

# Appendix

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## 1 Propositions

*Worlds* are ways for things to be—possible ones in *some* sense of “possible” (Notation:  $w$ .) *Logical space* is the set of worlds. (Notation:  $\mathbf{W}$ .) *Propositions* are subsets of logical space, or sets of worlds. (Notation:  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ ).<sup>1</sup> A proposition  $\mathbf{A}$  is *true* in world  $w$  iff  $w \in \mathbf{A}$ , otherwise *false*.<sup>2</sup> *Sentences* are, you know. (Notation:  $A, B, C, \dots$ .) *Expression* is the relation sentence  $X$  bears to proposition  $\mathbf{X}$  when for all  $w$ ,  $X$  is true in  $w$  iff  $w \in \mathbf{X}$ . (Notation:  $\mathbf{A}$  is the propositional content of  $A$ ,  $\mathbf{B}$  is the propositional content of  $B$ , ...)

### 1.1 Running Example: Propositional Calculus

*Worlds* are classical valuations of a fixed propositional language. (Notation:  $\nu$ .) *Logical space* is the set of all classical valuations of the language. (Notation:  $\mathbf{V}$ .) *Propositions* are sets of classical valuations. (Notation:  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ .) *Sentences* are atoms  $p, q, r, \dots$ , and truth-table combinations thereof. (Notation:  $A, B, C, \dots$ .) *Expression* is the relation sentence  $X$  bears to its propositional content = the set of its classical models (the classical valuations in which  $X$  is true). (Notation:  $A$  expresses  $\mathbf{A}$ ,  $B$  expresses  $\mathbf{B}$ , etc.)

## 2 Similarity and Equivalence

An *equivalence relation* on  $\mathbf{S}$  is a reflexive, transitive, symmetric relation on  $\mathbf{S}$ . (Notation:  $\approx$ .) A *similarity relation* on  $\mathbf{S}$  is a reflexive, symmetric relation on  $\mathbf{S}$ . ( $\sim$ .) A collection  $\mathbf{m}$  of subsets of  $\mathbf{S}$  is a *partition* of  $\mathbf{S}$  iff its members are disjoint and jointly exhaustive of  $\mathbf{S}$ . A partition’s members are its *cells*. Partitions determine equivalence relations and vice versa.  $x \approx_{\mathbf{m}} y$  iff  $x$  and  $y$  cohabit a cell of  $\mathbf{m}$ .  $\mathbf{m}$ ’s cells are maximal sets of equivalents. These are also called *equivalence classes*. A

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<sup>1</sup>Alternatively we could define propositions as functions from worlds to truth-values. This is the better approach when it comes to partial propositions—ones defined only on certain worlds.

<sup>2</sup>Alternatively  $\mathbf{A}$  is true or false in  $w$  according to whether  $\mathbf{A}$  maps  $w$  to truth or falsity.

collection  $\mathbf{m}$  of subsets of  $\mathbf{S}$  is a *decomposition* of  $\mathbf{S}$  iff its members are *incomparable* and jointly exhaustive of  $\mathbf{S}$ . These are again referred to as *cells*. Decompositions induce similarity relations in the obvious way:  $x \sim_{\mathbf{m}} y$  iff some  $\mathbf{m}$ -cell contains both of them. (Call that  *$\mathbf{m}$ -similarity*.) But the relation here is many-one; we focus therefore on a certain *kind* of decomposition. A decomposition is a *division* iff (i) each cell is closed under  $\mathbf{m}$ -similarity (it contains every  $x$   $\mathbf{m}$ -similar to all its members), and (ii) every set closed under  $\mathbf{m}$ -similarity is one of its cells.<sup>3</sup> Divisions determine similarity relations and vice versa.  $x \sim_{\mathbf{m}} y$  iff some  $\mathbf{m}$ -cell contains both.  $\mathbf{m}$ 's cells are maximal sets of similars. A division's cells are called *similarity classes*.<sup>4</sup>

### 3 Refinement

If  $\mathbf{m}$  and  $\mathbf{m}'$  are partitions of  $\mathbf{S}$ , then  $\mathbf{m}$  *refines*  $\mathbf{m}'$  (according to the usual definition) iff each cell  $\mathbf{C}$  of  $\mathbf{m}$  is contained in a cell  $\mathbf{C}'$  of  $\mathbf{m}'$ . Call that *textbook* refinement. A slight variant proves more useful here. Here,  $\mathbf{m}$  *refines*  $\mathbf{m}'$  iff each cell  $\mathbf{C}'$  of  $\mathbf{m}'$  contains a cell  $\mathbf{C}$  of  $\mathbf{m}$ . Likewise if  $\mathbf{m}$  and  $\mathbf{m}'$  are divisions of  $\mathbf{S}$ .  $\mathbf{m}$  refines  $\mathbf{m}'$  iff each cell  $\mathbf{C}'$  of  $\mathbf{m}'$  contains a cell  $\mathbf{C}$  of  $\mathbf{m}$ . Refinement of one equivalence relation (similarity relation) by another corresponds in the obvious way to refinement of one partition (division) by another.

### 4 Subject Matters

A *lewisian* subject matter is an equivalence relation on  $\mathbf{W}$  = the set of worlds. Alternatively, it is a partition of  $\mathbf{W}$ , or a set of pairwise incompatible, jointly exhaustive, propositions.<sup>5</sup> An example is  $\mathbf{m}$  = **the number of stars**. This is the relation one world bears to another iff it contains exactly as many stars as the other. Alternatively it is a partition of logical space into the worlds with no stars, the worlds with one star, and so on.<sup>6</sup>

A subject matter as *we* use the term is a *similarity* relation on  $\mathbf{W}$  = the set of worlds. Alternatively, it is a division of  $\mathbf{W}$ : a set of pairwise independent, jointly exhaustive, propositions meeting the above-mentioned closure condition.<sup>7</sup> **The number of stars give or take ten** is the relation one world bears to another iff the number of stars in the one is within ten of the number of stars in the other. (This is not an equivalence relation for transitivity reasons. A 0-star world

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<sup>3</sup>I am indebted here to Hazen and Humberstone, "Similarity Relations and The Preservation of Solidity." Divisions are *quasi-closed* decompositions in their terms.

<sup>4</sup>See Hazen and Humberstone for details.

<sup>5</sup>I say "exhaustive," but Lewis is friendly to partial subject matters as well, defined on less than the full set of worlds (see the next note). We will follow him in this.

<sup>6</sup>Another example he gives is **the 20th century**. This is defined only on worlds where time passes. In practice, then, a lewisian subject matter is an equivalence relation on a subset  $\mathbf{U}$  of  $\mathbf{W}$ ; this will be treated as understood.

<sup>7</sup>I say "exhaustive," but subject matters need not be defined on all worlds; this is again treated as understood.

$w$  bears it to a 10-star world  $w'$ , which bears it to a 20-star world  $w''$ ;  $w$  does not have as many stars give or take ten  $w''$ .) **the number of stars give or take ten** corresponds to a division of logical space into the worlds with 0-10 stars, the worlds with 1-11 stars, and so on.

## 4.1 Subject Matters in PC

A lewisian subject matter is an equivalence relation on  $\mathbf{V}$ . Equivalently: a partition of  $\mathbf{V}$ . A subject matter (sans phrase) is a similarity relation on  $\mathbf{V}$ . Equivalently: a division of  $\mathbf{V}$ .

## 5 Inclusion

The subject matter **the number of stars and their sizes** would seem intuitively to include the subject matter **the number of stars**. How can we make this precise?

Lewis defines subject-matter inclusion as follows:  $\mathbf{m}$  contains  $\mathbf{m}'$  iff each way  $\mathbf{W}$  things can be with respect to  $\mathbf{m}$  implies a way  $\mathbf{W}'$  for things to be with respect to  $\mathbf{m}'$ . Here we use a slight variant of Lewis's definition:  $\mathbf{m}$  includes  $\mathbf{m}'$  iff each way  $\mathbf{W}'$  things can be with respect to  $\mathbf{m}'$  is implied by a way  $\mathbf{W}$  they can be with respect to  $\mathbf{m}$ .<sup>8</sup> Looking back at the definition of refinement (section 3), this is equivalent to saying that  $\mathbf{m}$  includes  $\mathbf{m}'$  iff  $\mathbf{m}$  is a refinement of  $\mathbf{m}'$ .<sup>9</sup>

Assuming the standard theory of questions as the set of an interrogative's possible answers, each subject matter  $\mathbf{m}$  = the question of **how matters stand with respect to  $\mathbf{m}$** .  $\mathbf{m}$  includes  $\mathbf{m}'$  iff each answer to *how do matters stand with respect to  $\mathbf{m}'$ ?* is implied by an answer to *how do matters stand with respect to  $\mathbf{m}$ ?*

## 6 Aboutness

Now we know what subject matters are, considered as self-standing objects, and we know what it is for one subject matter to include another. But nothing has been done to associate sentences with the subject matters they are about; so we don't know what it is for  $A$ 's subject matter to include  $B$ 's.

