s_1)| = \frac{(\alpha_1 + \alpha_2 - 1)|s_1 - s_2|}{\lambda_1 + \lambda_2}, \tag{94}$$

which is the amount of trading (in absolute value) in the ex post efficient allocation.

Let $q^*(C) \equiv |x_1(p^*; s_1)|$ be the amount of trading (in absolute value) in an ex post equilibrium (x_1, x_2) from Proposition 1, where the constant C satisfies Conditions (5) and (6). Let us also define

$$f(y) \equiv C(\alpha_1 + \alpha_2)y^{\alpha_1+\alpha_2-1} - (\lambda_1 + \lambda_2)y. \tag{95}$$

In Lemma 2 (Section A.1.2) we show that $y = q^*(C)$ is the smaller solution to

$$(2 - \alpha_1 - \alpha_2)|s_1 - s_2| = f(y), \tag{96}$$

before $f(y)$ reaches its maximum.[9]

We show that $f(q^c) > (2 - \alpha_1 - \alpha_2)|s_1 - s_2|$, where f is defined in Equation (95). Since $y = q^*(C)$ is the smaller solution to $f(y) = (2 - \alpha_1 - \alpha_2)|s_1 - s_2|$, we must have $q^*(C) < q^c$, which proves the first part of the proposition.

[9] It is straightforward to show that given Conditions (5) and (6), there always exist two solutions to (96), one before and one after $f(y)$ reaches the maximum.

Clearly, $f(q^c) > (2-\alpha_1-\alpha_2)|s_1-s_2|$ is equivalent to:

$$C > \frac{|s_1-s_2|^{2-\alpha_1-\alpha_2}}{\alpha_1+\alpha_2}\left(\frac{\lambda_1+\lambda_2}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}. \tag{97}$$

Let us define:

$$\mathcal{C}_1 \equiv \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_2}\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}, \tag{98}$$

$$\mathcal{C}_2 \equiv \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1}\left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}, \tag{99}$$

$$\mathcal{C} \equiv \frac{(\bar{s}-\underline{s})^{2-\alpha_1-\alpha_2}}{\alpha_1+\alpha_2}\left(\frac{\lambda_1+\lambda_2}{\alpha_1+\alpha_2-1}\right)^{\alpha_1+\alpha_2-1}. \tag{100}$$

We claim that $\max(\mathcal{C}_1,\mathcal{C}_2) > \mathcal{C}$. For the sake of contradiction, suppose $\max(\mathcal{C}_1,\mathcal{C}_2) \le \mathcal{C}$; this implies:

$$\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} \le \frac{\alpha_2}{\alpha_1+\alpha_2}$$

$$\left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} \le \frac{\alpha_1}{\alpha_1+\alpha_2},$$

which implies

$$\left(\frac{\lambda_2\left(1-\frac{\alpha_1}{2}\right)+\lambda_1\frac{\alpha_2}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} + \left(\frac{\lambda_1\left(1-\frac{\alpha_2}{2}\right)+\lambda_2\frac{\alpha_1}{2}}{\lambda_1+\lambda_2}\right)^{\alpha_1+\alpha_2-1} \le 1,$$

which is clearly false given $0 < \alpha_1+\alpha_2-1 < 1$.

Hence Conditions (5) and (6), which state that $C \ge \max(\mathcal{C}_1,\mathcal{C}_2)$, imply that $C > \mathcal{C}$, which implies (97).

Appendix B. Proof of Proposition 3

B.1. *Existence of Linear Equilibrium*

We conjecture a strategy profile $(x_1,\ldots,x_n)$. For notational convenience, we define

$$\beta \equiv \frac{1-\alpha}{n-1}. \tag{101}$$

Given that all other bidders use this strategy profile and for a fixed profile of signals $(s_1,\ldots,s_n)$, the profit of bidder i at the price of p is

$$\Pi_i(p) = \left(\alpha s_i + \beta\sum_{j\neq i} s_j - p\right)\left(-\sum_{j\neq i} x_j(p;s_j)\right) - \frac{1}{2}\lambda\left(-\sum_{j\neq i} x_j(p;s_j)\right)^2.$$