$A$  is *wholly about*  $\mathbf{m}$ , according to Lewis, iff  $A$  always has the same truth-value in  $\mathbf{m}$ -equivalent worlds. Likewise, let's say, if  $\mathbf{m}$  is a division rather than a partition;  $A$  is wholly about  $\mathbf{m}$  iff

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<sup>8</sup>**Height in millimeters** does not lewis-include **height in inches**, because some (one in a thousand) cells of the former aren't included in cells of the latter. **The weight of each star** does not lewis-include **the sun's weight**, since the former has cells that are defined only on worlds where the sun does not exist. These inclusions do obtain on our definition. (Lewis's notion would actually serve us fine, except in connection subject *anti*-matters (see below). Subject anti-matters aside, "inclusion" might as well mean lewis-inclusion.)

<sup>9</sup> $\mathbf{m}$  includes  $\mathbf{m}'$  in Lewis's sense iff  $\mathbf{m}$  is a textbook refinement of  $\mathbf{m}'$ .

when two worlds are **m**-similar, *A* has the same truth-value in both. *A*'s truth-value supervenes, in other words, on which cell of the subject matter we're in. (**A** is wholly about whatever *A* is wholly about.)

The notion Lewis gives us is weak and indiscriminating. Any ordinary *A* is wholly about a huge range of subject matters—for if *A* is wholly about **m**, then it is wholly about any subject matter that includes **m**. Any two sentences, however intuitively unrelated, have a subject matter in common, viz. **how matters stand in every respect**.

## 7 Sentential subject matters

Which of the subject matters *A* is wholly about is “the” subject matter of *A*? One might think of nominating the *weakest* subject matter *A* is wholly about for this role. But the weakest such subject matter is the two-cell subject matter **whether *A* is true**. And intuitively worlds differ with respect to *A*'s subject matter (not only when *A*'s truth-value changes but) when the *reasons* for *A*'s truth-value change.

Changes in the reasons for *A*'s truth are changes in how things stand with respect to *A*'s subject matter. If *A* is true for a certain reason in *w* but not in *w'*, then something has changed with respect to *A*'s subject matter and *w* and *w'* should not belong to exactly the same cells. That gives us a *lower* bound on *A*'s subject matter.

Conversely, nothing pertains to *A*'s subject matter which is of no possible relevance to why *A* is true. *A*'s subject matter will put *w* and *w'* into different cells *only if* there is a reason *A* is true in *w* that does not extend to *w'*. That gives us an *upper* bound on *A*'s subject matter.

The upper bound = the lower bound. A sentence *A*'s subject matter is the relation that counts worlds similar iff *A* is true in the two worlds for a shared reason, and dissimilar iff its reasons for being true are completely different in the two worlds. This corresponds to the division of logical space whose cells are the sets of worlds in which *A* is true for such and such a reason.

Cells are sets of worlds are propositions. *A*'s subject matter construed as a set of propositions is the set of all possible answers to *why is A true?* *A*'s subject matter, then, is not **whether *A* is true**, it is **why *A* is true**.

*A*'s subject matter so understood is not total; it is defined only on worlds where *A* is true. It may not be exclusive either—because a sentence's truth can be overdetermined, as when a disjunction has two true disjuncts.

By a sentence's truthmakers, we mean just the possible answers to *why is it true?* But the question has to be understood in a certain way. The next two sections try to explain the kind of thing we are looking for.

## 8 Truthmakers

A *fact obtaining in world  $w$*  is a proposition true in  $w$ . A proposition true in  $w$  is a set of worlds to which  $w$  belongs. So  $X$ 's subject matter can be thought of as a function assigning to each  $w$  a certain set of facts. These facts are  $X$ 's truthmakers in the relevant world.

A truthmaker for  $X$  in  $w$  is, at a minimum, a fact obtaining in  $w$  that necessitates  $X$ 's truth. Of course there will be many such facts. Truthmakers have the further feature that  $X$  is true in  $w$  “because of,” or “in virtue of,” the fact that its truthmaker obtains.<sup>10</sup>

I do not mean anything terribly heavy duty by “truthmaker” or “in virtue of,” however. I mean only that the truthmaker tells us *why*  $X$  is true. To say why  $X$  is true in  $w$  is to subsume its truth in  $w$  under some kind of regularity:  $X$  is true in all worlds of a certain type, and  $w$  is a world of that type. Insofar as the regularities affording the best line of sight on  $X$ 's truth vary with context, what counts as a truthmaker for  $X$  varies with context.

One can still ask, what are the principal factors bearing on a fact's status as truthmaker? The principal factors are naturalness and proportionality.

## 9 Naturalness and Proportionality

Suppose that  $\mathbf{T}$  and  $\mathbf{T}'$  both imply  $X$ , and are to that extent candidates for the role of truthmaker.

*Naturalness:*  $\mathbf{T}$  is more natural if it obtains in a more compact, principled set of worlds. **I am a man** is preferred to **I am a man or a mouse** as truthmaker for *I am a man or a mouse*.  $\mathbf{T}$  is also more natural if goings-on in a more compact, well-defined region determine whether it obtains. **That chair is empty** is preferred to **No one is ten feet tall** as truthmaker for *No one in that chair is ten feet tall*.

*Proportionality:*  $\mathbf{T}$  is more proportional to  $X$  if it involves fewer irrelevant extras in whose absence it would still imply  $X$ . Proportionality favors **Sparky weighs 16 pounds** over **Sparky is a black and white dog weighing 16 pounds** as truthmaker for *Sparky weighs under 20 pounds*.

For  $X$  to hold in virtue of  $\mathbf{T}$  is *something* in the neighborhood of  $\mathbf{T}$  effecting an optimal tradeoff between naturalness and proportionality. Any gains in naturalness to be made by strengthening  $\mathbf{T}$  are not worth the losses in proportionality, and any proportionality gains to be made by weakening  $\mathbf{T}$  are not worth the losses in naturalness.

Putting all this together, a truthmaker for  $X$  in world  $w$  is a fact that (i) obtains in  $w$ , (ii) implies  $X$ , and (iii) is as natural and proportional to  $X$  as a fact satisfying (i) and (ii) can be. A *falsemaker* for  $X$  in  $w$  is a truthmaker for  $X$ 's negation that obtains in  $w$ . In worlds where

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<sup>10</sup>This is assuming that  $X$  is fully defined, that is, it has a truth-value in every world. Later we'll extend the notion of truthmaker to sentences  $Y$  that are defined only on certain worlds. The truthmakers (in the extended sense) of such a  $Y$  will be expected to necessitate, not that  $Y$  is true, but that it is true-if-defined.

$X$ 's truth (falsity) is overdetermined, it has more than one truthmaker (falsemaker). A *potential truthmaker* for  $X$  is a  $\mathbf{T}$  which makes  $X$  true in some  $w$ . Potential falsemakers are defined similarly. (Notation:  $\mathbf{T}^X$ ,  $\mathbf{F}^X$ )

## 9.1 Truthmakers in PC

A *partial valuation*  $\nu$  of a given propositional language is an assignment of truth-values to some but not necessarily all of the language's atoms. A *partial model of  $A$*  is a partial valuation each of whose classical extensions makes  $A$  true. A *minimal model*  $\chi$  of  $X$  is a partial model of  $X$  none of whose proper submodels are partial models of  $X$ . (Notation:  $\alpha$  is a minimal model of  $A$ ,  $\beta$  is a minimal model of  $B$ , and so on.) A truthmaker for sentence  $X$  is a minimal model of  $X$ . A falsemaker for  $X$  is a minimal model of  $\neg X$ , aka a minimal *countermodel* of  $X$ . A truthmaker for  $X$  in  $\nu$  is a minimal model  $\chi$  of  $X$  such that  $\chi \subseteq \nu$ . A falsemaker for  $X$  in  $\nu$  is a minimal countermodel  $\chi$  of  $X$  such that  $\chi \subseteq \nu$ .

## 10 Aboutness

$X$ 's *subject matter* is the the relation  $w$  bears to  $w'$  iff  $X$  has a truthmaker in  $w$  that it also has in  $w'$ . Considered as a division of logical space, this is just the set of  $X$ 's potential truthmakers.<sup>11</sup> (Notation:  $A$ 's subject matter is  $\vec{\mathbf{a}}$ ,  $B$ 's is  $\vec{\mathbf{b}}$ , and so on.)  $X$ 's *subject anti-matter* is the same with “falsemaker” substituted for “truthmaker.” (Notation:  $A$ 's subject anti-matter is  $\overleftarrow{\mathbf{a}}$ ,  $B$ 's is  $\overleftarrow{\mathbf{b}}$ , and so on.)  $X$ 's *overall subject matter* is its subject matter and subject anti-matter taken together. (Notation:  $A$ 's overall subject matter is  $\mathbf{a}$ ,  $B$ 's overall subject matter is  $\mathbf{b}$ .)

*Aboutness* is the relation a sentence bears to its overall subject matter—also, in an extended sense, the relation it bears to its subject matter, and to its subject anti-matter. Note:  $X$  and  $\neg X$  have the same overall subject matter. The one's subject matter is the other's anti-matter and vice versa; and overall subject matter lumps matter and anti-matter together.

### 10.1 Aboutness in PC

$X$ 's subject matter is the  $\mathbf{m}$  whose cells are made up, for each minimal model  $\chi$  of  $X$ , of the classical models “above” (including)  $\chi$ .<sup>12</sup> (Notation:  $A$ 's subject matter is  $\vec{\mathbf{a}}$ ,  $B$ 's is  $\vec{\mathbf{b}}$ , and so

<sup>11</sup>A more fine-grained option to be considered later:  $X$ 's subject matter is a function taking each world  $w$  to  $X$ 's truthmakers in  $w$ . This allows for the possibility of worlds where a certain truthmaker obtains yet is not in that world a truthmaker, perhaps because some more proportional alternative beats it out.

<sup>12</sup>In practice it's often easier to think of  $\mathbf{m}$  as made up of the minimal models themselves. Each is recoverable from the other so nothing deep hangs on this.

on.)  $X$ 's subject anti-matter is the  $\mathbf{m}$  whose cells are the classical models above  $X$ 's minimal countermodels. (Notation:  $A$ 's subject anti-matter is  $\overleftarrow{\mathbf{a}}$ ,  $B$ 's is  $\overleftarrow{\mathbf{b}}$ , and so on.)  $X$ 's overall subject matter = its subject matter and anti-matter taken together. (Notation:  $A$ 's overall subject matter is  $\mathbf{a}$ ,  $B$ 's overall subject matter is  $\mathbf{b}$ , and so on.)

*Aboutness* is the relation  $X$  bears to (the classical models above) its minimal models and countermodels.<sup>13</sup>

## 11 Intensional and Directed Content

$X$ 's *intensional* content  $\mathbf{X}$  is the proposition it expresses, that is, the set of worlds where  $X$  is true, or (when we want to leave the door open to partial propositions) the function mapping worlds where  $X$  is true (false) to  $\mathbf{t}(\mathbf{f})$ . (These will sometimes be called *thin* propositions.)

$X$ 's *directed* content is (i) its intensional content  $\mathbf{X}$  —which specifies truth-value—together with (ii) its subject matter  $\mathbf{x}$  —which specifies the reasons for truth-value. (Notation:  $A$ 's directed content is  $\mathbf{A}$ ,  $B$ 's is  $\mathbf{B}$ , and so on.) (These will sometimes be called *thick* propositions.)

## 12 Content-Parts

The inference  $A$ , *therefore*  $B$  is *truth preserving* iff  $B$  is true in every world where  $A$  is true. It is *aboutness preserving* iff  $A$ 's subject matter includes  $B$ 's and  $A$ 's subject anti-matter includes  $B$ 's subject anti-matter.

$A$  *includes*  $B$ , or has  $B$  as a *part*, iff the inference  $A$ , *therefore*  $B$  is both truth preserving and aboutness preserving. (Notation:  $A \geq B$ .)  $\mathbf{A}$  *includes*  $\mathbf{B}$  iff  $\mathbf{A}$  *implies*  $\mathbf{B}$  and  $\mathbf{a}$  includes  $\mathbf{b}$  —here  $\mathbf{a}$  and  $\mathbf{b}$  are the subject matters, and  $\mathbf{A}$  and  $\mathbf{B}$  are the intensional contents, of  $\mathbf{A}$  and  $\mathbf{B}$ . (Notation:  $\mathbf{A} \geq \mathbf{B}$ .)  $\mathbf{A}$  includes  $\mathbf{B}$  under the same conditions, except that  $\mathbf{a}$  and  $\mathbf{b}$  are provided by context.

### 12.1 Content-Parts in PC

Given two PC-sentences  $A$  and  $B$ ,  $A \geq B$  iff

1.  $A$  implies  $B$ ,
2.  $A$ 's subject matter includes  $B$ 's subject matter, and
3.  $A$ 's subject anti-matter includes  $B$ 's subject anti-matter.

In full gory detail,  $A$ 's subject matter includes  $B$ 's iff for each minimal valuation  $\beta$  of  $B$ ,  $A$  has a minimal valuation  $\alpha$  such that every total valuation above  $\alpha$  is a total valuation above  $\beta$ . This

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<sup>13</sup>Also, in a looser sense, the relation it bears just to the minimal models, or minimal countermodels.

simplifies to: each minimal model  $\beta$  of  $B$  is included in a minimal model  $\alpha$  of  $A$ . Similarly  $A$ 's subject anti-matter includes  $B$ 's iff each minimal countermodel  $\bar{\beta}$  of  $B$  is included in a minimal countermodel  $\bar{\alpha}$  of  $A$ . If  $A$  implies  $B$ , this simplifies to: each minimal countermodel  $\bar{\beta}$  of  $B$  is a minimal countermodel  $\bar{\alpha}$  of  $A$ . So,  $A \geq B$  iff

1.  $A$  implies  $B$  (= each  $\alpha$  includes some  $\beta$ )
2. each  $\beta$  is included in some  $\alpha$ , and
3. each  $\bar{\beta}$  is identical to some  $\bar{\alpha}$ .

Examples of implications that are content-parts and implications that are not content-parts ( $\dot{\vee}$  is exclusive disjunction):

parts	non-parts
$p \wedge q \geq p$	$p \wedge q \not\geq p \vee q$
$p \wedge q \geq q$	$p \wedge q \not\geq p \leftrightarrow q$
$p \leftrightarrow q \geq p \rightarrow q$	$p \wedge q \not\geq p \rightarrow q$
$p \leftrightarrow q \geq q \rightarrow p$	$p \wedge q \not\geq q \rightarrow p$
$p \dot{\vee} q \geq p \vee q$	$p \wedge q \not\geq p \dot{\vee} q$
$pq \vee r \geq p \vee r$	$pq \vee pr \not\geq q \vee pr$
$pq \vee rs \geq q \vee s$	$pq \vee pr \not\geq pq \vee r$

Comparison with Gemes's notion of content-part.  $A \geq_g B$  iff each relevant model of  $B$  is contained in a relevant model of  $A$ —where a relevant model of  $X$  is a partial model of  $X$  evaluating all and only atoms  $p$  in  $X$  toggling which can change  $X$ 's truth-value. Asterisks in the following table indicates a difference in verdict with the present account:

parts	non-parts
$p \wedge q \geq_g p$	$p \wedge q \not\geq_g p \vee q$
$p \wedge q \geq_g q$	$p \wedge q \not\geq_g p \leftrightarrow q$
** $p \leftrightarrow q \not\geq_g p \rightarrow q$ **	$p \wedge q \not\geq_g p \rightarrow q$
** $p \leftrightarrow q \not\geq_g q \rightarrow p$ **	$p \wedge q \not\geq_g q \rightarrow p$
** $p \dot{\vee} q \not\geq_g p \vee q$ **	$p \wedge q \not\geq_g p \dot{\vee} \neg q$
$pq \vee r \geq_g p \vee r$	$pq \vee r \not\geq_g p \vee q \vee r$
$pq \vee rs \geq_g p \vee r$	$pq \vee pr \not\geq_g q \vee pr$
$pq \vee rs \geq_g q \vee s$	$pq \vee pr \not\geq_g pq \vee r$

## 13 Context

Whether  $A$  includes  $B$  is sensitive to the two sentences' subject matters. A sentence's subject matter can vary with context. Whether  $A$  has  $B$  as a part can, to that extent, vary with context



too. Take *Nothing is both F and G*. Coming after *Tell me about the Fs*, it is made true by facts like the following: **a, b, and c are not G** (here *a*, *b*, and *c* are the Fs). Coming after *Tell me about the Gs*, it is made true by facts of a different sort: **d and e are not F** (where *d* and *e* are the Gs). *a is not G* is part of *Nothing is F and G* in the first context but not the second; *d is not F* is part of *Nothing is F and G* in the second context but not the first.