We can see that bidder i is effectively selecting an optimal price p. Taking the first-order condition of $\Pi_i(p)$ at $p=p^*$, we have, for all i,

$$0=\Pi_i'(p^*)=-x_i(p^*;s_i)+\left(\alpha s_i+\beta\sum_{j\neq i}s_j-p^*-\lambda x_i(p^*;s_i)\right)\left(-\sum_{j\neq i}\frac{\partial x_j}{\partial p}(p^*;s_j)\right). \tag{102}$$

Therefore, an ex post equilibrium corresponds to a solution $\{x_i\}$ to the first-order condition (102), such that for each i, x_i depends only on s_i and p.

We conjecture a symmetric linear demand schedule:

$$x_j(p;s_j)=as_j-bp+c, \tag{103}$$

where $a\neq 0$, b, and c are constants. In this conjectured equilibrium, all bidders $j\neq i$ use the strategy (103). Thus, we can rewrite the each bidder j's signal s_j in terms of his demand x_j:

$$\sum_{j\neq i}s_j=\sum_{j\neq i}\frac{x_j(p^*;s_j)+bp^*-c}{a}=\frac{1}{a}\left(-x_i(p^*;s_i)+(n-1)(bp^*-c)\right),$$

where we have also used the market clearing condition. Substituting the above equation into bidder i's first order condition (102) and rearranging, we have

$$\begin{aligned}x_i(p^*;s_i)&=\frac{\alpha(n-1)bs_i-(n-1)b\left[1-\beta(n-1)b/a\right]p^*-(n-1)c\beta(n-1)b/a}{1+\lambda(n-1)b+\beta(n-1)b/a}\\&\equiv as_i-bp^*+c.\end{aligned}$$

Matching the coefficients and using the normalization that $\alpha+(n-1)\beta=1$, we solve

$$a=b=\frac{1}{\lambda}\cdot\frac{n\alpha-2}{n-1},\qquad c=0.$$

It is easy to verify that under this linear strategy, $\Pi_i''(\cdot)=-n(n-1)\alpha b<0$ if $n\alpha>2$. We thus have a linear ex post equilibrium.

B.2. Uniqueness of Equilibrium

We fix an ex post equilibrium strategy $(x_1,\ldots,x_n)$ such that for every i, x_i is twice continuously differentiable, $\frac{\partial x_i}{\partial p}(p;s_i)<0$ and $\frac{\partial x_i}{\partial s_i}(p;s_i)>0$ for every $(p,s_1,\ldots,s_n)\in(\underline{s},\overline{s})^{n+1}$.

Fix an arbitrary $s=(s_1,\ldots,s_n)\in(\underline{s},\overline{s})^n$. It is easy to see that $p^*(s)\in(\underline{s},\overline{s})$.[10] We will prove that there exist a $\delta'>0$ sufficiently small and constants a, b, and c such that

$$x_i(p;s_i')=as_i'-bp+c \tag{104}$$

holds for every $p\in(p^*(s)-\delta',p^*(s)+\delta')$, $s_i'\in(s_i-\delta',s_i+\delta')$, and $i\in\{1,\ldots,n\}$.

Once (104) is established, the values of a, b and c are pinned down by the construction of linear equilibrium in Section B.1; in particular, the values of a, b and c are independent of $(s_1,\ldots,s_n)$ and of δ'. Since $s=(s_1,\ldots,s_n)$ is arbitrary, the same constants a, b, and c in (104) apply to any $s=(s_1,\ldots,s_n)\in(\underline{s},\overline{s})^n$ and $p=p^*(s)$. Finally for s on the boundary of $[\underline{s},\overline{s}]^n$, we take an approximating sequence of signal profiles from the interior and uses the continuity of $x_i(p;s_i)$. This proves the uniqueness in Proposition 3.

[10] For the sake of contradiction suppose $p^*=\underline{s}$. Then the trader i with $x_i(p^*;s_i)\leq 0$ would strictly prefer a higher price, which contradicts the ex post optimality. Likewise for $p^*=\overline{s}$.