**A**'s subject matter is subject to an additional sort of contextual variation: variation in the sentence whose intensional content **A** is understood to be. Inclusion relations among intensional contents are thus doubly variable. *All Fs are G* and *All non-Gs are non-Fs* are true in the same worlds, so they have the same intensional content, call it **X**. **X** will include **a is G** in contexts where the sentence employed is *All Fs are G* but not where the sentence is *All non-Gs are non-Fs*.

There is nothing context-sensitive, however, about  $\mathbf{A} \geq \mathbf{B}$ —for **B** and **A** have their subject matters written into them. **A** includes **B** iff **A** implies **B** and **a** includes **b**—here **a** and **b** are the subject matters which with **A** and **B** constitute the directed contents **A** and **B**. (Notation:  $\mathbf{A} \geq \mathbf{B}$ .)

Content-part is thus an external relation on sentences, and a doubly external relation on intensional contents; but it is an internal relation on directed contents.

Unless otherwise indicated we're assuming that *context is held fixed*, so that **S** is determined by **S** is determined by *S*. Which sentence *S* is used to express **S** is counted part of context, so the determination goes the other way too: *S* is determined by **S** is determined by **S**.  $A \geq B$ ,  $\mathbf{A} \geq \mathbf{B}$ , and  $\mathbf{A} \geq \mathbf{B}$  thus become equivalent claims, between which we can move freely back and forth. This is a freedom we will be taking advantage of in what follows.

## 14 Truthmaker Laws

“Truthmaker Laws” is a fancy name for presumed facts about truthmakers that don't fall immediately out of the formal structure but will be appealed to in proofs. I offer in their defense that they are provable in the simplified PC setting.

**1st Law** A statement's truthmakers are independent, that is, none implies any other.

*PC Version:* A statement's minimal models are incomparable, none includes any other. *Proof:* By definition of “minimal.”  $\square$

**2nd Law** If *A* implies *B*, then each of *A*'s truthmakers implies a truthmaker for *B*.

*PC Version:* If *A* implies *B*, then each minimal model of *A* includes a minimal model of *B*. *Proof:* Any minimal model  $\alpha$  of *A* is a model of *B* since *A* implies *B*. If  $\alpha$  is not itself a minimal model of *B*, then there is a truth-value assignment  $p\mathbf{v}$  in  $\alpha$  such that  $\alpha - p\mathbf{v}$  is still a model of *B*. Repeat the operation as necessary until arriving at a minimal model of *B*.  $\square$

**3rd Law** If  $B$  is part of  $A$ , then each falsemaker for  $B$  is a falsemaker for  $A$ .

*PC Version:* If  $B$  is part of  $A$ , each minimal countermodel  $\bar{\beta}$  of  $B$  is a minimal countermodel  $\bar{\alpha}$  of  $A$ . *Proof:*  $\bar{\beta}$  is a countermodel of  $A$  since  $A$  implies  $B$ .  $\bar{\beta}$  is included in an  $\bar{\alpha}$  since  $B$  is part of  $A$ .  $\bar{\alpha}$  wouldn't be minimal if the inclusion was proper. So  $\bar{\beta} = \bar{\alpha}$ .  $\square$

## 15 Content-Part is a Partial Order

$A \geq B$  (according to our definitions) iff (i)  $A$  implies  $B$ . (ii)  $A$ 's subject matter includes  $B$ 's subject matter, and (iii)  $A$ 's subject anti-matter includes  $B$ 's subject anti-matter.

Now we rewrite this definition in light of our account of subject matter and anti-matter. Let  $\mathbf{T}^X$  range over  $X$ 's truthmakers and  $\mathbf{F}^X$  over  $X$ 's falsemakers, for any sentence  $X$ .

$A \geq B$  iff (i)  $A$  implies  $B$ , (ii) each  $\mathbf{T}^B$  is implied by a  $\mathbf{T}^A$ , and (iii) each  $\mathbf{F}^B$  is implied by an  $\mathbf{F}^A$ .<sup>14</sup>

**Theorem 1**  $A \geq A$  (reflexivity).

*Proof* Obvious. [(i)]  $A$  implies  $A$ . [(ii)(iii)] Each of  $A$ 's truthmakers (falsemakers) is implied by one of  $A$ 's truthmakers (falsemakers).  $\square$

**Theorem 2** If  $A \geq B$  and  $B \geq A$ , then  $A = B$  (antisymmetry).<sup>15</sup>

*Proof* Suppose  $A$  includes  $B$  and vice versa. We must show that  $A$  and  $B$  are true in the same worlds ( $\mathbf{A} = \mathbf{B}$ ) and that they have the same truthmakers and falsemakers. By (i),  $A$  implies  $B$  and vice versa; hence  $A$  and  $B$  are true in the same worlds. From (ii) we have that each  $\mathbf{T}^A$  is implied by a  $\mathbf{T}^B$  and vice versa. Suppose for contradiction that some  $\mathbf{T}^A$  is not equivalent to any  $\mathbf{T}^B$ .  $\mathbf{T}^A$  must then be properly implied by some  $\mathbf{T}^B$ . This  $\mathbf{T}^B$  must be implied in turn by a  $\mathbf{T}^A$ . But then one of  $A$ 's truthmakers properly implies another, contrary to the 1st Law (section 8).  $\mathbf{T}^A$  is thus equivalent to a  $\mathbf{T}^B$  after all and (by the same reasoning) each  $\mathbf{T}^B$  is equivalent to a  $\mathbf{T}^A$ . The argument on the falsemaker side is similar.  $\square$

**Theorem 3** If  $A \geq B$  and  $B \geq C$  then  $A \geq C$  (transitivity).

*Proof* [(i)] Clearly  $A$  implies  $C$ . [(ii)(iii)] Each  $\mathbf{T}^C$  is implied by a  $\mathbf{T}^B$  since  $B \geq C$ , which is in turn implied by a  $\mathbf{T}^A$  since  $A \geq B$ ; so each  $\mathbf{T}^C$  is implied by a  $\mathbf{T}^A$ . Similar reasoning applies on the falsemaker side.  $\square$

<sup>14</sup>By the 3rd Truthmaker Law, this can be read as: each  $\mathbf{F}^B$  is an  $\mathbf{F}^A$ .

<sup>15</sup>Remember,  $A = B$  here expresses identity up to directed content.

## 16 Partial Truth

A directed content  $\mathbf{A}$  is *partly true* iff it has a (non-trivial) true part. A sentence  $A$  with  $\mathbf{A}$  as its directed content is partly true iff  $\mathbf{A}$  is partly true. An intensional content  $\mathbf{A}$  is partly true iff  $\mathbf{A} = \langle \mathbf{A}, \mathbf{a} \rangle$  is partly true, where  $\mathbf{a}$  is the contextually indicated subject matter .

## 17 The Part of $A$ about BLAH

Given a sentence  $A$  and a lewisian subject matter  $\mathbf{m}$ , we now define  $A_{\mathbf{m}}$  = the part of  $A$  that concerns  $\mathbf{m}$ .<sup>16</sup> This is done by specifying first  $A_{\mathbf{m}}$ 's intensional content  $\mathbf{A}_{\mathbf{m}}$ , and then its subject matter  $\vec{\mathbf{a}}_{\mathbf{m}}$  and subject anti-matter  $\overleftarrow{\mathbf{a}}_{\mathbf{m}}$ .

Intensional content:  $\mathbf{A}_{\mathbf{m}}$  is the union of all  $\mathbf{m}$ -cells overlapping  $\mathbf{A}$ , that is, the set of all worlds  $w$  that are  $\mathbf{m}$ -equivalent to some  $\mathbf{A}$ -world.  $A$  is *true about  $\mathbf{m}$*  in  $w$  iff  $\mathbf{A}_{\mathbf{m}}$  is true in  $w$ , that is, iff  $w$  is  $\mathbf{m}$ -equivalent to a world where  $A$  is simply true.

Let  $A$  be  $\#(\textit{sheep}) = 3 \times \#(\textit{goats})$ . Let  $\mathbf{m}$  be **concreta**, that is, worlds are  $\mathbf{m}$ -equivalent iff their concrete parts are exactly alike.  $\mathbf{A}_{\mathbf{m}}$  is the set of all worlds concretely indiscernible from some number-containing world where  $\#(\textit{sheep}) = 3 \times \#(\textit{goats})$ .  $\mathbf{A}_{\mathbf{m}}$  is thus the set of worlds with three times as many sheep as goats.  $\#(\textit{sheep}) = 3 \times \#(\textit{goats})$  is true about **concreta** in worlds like that whether numbers exist or not.

Subject matter:  $\vec{\mathbf{a}}_{\mathbf{m}}$  is the set of  $A_{\mathbf{m}}$ 's truthmakers and  $\overleftarrow{\mathbf{a}}_{\mathbf{m}}$  is the set of its falsemakers. Truthmakers are obtained as follows. Given a truthmaker  $\mathbf{T}^A$  for  $A$ , the corresponding truthmaker  $\mathbf{T}_{\mathbf{m}}^A$  for  $A_{\mathbf{m}}$  is the union of all  $\mathbf{m}$ -cells overlapping  $\mathbf{T}^A$ .  $A_{\mathbf{m}}$ 's falsemakers are those of  $A$ 's falsemakers containing no worlds  $\mathbf{m}$ -equivalent to worlds where  $A$  is true.

Let  $A$  be  $\#(\textit{sheep}) = 3 \times \#(\textit{goats})$  and let  $\mathbf{m}$  be **concreta**. A truthmaker for  $A_{\mathbf{m}}$  is a truthmaker for  $A$  closed under concrete indiscernibility. Say the truthmaker for  $A$  is  **$\#(\textit{sheep}) = 21$  and  $\#(\textit{goats}) = 7$** ; the corresponding truthmaker for  $A_{\mathbf{m}}$  is the set of all worlds concretely indiscernible from some world where  $\#(\textit{sheep}) = 21$  and  $\#(\textit{goats}) = 7$ , that is, the set of worlds with twenty-one sheep and seven goats.  $A_{\mathbf{m}}$ 's truthmakers are thus all and only propositional contents to the effect that there are so and so many sheep and thus and so many goats, where so and so many is three times as many as thus and so many.

A falsemaker for  $A_{\mathbf{m}}$  is a falsemaker for  $A$  containing no worlds concretely indiscernible from worlds where  $A$  is true.  $A$ 's falsemakers are **Numbers don't exist** and facts of the form **There are so and so many goats and thus and so many sheep**, where so and so many is *not* three times as many as thus and so many. **Numbers don't exist** is *not* a falsemaker for  $A_{\mathbf{m}}$  because it contains worlds concretely indiscernible from  $\#(\textit{sheep}) = 3 \times \#(\textit{goats})$ -worlds, for instance, num-

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<sup>16</sup>Strictly we are defining a directed content  $\mathbf{A}_{\mathbf{m}}$  from  $\mathbf{A}$  and  $\mathbf{m}$ , not a sentence as the notation  $A_{\mathbf{m}}$  suggests, but it is easier to think in terms of sentences.

berless worlds with twenty-one goats and seven sheep. **There are six goats and five sheep** is a falsemaker for  $A_{\mathbf{m}}$  because it contains no world that is the same in concrete respects as a world where  $\#(\text{sheep}) = 3 \times \#(\text{goats})$ .

$A_{\mathbf{m}}$ 's falsemakers are thus all and only propositional contents to the effect that there are so and so many sheep and thus and so many goats, where so and so many is *not* three times as many as thus and so many.

## 18 The State of Things Where BLAH is Concerned

A world  $w$ 's  $\mathbf{m}$ -condition (intuitively, what  $w$  is like where subject matter  $\mathbf{m}$  is concerned) is the set of worlds  $\mathbf{m}$ -equivalent to  $w$ . Notation:  $\mathbf{m}(w)$ .

$w$ 's  $\mathbf{m}$ -condition *precludes*  $A$  iff  $\mathbf{m}(w)$  implies  $\neg A$ , that is,  $A$  is false in all worlds  $\mathbf{m}$ -equivalent to  $w$ . Otherwise  $w$ 's  $\mathbf{m}$ -condition *allows*  $A$ .  $w$ 's  $\mathbf{m}$ -condition *forces*  $A$  iff it precludes  $\neg A$ .

Let  $\mathbf{m}$  be **concreta** as before. If  $w$  is a numberless world with twice as many sheep as goats, then  $w$ 's  $\mathbf{m}$ -condition precludes  $\#(\text{sheep}) = 3 \times \#(\text{goats})$ . If  $w$  is a numberless world with three times as many sheep as goats, then its  $\mathbf{m}$ -condition allows  $\#(\text{sheep}) = 3 \times \#(\text{goats})$ . A world's  $\mathbf{m}$ -condition (what it is like concretely) never forces  $\#(\text{sheep}) = 3 \times \#(\text{goats})$ .

Here is a useful recharacterization of the intensional content  $\mathbf{A}_{\mathbf{m}}$  of the part of  $A$  that concerns  $\mathbf{m}$ :  $\mathbf{A}_{\mathbf{m}}$  is the set of worlds in an  $\mathbf{m}$ -condition that allows  $A$ .

Here is a useful recharacterization of  $A_{\mathbf{m}}$ 's subject matter, aka its truthmakers, and its subject anti-matter, aka its falsemakers: The truthmaker  $\mathbf{T}_{\mathbf{m}}^A$  for corresponding to  $A$ 's truthmaker  $\mathbf{T}^A$  is the set of worlds in an  $\mathbf{m}$ -condition that allows  $\mathbf{T}^A$ .  $A_{\mathbf{m}}$ 's subject matter is thus the subject matter of  $A$ , but with each cell  $\mathbf{T}^A$  expanded to include worlds  $w$  where, although  $\mathbf{T}^A$  is false, it is not disallowed by  $w$ 's  $\mathbf{m}$ -condition. A falsemaker  $\mathbf{F}_{\mathbf{m}}^A$  for  $A_{\mathbf{m}}$  is a falsemaker  $\mathbf{F}^A$  for  $A$  none of whose member worlds is in an  $\mathbf{m}$ -condition that allows  $A$ .  $A_{\mathbf{m}}$ 's subject anti-matter is thus the subject anti-matter of  $A$ , minus any falsemakers obtaining in worlds whose  $\mathbf{m}$ -condition allows  $A$ .

$A$  is true about  $\mathbf{m}$  iff the part of  $A$  about  $\mathbf{m}$  is true simpliciter.<sup>17</sup>

## 19 Properties of Directed Truth

Directed truth is more possibility-like than necessity-like. That a sentence is true about  $\mathbf{m}$  in  $w$  means it *can* be true compatibly with  $w$ 's  $\mathbf{m}$ -condition, not that it *must* be true given  $w$ 's  $\mathbf{m}$ -condition.<sup>18</sup>

<sup>17</sup>Note, we have still to prove that what I am calling the part of of  $A$  about  $\mathbf{m}$  really is part of  $A$  by our definitions. Only then can we say that truth about  $\mathbf{m}$  entails partial truth.

<sup>18</sup>One could also define a necessity-like notion:  $A$  is true "with respect to"  $\mathbf{m}$  in  $w$  iff  $w$ 's  $\mathbf{m}$ -condition precludes  $\neg A$ . So, for instance, if the actual world's concrete condition precludes *The number of planets  $\neq 8$* , then *The*

Just as  $A$ 's possibility doesn't rule out that  $\neg A$  is also possible, the fact that  $A$  is true about  $\mathbf{m}$  does not rule out that  $\neg A$  is also true about  $\mathbf{m}$ . From this we see that truth about  $\mathbf{m}$  is not closed under conjunction.<sup>19</sup>

Example:  $\mathbf{m}$  is **the number of stars** and  $A$  is *There are more stars than planets*.  $A$  is true about  $\mathbf{m}$  in the actual world @ since @ has a number-of-stars duplicate with fewer planets than stars.  $\neg A$  is also true of  $\mathbf{m}$  in @, since we could pile on enough additional planets to outnumber the stars with changing the number of stars. As a rule, the more worlds there are in which  $A$  and its negation are both true about  $\mathbf{m}$ , the less  $A$  and its negation are saying about  $\mathbf{m}$ .<sup>20</sup> *There are more stars than planets* and its negation are both true about **the number of stars**. This is because the two sentences don't say much about how many stars there are. They say a little, since *There are more stars than planets* tells us the universe is not completely starless. And they say a lot when combined with information about how many planets there are. But *There are more stars than planets*'s bearing all by itself on the subject matter of how many stars there are is very slight.