To prove (104), we work with the inverse function of $x_i(p;\,\cdot)$, to which we refer as $\tilde{s}_i(p;\,\cdot)$. That is, for any realized allocation $y_i \in \mathbb{R}$, we have $x_i(p; \tilde{s}_i(p; y_i)) = y_i$. Because $x_i(p; s_i)$ is strictly increasing in s_i, $\tilde{s}_i(p; y_i)$ is strictly increasing in y_i. Throughout the proof, we will denote trader i's realized allocation by y_i and his demand schedule by $x_i(\cdot\,;\,\cdot)$. With an abuse of notation, we denote $\frac{\partial x_i}{\partial p}(p; y_i) \equiv \frac{\partial x_i}{\partial p}(p; \tilde{s}_i(p; y_i))$.

Fix $s = (s_1, \ldots, s_n) \in (\underline{s}, \bar{s})^n$. Let $\bar{p} = p^*(s)$ and $\bar{y}_i = x_i(p^*(s); s_i)$. By continuity, there exists some $\delta > 0$ such that, for any i and any $(p, y_i) \in (\bar{p} - \delta, \bar{p} + \delta) \times (\bar{y}_i - \delta, \bar{y}_i + \delta)$, there exists some $s_i' \in (\underline{s}, \bar{s})$ such that $x_i(p; s_i') = y_i$. In other words, every price and allocation pair in $(\bar{p} - \delta, \bar{p} + \delta) \times (\bar{y}_i - \delta, \bar{y}_i + \delta)$ is "realizable" given some signal.

We will prove that there exist constants $A \neq 0$, $B \neq 0$, and C such that

$$\tilde{s}_i(p; y_i) = Ay_i + Bp + C \tag{105}$$

for every $(p, y_i) \in (\bar{p} - \delta, \bar{p} + \delta) \times (\bar{y}_i - \delta/n, \bar{y}_i + \delta/n)$, $i \in \{1, \ldots, n\}$. Clearly, this implies (104). We now proceed to prove (105). There are two cases. In Case 1, $\alpha < 1$ and $n \geq 4$. In Case 2, $\alpha = 1$ and $n \geq 3$.

B.2.1. Case 1: $\alpha < 1$ and $n \geq 4$

The proof for Case 1 consists of two steps.

Step 1 of Case 1: Lemma 3 and Lemma 4 below imply equation (105).

We now restrict y_j to $(\bar{y}_j - \delta/n, \bar{y}_j + \delta/n)$, $j \in \{1, \ldots, n-1\}$, so that $y_n = -\sum_{j=1}^{n-1} y_j \in (\bar{y}_n - \delta, \bar{y}_n + \delta)$, and as a result $\tilde{s}(p; y_n)$ and $\frac{\partial x_n}{\partial p}(p; y_n)$ are well-defined.

Lemma 3. *There exist functions $A(p)$, $\{B_i(p)\}$ such that*

$$\tilde{s}_i(p; y_i) = A(p)y_i + B_i(p), \tag{106}$$

holds for every $p \in (\bar{p} - \delta, \bar{p} + \delta)$ and every $y_i \in (\bar{y}_i - \delta/n, \bar{y}_i + \delta/n)$, $1 \leq i \leq n$.

Proof. This lemma is proved in Step 2 of Case 1. For this lemma we need the condition that $n \geq 4$; in the rest of the proof $n \geq 3$ suffices. □

Lemma 4. *Suppose that $l \geq 2$ and for every $i \in \{1, \ldots, l\}$, Y_i is an open subset of $\mathbb{R}$, P is an arbitrary set, and $f_i(p; y_i)$ is a differentiable function of $y_i \in Y_i$ for every $p \in P$. Moreover, suppose that*

$$\sum_{i=1}^{l} f_i(p; y_i) = f_{l+1}\left(p; \sum_{i=1}^{l} y_i\right), \tag{107}$$

for every $p \in P$ and $(y_1, \ldots, y_l) \in \prod_{i=1}^{l} Y_i$. Then there exist functions $G(p)$ and $\{H_i(p)\}$ such that

$$f_i(p; y_i) = G(p)y_i + H_i(p)$$

holds for every $i \in \{1, \ldots, l\}$, $p \in P$ and $y_i \in Y_i$.

Proof. We differentiate (107) with respect to y_i and to y_j, where $i, j \in \{1, 2, \ldots, l\}$, and obtain

$$\frac{\partial f_i}{\partial y_i}(p; y_i) = \frac{\partial f_{l+1}}{\partial y_i}\left(p; \sum_{j=1}^{l} y_j\right) = \frac{\partial f_j}{\partial y_j}(p; y_j)$$

for any $y_i \in Y_i$ and $y_j \in Y_j$. Because $(y_1, \ldots, y_l)$ are arbitrary, the partial derivatives above cannot depend on any particular y_i. Thus, there exists some function $G(p)$ such that $\frac{\partial f_i}{\partial y_i}(p; y_i) = G(p)$ for all y_i. Lemma 4 then follows. □

In Step 1 of the proof of Case 1 of Proposition 3, we show that Lemma 3 and Lemma 4 imply equation (105). Define

$$\beta \equiv \frac{(1-\alpha)}{n-1}, \tag{108}$$

and rewrite trader i's ex post first-order condition as:

$$-y_i + \left(\alpha \tilde{s}_i(p; y_i) + \beta \sum_{j \neq i} \tilde{s}_j(p; y_j) - p - \lambda y_i\right)\left(-\sum_{j \neq i} \frac{\partial x_j}{\partial p}(p; y_j)\right) = 0, \tag{109}$$

where $y_n = -\sum_{j=1}^{n-1} y_j$, $p \in (\bar{p} - \delta, \bar{p} + \delta)$ and $y_j \in (\bar{y}_j - \delta/n, \bar{y}_j + \delta/n)$.

Our strategy is to repeatedly apply Lemma 3 and Lemma 4 to (109) in order to arrive at (105).

First, we plug the functional form of Lemma 3 into (109). Without loss of generality, we let $i = n$ and rewrite (109) as

$$\underbrace{\sum_{j=1}^{n-1} \frac{\partial x_j}{\partial p}(p; y_j)}_{\text{left-hand side of (107)}} = \underbrace{-\frac{y_n}{\alpha(A(p)y_n + B_n(p)) + \beta \sum_{j=1}^{n-1}(A(p)y_j + B_j(p)) - p - \lambda y_n}}_{\text{right-hand side of (107)}}.$$

Applying Lemma 4 to the above equation, we see that there exist functions $G(p)$ and $\{H_j(p)\}$ such that

$$\frac{\partial x_j}{\partial p}(p; y_j) = G(p) y_j + H_j(p), \tag{110}$$

for $j \in \{1, \ldots, n-1\}$. Note that we have used the condition $n \geq 3$ when applying Lemma 3.

By the same argument, we apply Lemma 4 to (109) for $i = 1$, and conclude that (110) holds for $j = n$ as well.

Using (106) and (110), we rewrite trader i's ex post first-order condition as:

$$\left((\alpha - \beta)\tilde{s}_i(p; y_i) + \beta\left(\sum_{j=1}^{n} B_j(p)\right) - p - \lambda y_i\right)\left(-G(p)(-y_i) - \sum_{j \neq i} H_j(p)\right) - y_i = 0. \tag{111}$$

Solving for $\tilde{s}_i(p; y_i)$ in terms of p and y_i from equation (111), we see that for the solution to be consistent with (106), we must have $G(p) = 0$. Otherwise, i.e. if $G(p) \neq 0$, then (111) implies that $\tilde{s}_i(p; y_i)$ contains the term $y_i / \left(-G(p)(-y_i) - \sum_{j \neq i} H_j(p)\right)$, contradicting the linear form of Lemma 3.

Inverting (106), we see that $x_i(p; s_i) = (s_i - B_i(p))/A(p)$. Therefore, for $\frac{\partial x_i}{\partial p}(p; s_i)$ to be independent of s_i (i.e., $G(p) = 0$), $A(p)$ must be a constant function, i.e. $A(p) = A$ for some constant $A \in \mathbb{R}$. This implies that

$$H_i(p) = -\frac{B_i'(p)}{A}, \tag{112}$$

by the definition of $H_i(p)$ in (110).