Sometimes  $A$  and its negation say nothing at all about about  $\mathbf{m}$ . *There are numbers* is true about how matters stand concretely in any world  $w$ , for it is true in a concrete duplicate of  $w$  to which numbers have been (if necessary) added. *There are no numbers* is true about how matters stand concretely in  $w$  too, for it is true in a concrete duplicate of  $w$  from which numbers have been (if necessary) removed. If it seems strange that *There are numbers* and its negation should both be true about the concrete world, it should be remembered that *There are numbers* would be a silly thing to say in a discussion of how matters stand concretely. More generally a discussion  $\mathbf{m}$  is unlikely to include sentences that say absolutely nothing about  $\mathbf{m}$ .

**Theorem 4** *If  $A$  implies  $B$ , then  $B$  is true about  $\mathbf{m}$  if  $A$  is.* (Single Premise Closure)

*Proof* Suppose  $A$  implies  $B$ , and  $A$  is true about  $\mathbf{m}$  in world  $w$ . Then  $w$ 's  $\mathbf{m}$ -cell contains an  $A$ -world.  $A$  implies  $B$  so that  $A$ -world is also a  $B$ -world. It follows that  $B$  is true in a world  $\mathbf{m}$ -equivalent to  $w$  and hence that  $B$  is true about  $\mathbf{m}$  in  $w$ .  $\square$

$A_1, \dots, A_k$  are  $\mathbf{m}$ -compatible iff  $\mathbf{m}$ -cells containing worlds where  $A_1$  is true and worlds where  $A_2$  is true...and worlds where  $A_k$  is true also contain worlds in which the  $A_i$ s are true together.

**Theorem 5** *Implications of statements true about  $\mathbf{m}$  are also true about  $\mathbf{m}$ , provided the implying statements are  $\mathbf{m}$ -compatible.* (Multi-Premise Quasi-Closure)

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*number of planets = 8* is true with respect to concreta. (I assume that numbers are presupposed here and that the presupposition is preserved under negation.) This alternative notion is worth exploring, but not here.

<sup>19</sup>Truth with respect to  $\mathbf{m}$  is closed under conjunction.

<sup>20</sup>This does not mean  $A$  is not informative about goings on in our little corner of logical space, for the actual world may be one in which  $A$  is true about  $\mathbf{m}$  and  $\neg A$  is false.

*Proof* Suppose  $A_1, \dots, A_k$  are  $\mathbf{m}$ -compatible and imply  $B$ . Suppose each of the  $A_i$ s is true about  $\mathbf{m}$  in world  $w$ . Then  $w$ 's  $\mathbf{m}$ -cell contains  $A_1$ -worlds and...and  $A_k$ -worlds. It follows by  $\mathbf{m}$ -compatibility that  $w$ 's  $\mathbf{m}$ -cell contains a world  $u$  in which all the  $A_i$ s are true.  $B$  is implied by the  $A_i$ s, so  $B$  is true in  $u$  too.  $u$  is a  $B$ -world  $\mathbf{m}$ -equivalent to  $w$ , so  $B$  is true about  $\mathbf{m}$  in  $w$ .  $\square$

## 20 Directed Truth and Partial Truth

**Theorem 6**  $A_{\mathbf{m}}$  is part of  $A$ .

*Proof* The inference from  $A$  to  $A_{\mathbf{m}}$  is [Truth-preserving] If  $A$  is true in  $w$ , then  $w$  is  $\mathbf{m}$ -equivalent to an  $A$ -world, hence  $A_{\mathbf{m}}$  is true in  $w$ . [Aboutness-preserving] *Subject matter inclusion*: Every truthmaker  $\mathbf{T}_{\mathbf{m}}^A$  for  $A_{\mathbf{m}}$  is obtained from a  $\mathbf{T}^A$  that implies it. *Subject anti-matter inclusion*: Every  $\mathbf{F}_{\mathbf{m}}^A$  is identical to an  $\mathbf{F}^A$ , hence implied by an  $\mathbf{F}^A$ .  $\square$

**Theorem 7** If  $A$  is true about  $\mathbf{m}$ , then  $A$  is partly true.

*Proof* Suppose  $A$  is true about  $\mathbf{m}$ . Then  $A_{\mathbf{m}}$  is true (by definition of “true about  $\mathbf{m}$ ”).  $A_{\mathbf{m}}$  is part of  $A$  (just proved). So part of  $A$  is true. So  $A$  is partly true (by definition of partial truth as truth of a part).  $\square$

## 21 Value Added

Consider the relation between  $p \wedge q$  and  $p$ , in a world where  $p$  is false and  $q$ 's truth-value is unspecified.  $p \wedge q$  and  $p$  are both false in this world, and neither is any falsier than the other. But there is a clear sense in which  $p \wedge q$  *adds falsity* to  $p$  if  $q$  is false (it commits a further offense against truth beyond that committed already by  $p$ ) and *adds truth* if  $q$  is true (it is true where it goes beyond  $p$ ).<sup>21</sup>

This much seems clear—if  $B$  is true in  $w$ , then  $A$  adds truth or falsity according to whether  $A$  is itself true in  $w$  or false there. The question is what  $A$  adds to  $B$  in  $w$  if  $B$  is false in  $w$ . (Don't assume something is always added.  $x$  is red adds truth to  $x$  is colored if  $x$  is red, and it adds falsity if  $x$  is some other color. But if  $x$  is not colored at all (if it is, say, a natural number), then  $x$  is red does not add truth OR falsity to  $x$  is colored.)

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<sup>21</sup>“But  $q$  might contain some truth; it might be the conjunction of a truth and a falsehood.” It doesn't matter;  $q$  is still false, and  $p \wedge q$  still adds falsity to  $p$ . This is the same asymmetry we see with outright truth and falsity; a truth conjoined with a falsehood is false rather than true. Adding falsity is (to a first approximation) adding *at least some* falsity. Adding truth is adding *only* truth.

The idea is simple:  $A$  adds truth to  $B$  in  $w$  iff  $B \supset A$  has in  $w$  a certain kind of truthmaker — what I'll call a *targeted* truthmaker. A targeted truthmaker  $\mathbf{T}$  for  $B \supset A$  is a fact that (as far as possible, see below) rules out the *combination* of  $B$  true with  $A$  false *as such*—not, in other words, (1) by ruling  $B$  out, nor (2) by ruling  $A$  in.

What (1) requires of targeted truthmakers  $\mathbf{T}$  is that they obtain in at least some  $B$ -worlds. This makes sense since a targeted truthmaker is to be that in virtue of which  $A$  *adds* truth, or falsity, to  $B$ , and the fact that  $B$  is false in a world  $w$  should be completely irrelevant to what  $A$  *adds* to  $B$ . Any reason  $A$  has for adding truth or falsity in  $w$  should be a reason that obtains also in  $B$ -worlds.

Should we require conversely that any reason  $A$  has for adding truth (falsity) in a  $B$ -world is a reason that obtains also in  $\neg B$ -worlds? No. The difference is that  $A$  quite definitely adds truth or falsity to  $B$  in each  $B$ -world (according to whether  $A$  is itself true or false). But we cannot assume that  $A$  adds either truth or falsity in each  $\neg B$ -world; indeed  $A$  may not add *either* in *any*  $\neg B$ -world. (As  *$x$  is red* does not add truth or falsity to  *$x$  is colored* in worlds where  $x$  is not colored.) What we can require of  $\mathbf{T}$  is that it should assign to  $B$  as much responsibility as possible for the fact that combined with  $B$ , it implies  $A$ .

Any truthmaker  $\mathbf{T}$  for  $B \supset A$  combines with  $B$  to imply  $A$ . A targeted truthmaker should minimize the extent to which proper parts  $B^-$  of  $B$  *also* combine with  $\mathbf{T}$  to imply  $A$ . ( $\mathbf{T}$  and  $\mathbf{T}'$  range over truthmakers for  $B \supset A$ .)