Given $G(p) = 0$ and $A(p) = A$, (111) can be rewritten as

$$(\alpha - \beta)\tilde{s}_i(p; y_i) + \beta\left(\sum_{j=1}^{n} B_j(p)\right) - p - \lambda y_i - \frac{y_i}{-\sum_{j\neq i} H_j(p)} = 0. \tag{113}$$

For (113) to be consistent with $\tilde{s}_i(p; y_i) = Ay_i + B_i(p)$, we must have that $H_j(p) = H_j$ for some constants H_j, $j \in \{1, \ldots, n\}$, and that

$$\frac{1}{\sum_{j\neq i} H_j} = \frac{1}{\sum_{j\neq i'} H_j}, \quad \text{for all } i \neq i',$$

which implies that for all i, $H_i = H$ for the same constant H.

By (112), this means that $B_i(p) = Bp + C_i$, where $B = -HA$, and $\{C_i\}$ are some constants. Finally, (113) implies that for all i, $C_i = C$ for the same constant C.

Hence, we have shown that Lemma 3 implies (105). This completes Step 1 of the proof of Case 1 of Proposition 3. In Step 2 below, we prove Lemma 3.

Step 2 of Case 1: Proof of Lemma 3. Trader n's ex post first order condition can be written as:

$$\sum_{j=1}^{n-1} \frac{\partial x_j}{\partial p}(p; y_j) = -\frac{y_n}{\alpha\tilde{s}_n(p; y_n) + \beta\sum_{j=1}^{n-1}\tilde{s}_j(p; y_j) - p - \lambda y_n}, \tag{114}$$

where $y_n = -\sum_{j=1}^{n-1} y_j$. Differentiating (114) with respect to y_i, $i \in \{1, \ldots, n-1\}$, gives:

$$\frac{\partial}{\partial y_i}\left(\frac{\partial x_i}{\partial p}(p; y_i)\right) = \frac{\Gamma(y_1, \ldots, y_{n-1}) + y_n\left(-\alpha\frac{\partial \tilde{s}_n}{\partial y_n}(p; y_n) + \beta\frac{\partial \tilde{s}_i}{\partial y_i}(p; y_i) + \lambda\right)}{\Gamma(y_1, \ldots, y_{n-1})^2}, \tag{115}$$

where

$$\Gamma(y_1, \ldots, y_{n-1}) = \alpha\tilde{s}_n(p; y_n) + \beta\sum_{j=1}^{n-1}\tilde{s}_j(p; y_j) - p - \lambda y_n. \tag{116}$$

Solving for $\Gamma(y_1, \ldots, y_{n-1})$ in (115), we get

$$\Gamma(y_1, \ldots, y_{n-1}) = \rho_i\left(y_i, \sum_{j=1}^{n-1} y_j\right) \tag{117}$$

for some function ρ_i, $i \in \{1, \ldots, n-1\}$.

We let $\rho_{i,1}$ be the partial derivative of ρ_i with respect to its first argument, and let $\rho_{i,2}$ be the partial derivative of ρ_i with respect to its second argument. For each pair of distinct $i, k \in \{1, \ldots, n-1\}$, differentiating $\Gamma(y_1, \ldots, y_{n-1}) = \rho_i\left(y_i, \sum_{j=1}^{n-1} y_j\right) = \rho_k\left(y_k, \sum_{j=1}^{n-1} y_j\right)$ with respect to y_i and to y_k, we have

$$\frac{d\Gamma(y_1, \ldots, y_{n-1})}{dy_i} = \rho_{i,1} + \rho_{i,2} = \rho_{k,2},$$

$$\frac{d\Gamma(y_1, \ldots, y_{n-1})}{dy_k} = \rho_{k,1} + \rho_{k,2} = \rho_{i,2},$$

which imply that for all $i \neq k \in \{1, \dots, n-1\}$,

$$\rho_{i,1} + \rho_{k,1} = 0. \tag{118}$$

Choose any three distinct i, j and k from $\{1, \dots, n-1\}$, we have $\rho_{i,1} = -\rho_{j,1} = \rho_{k,1} = -\rho_{i,1}$; here we have used $n \geq 4$. Thus we have $\rho_{i,1} = 0$ for all $i \in \{1, \dots, n-1\}$. That is, each ρ_i is only a function of its second argument:

$$\rho_i \left(y_i, \sum_{j=1}^{n-1} y_j \right) = \rho_i \left(\sum_{j=1}^{n-1} y_j \right). \tag{119}$$