1.  $\mathbf{T}'$  uses more of  $B$  than  $\mathbf{T}$  does iff  $\{B^- \mid \mathbf{T}', B^- \not\equiv A\} \subsetneq \{B^- \mid \mathbf{T}, B^- \not\equiv A\}$
2.  $\mathbf{T}$  is *B-wasteful* in  $w$  iff some  $\mathbf{T}'$  obtaining in  $w$  uses more of  $B$  than  $\mathbf{T}$  does.
3.  $\mathbf{T}$  is *B-wasteful* (period) iff it is *B-wasteful* in every  $B$ -world where it holds.
4.  $\mathbf{T}$  is *B-efficient* iff it is not *B-wasteful*.<sup>22,23</sup>

A truthmaker  $\mathbf{T}$  for  $B \supset A$  in  $w$  is *targeted* iff it is *B-compatible* and *B-efficient*.<sup>24</sup>

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<sup>22</sup> $B$ -wasteful truthmakers for  $B \supset \neg A$  are defined similarly.  $B \supset \neg A$ 's truthmakers are never  $B$ -wasteful if  $A$  implies  $B$ , since  $B \supset \neg A$  is in that case equivalent to  $\neg A$ , and any  $\mathbf{T}$  that implies  $\neg A$  trivially implies  $B^- \supset A$  for all  $B^-$ .

<sup>23</sup>Suppose, for instance, that  $B$  is *snow is white* and  $A$  is *snow is white and cold*. The fact  $\mathbf{T}$  that snow is white and cold is  $B$ -wasteful, because  $\mathbf{T}' =$  the fact that snow is cold holds in every  $B \wedge \mathbf{T}$ -world and uses more of  $B$  in all of them.

<sup>24</sup>Why define  $B$ -wastefulness as the property  $\mathbf{T}$  has if *in every*  $\mathbf{T}$ -world, some  $\mathbf{T}^{B \supset A}$  obtaining in  $w$  uses more of  $B$  than  $\mathbf{T}$  does? Wouldn't it have been simpler to leave out the world-quantifier, and let  $\mathbf{T}$  be  $B$ -wasteful iff some  $\mathbf{T}^{B \supset A}$  uses more of  $B$  than it does, regardless of which worlds it obtains in? The problem is that a less wasteful  $\mathbf{T}^{B \supset A}$  may not be available in every  $B$ -world where  $A$  holds— and we are agreed that  $A$  adds truth (falsity) to  $B$  in *every*  $B$ -world where  $A$  is true (false). Suppose, for instance, that  $A$  is  $p$  and  $B = p \vee q$ .  $\mathbf{p}$  is a  $p \vee q$ -wasteful (in the suggested alternative sense) truthmaker for  $(p \vee q) \supset p$ , since  $\mathbf{p}$  unlike  $\bar{\mathbf{q}}$  continues to imply  $B^- \supset p$  whatever we put in for  $B^-$ . But we can't throw  $\mathbf{p}$  out, or  $p$  will not add truth to  $p \vee q$  in  $p$ -worlds where  $q$  is also true,  $\mathbf{p}$  being the only available truthmaker for  $(p \vee q) \supset p$  in such worlds.

**Theorem 8** *If  $A$  implies  $B$ ,  $B \supset A$  has a targeted  $t$ -maker in  $B$ -worlds where it has any  $t$ -maker.*

*Proof:* Suppose  $w$  is a  $B$ -world and let  $\mathbf{T}$  make  $B \supset A$  true in  $w$ . If  $\mathbf{T}$  is  $B$ -efficient, we're done. Otherwise there is a  $\mathbf{T}'$  holding in  $w$  using more of  $B$  than  $\mathbf{T}$  does. The same holds of  $\mathbf{T}'$ , etc. Continuing in this way, we eventually reach a  $\mathbf{T}^{(n)}$  holding in  $w$  such that no  $\mathbf{T}^{(n+1)}$  holding in  $w$  uses more of  $B$  than  $\mathbf{T}^{(n)}$  does.  $\mathbf{T}^{(n)}$  is by definition not  $B$ -wasteful in  $w$ ; hence it is not  $B$ -wasteful in at least one  $B$ -world; hence it is not  $B$ -wasteful period. It is  $B$ -compatible since  $w$  is a  $B$ -world. Hence  $B \supset A$  has a targeted truthmaker in  $w$ .  $\square$

**Theorem 9** *If  $A$  implies  $B$ , then (i)  $A$  adds truth to  $B$  in  $w$  iff  $B \supset A$  has a  $B$ -compatible truthmaker in  $w$ , and (ii)  $A$  adds falsity to  $B$  in  $w$  iff  $B \supset \neg A$  has a  $B$ -compatible truthmaker in  $w$ .*

*Proof:*  $[\Rightarrow]$  Obvious.  $[(ii), \Leftarrow]$   $B \supset \neg A$  is equivalent to  $\neg A$ , since  $A$  implies  $B$ . Any  $\mathbf{T}$  that implies  $\neg A$  trivially implies  $B^- \supset A$  for all  $B^-$ . But then no  $\mathbf{T}^{B \supset \neg A}$  uses more of  $B$  than any other, making  $B \supset \neg A$ 's truthmakers all  $B$ -efficient.  $B \supset \neg A$ 's  $B$ -compatible truthmaker is thus also a  $B$ -efficient truthmaker, hence a targeted truthmaker; so  $A$  adds falsity to  $B$  in  $w$ .  $[(i), \Leftarrow]$  This is immediate from the previous theorem, if  $w$  is a  $B$ -world. Suppose, then, that  $w$  is a  $\neg B$ -world. If  $\mathbf{T}$  is a  $B$ -compatible truthmaker for  $B \supset A$  in  $w$ , then  $\mathbf{T}$  obtains also in some  $B$ -world. Now we show that every  $\neg B$ -world covered by a  $\mathbf{T}^{B \supset A}$  obtaining in some  $B$ -world is covered by a  $B$ -efficient such  $\mathbf{T}^{B \supset A}$ .  $B^-$  holds in a world iff some  $B \vee \mathbf{fB}$  holds there, where  $\mathbf{fB}$  = a conjunction of one or more falsemakers for  $B$  with the negations of  $B$ 's other falsemakers. Thus  $\mathbf{T}'$  uses more of  $B$  than  $\mathbf{T}$  does iff there are  $\mathbf{fBs}$  which combine with  $\mathbf{T}$ , but not  $\mathbf{T}'$ , to imply  $A$ . Now, each  $\mathbf{fB}$  is inconsistent with  $A$ , since  $A$  implies  $B$  and  $\mathbf{fB}$  implies  $\neg B$ ; so  $\mathbf{T}$  combines with more  $\mathbf{fBs}$  than  $\mathbf{T}'$  to imply  $A$  iff it is consistent with fewer  $\mathbf{fBs}$  than  $\mathbf{T}'$  is. But this can be true only if  $\mathbf{T}'$  holds in more  $\neg B$ -worlds than  $\mathbf{T}$  does, for the  $\mathbf{fBs}$  partition the  $\neg B$ -worlds. Thus the  $B$ -compatible,  $B$ -efficient  $\mathbf{T}^{B \supset A}$ s extend collectively *at least* as far into the  $\neg B$ -region as the  $B$ -compatible,  $B$ -wasteful ones. If any  $B$ -compatible  $\mathbf{T}^{B \supset A}$  obtains in  $w$ , then, a  $B$ -efficient (hence a targeted) one does—which is what it takes for  $A$  to add truth to  $B$  in  $w$ .  $\square$

Targeted truthmakers for  $B \supset A$  ( $B \supset \neg A$ ) can thus be understood, for added-value purposes, just as  $B$ -compatible truthmakers—until section 31 when we drop the assumption that  $A$  implies  $B$ .

## 21.1 Value Added in PC

Let  $X$  and  $Y$  be sentences of the propositional calculus such that  $X$  implies  $Y$ ; and let  $\mu$  be a minimal model of  $Y \supset X$ .  $\mu$  is  *$Y$ -compatible* =<sub>df</sub> some classical model  $\eta$  of  $X$  contains  $\mu$ .