Then, using (116), (117) and (119) for $i = 1$, we have

$$\beta \sum_{j=1}^{n-1} \tilde{s}_j(p; y_j) = \rho_1 \left(\sum_{j=1}^{n-1} y_j \right) + p + \lambda y_n - \alpha \tilde{s}_n(p; y_n). \tag{120}$$

Applying Lemma 4 to (120) (recall that $y_n = -\sum_{j=1}^{n-1} y_j$), we conclude that, for all $j \in \{1, \dots, n-1\}$,

$$\tilde{s}_j(p; y_j) = A(p) y_j + B_j(p). \tag{121}$$

Finally, we repeat this argument to trader 1's ex post first-order condition and conclude that (121) holds for $j = n$ as well. This concludes the proof of Lemma 3.

B.2.2. *Case 2: $\alpha = 1$ and $n \geq 3$*

We now prove Case 2 of Proposition 3. Trader n's ex post first order condition in this case is:

$$\sum_{j=1}^{n-1} \frac{\partial x_j}{\partial p}(p; y_j) = \frac{-y_n}{\tilde{s}_n(p; y_n) - p - \lambda y_n}, \tag{122}$$

for every $p \in (\bar{p} - \delta, \bar{p} + \delta)$ and $(y_1, \dots, y_{n-1}) \in \prod_{j=1}^{n-1} (\bar{y}_j - \delta/n, \bar{y}_j + \delta/n)$, and where $y_n = -\sum_{j=1}^{n-1} y_j$.

Applying Lemma 4 to (122) gives:

$$\frac{\partial x_j}{\partial p}(p; y_j) = G(p) y_j + H_j(p), \tag{123}$$

for $j \in \{1, \dots, n-1\}$. Applying Lemma 4 to the ex post first-order condition of trader 1 shows that (123) holds for $j = n$ as well.

Substituting (123) back into the first-order condition (122), we obtain:

$$(\tilde{s}_i(p; y_i) - p - \lambda y_i) \left(-G(p)(-y_i) - \sum_{j \neq i} H_j(p) \right) - y_i = 0,$$

which can be rewritten as:

$$\frac{\partial x_i}{\partial p}(p; y_i) = G(p) y_i + H_i(p) = \frac{y_i}{\tilde{s}_i(p; y_i) - p - \lambda y_i} + \sum_{j=1}^{n} H_j(p). \tag{124}$$

We claim that $G(p) = 0$. Suppose for contradiction that $G(p) \neq 0$. Then matching the coefficient of y_i in (124), we must have $\tilde{s}_i(p; y_i) = \lambda y_i + B_i(p)$ for some function $B_i(p)$. But this implies that $\frac{\partial x_i}{\partial p}(p; y_i) = -B_i'(p)/\lambda$, which is independent of y_i. This implies $G(p) = 0$, a contradiction. Thus, $G(p) = 0$.

Then, (124) implies that $\tilde{s}_i(p; y_i) - p = A_i(p)y_i$ for some function $A_i(p)$. And since $\frac{\partial x_i}{\partial p}(p; y_i)$ is independent of y_i, $A_i(p)$ must be a constant function, i.e., $\tilde{s}_i(p; y_i) - p = A_i y_i$ for some $A_i \in \mathbb{R}$. Substitute this back to (124) gives:

$$\frac{\partial x_i}{\partial p}(p; y_i) = -\frac{1}{A_i} = \frac{1}{A_i - \lambda} - \sum_{j=1}^{n} \frac{1}{A_j},$$

which implies

$$\frac{1}{A_i - \lambda} - \frac{1}{A_j - \lambda} = \frac{1}{A_j} - \frac{1}{A_i}, \qquad \text{for all } i \neq j,$$

which is only possible if $A_i = A_j \equiv A \in \mathbb{R}$ for all $i \neq j$. Thus, $\tilde{s}_i(p; y_i) - p = Ay_i$, which concludes the proof of this case.

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