$X$  adds  $\left\{ \begin{array}{l} \text{truth} \\ \text{falsity} \end{array} \right\}$  to  $Y$  in  $\eta$  iff  $\eta$  contains a  $Y$ -compatible model of  $\left\{ \begin{array}{l} Y \supset X \\ Y \supset \neg X \end{array} \right\}$ .<sup>25</sup>

A  $Y$ -compatible model of  $Y \supset X$  (...) *witnesses* the fact that  $X$  adds truth (falsity) to  $Y$ . To determine whether  $X$  adds falsity (truth) to  $Y$  in  $\eta$ , we first make truth-tables for  $Y \supset X$  and  $Y \supset \neg X$ . Then we proceed as follows

1. Is  $Y \supset X$  true for a  $Y$ -compatible reason?<sup>26</sup>
  - (a) If so, then  $X$  ADDS TRUTH TO  $Y$  in  $\eta$ .
  - (b) If not, then  $X$  ADDS NO TRUTH TO  $Y$  in  $\eta$ .
2. Is  $Y \supset \neg X$  true for a  $Y$ -compatible reason?<sup>27</sup>
  - (a) If so, then  $X$  ADDS TRUTH TO  $Y$  in  $\eta$ .
  - (b) If not, then  $X$  ADDS NO TRUTH TO  $Y$  in  $\eta$ .

The above method is now used to determine under what conditions

1.  $p \wedge q$  adds truth or falsity to  $q$
2.  $p$  adds truth or falsity to  $p \vee q$
3.  $p \leftrightarrow q$  adds truth or falsity to  $p \rightarrow q$
4.  $p \dot{\vee} q$  adds truth or falsity to  $p \vee q$  ( $\dot{\vee}$  is exclusive disjunction)
5.  $p \wedge q$  adds truth or falsity to  $p \leftrightarrow q$
6.  $p \wedge q$  adds truth or falsity to  $p \vee q$
7.  $pq \vee r$  adds truth to  $p \vee r$
8.  $pq \vee rs$  adds truth or falsity to  $p \vee r$

$X+\mathbf{f}$  ( $X+\mathbf{t}$ ) means “ $X$  adds falsity (truth).”  $w$ 's are “witnesses.”  $\overline{pq}$  is  $\neg(p \wedge q)$ .  $\overline{p\overline{q}}$  is  $\neg p \wedge \neg q$ .

$p$	$q$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X+\mathbf{f}$	$X+\mathbf{t}$
$p$	$q$	$p \wedge q$	$q$	$q \supset p$	$q \supset \neg p$	$w$ 's	$w$ 's
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{p\mathbf{t}\}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	—	$\{p\mathbf{t}\}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{p\mathbf{f}\}$	—
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\{p\mathbf{f}\}$	—

<sup>25</sup>Officially,  $X$  adds truth to  $Y$  iff a *targeted* ( $Y$ -compatible and  $Y$ -efficient) model of  $Y \supset X$  exists. But we know by Theorem 9 that  $Y$ -compatibility is enough when  $X$  implies  $Y$ ; where models exist at all, a  $Y$ -efficient model exists.

<sup>26</sup>Meaning, does  $\eta$  contain a  $Y$ -compatible model of  $Y \supset X$ ?

<sup>27</sup>Meaning, does  $\eta$  contain a  $Y$ -compatible model of  $Y \supset \neg X$ ?

2.

$p$	$q$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X + \mathbf{f}$	$X + \mathbf{t}$
$p$	$q$	$p$	$p \vee q$	$q \supset p$	$\neg q \supset p$	$w's$	$w's$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	—	$\{\mathbf{pt}\}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	—	$\{\mathbf{pt}\}, \{\mathbf{qf}\}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{pf}\}$	—
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\{\mathbf{pf}\}$	$\{\mathbf{qf}\}$

3.

$p$	$q$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X + \mathbf{f}$	$X + \mathbf{t}$
$p$	$q$	$p \leftrightarrow q$	$p \supset q$	$q \supset p$	$p \leftrightarrow q$	$w's$	$w's$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{\mathbf{pt}\}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	—	$\{\mathbf{pt}\}, \{\mathbf{qf}\}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{pf}, \mathbf{qt}\}$	—
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{\mathbf{qf}\}$

4.

$p$	$q$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X + \mathbf{f}$	$X + \mathbf{t}$
$p$	$q$	$p \dot{\vee} q$	$p \vee q$	$q \supset \neg p$	$p \leftrightarrow q$	$w's$	$w's$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{pt}, \mathbf{qt}\}$	—
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{\mathbf{qf}\}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{\mathbf{pf}\}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	—	$\{\mathbf{pf}\}, \{\mathbf{qf}\}$

5.

$p$	$q$	$p \wedge q$	$p \leftrightarrow q$	$p \vee q$	$\neg p \vee \neg q$	$w's$	$w's$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{\mathbf{pt}\}, \{\mathbf{qt}\}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{qf}\}$	$\{\mathbf{pt}\}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{pf}\}$	$\{\mathbf{qt}\}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{pf}\}, \{\mathbf{qf}\}$	—

6.

$p$	$q$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X + \mathbf{f}$	$X + \mathbf{t}$
$p$	$q$	$p \wedge q$	$p \vee q$	$p \leftrightarrow q$	$\neg p \vee \neg q$	$w's$	$w's$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	—	$\{\mathbf{pt}, \mathbf{qt}\}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{qf}\}$	—
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\{\mathbf{pf}\}$	—
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\{\mathbf{pf}\}, \{\mathbf{qf}\}$	—

7.

$p$	$q$	$r$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X + \mathbf{f}$	$X + \mathbf{t}$
$p$	$q$	$r$	$pq \vee r$	$p \vee r$	$\bar{p}\bar{r} \vee pq \vee r$	$\bar{p}q \wedge \bar{r}$	$w's$	$w's$
<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{pt, qt\}, \{rt\}$
<b>t</b>	<b>t</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{pt, qt\}$
<b>t</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{rt\}$
<b>t</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	$\{qf, rf\}$	—
<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{rt\}$
<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	—	—
<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{rt\}$
<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	$\{qf, rf\}$	—

8.

$p$	$q$	$r$	$s$	$X$	$Y$	$Y \supset X$	$Y \supset \neg X$	$X + \mathbf{f}$	$X + \mathbf{t}$
$p$	$q$	$r$	$s$	$pq \vee rs$	$p \vee r$	$\bar{p}\bar{r} \vee pq \vee rs$	$\bar{p}q \vee \bar{r}s$	$w's$	$w's$
<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{pt, qt\}$
<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{pt, qt\}$
<b>t</b>	<b>t</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{pt, qt\}$
<b>t</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{pt, qt\}$
<b>t</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{rt, st\}$
<b>t</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	$\{qf, sf\}$	—
<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	$\{qf, rf\}$	—
<b>t</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	$\{qf, sf\}$	—
<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{rt, st\}$
<b>f</b>	<b>t</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	$\{pf, sf\}$	—
<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	—	$\{qt, st\}$
<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	$\{pf, sf\}$	—
<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>t</b>	<b>f</b>	—	$\{rt, st\}$
<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>t</b>	$\{qf, sf\}$	—
<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	$\{qf, rf\}$	—
<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>f</b>	<b>t</b>	<b>t</b>	$\{qf, sf\}$	—

Summarizing these results:

1.  $p \wedge q$  adds truth to  $q$  iff  $p$  is true; it adds falsity iff  $p$  is false.
2.  $p$  adds truth to  $p \vee q$  iff  $p$  is true or  $q$  is false; it adds falsity iff  $p$  is false.
3.  $p \leftrightarrow q$  adds truth to  $p \rightarrow q$  iff  $p$  is true or  $q$  is false; otherwise it adds falsity.
4.  $p \dot{\vee} q$  adds truth to  $p \vee q$  iff  $p$  or  $q$  is false; it adds falsity iff both are true.
5.  $p \wedge q$  adds truth to  $p \leftrightarrow q$  iff  $p$  or  $q$  is true; it adds falsity iff either is false.
6.  $p \wedge q$  adds truth to  $p \vee q$  iff  $p$  and  $q$  are true; it adds falsity iff either is false.

7.  $pq \vee r$  adds truth to  $p \vee r$  iff  $q$  is true or  $r$  is true; it adds falsity iff  $q$  and  $r$  are false.
8.  $pq \vee rs$  adds truth to  $p \vee r$  iff  $pq$ ,  $rs$ , or  $qs$  is true; otherwise it adds falsity.

## 22 Subtraction, when $A$ implies $B$

Can we develop a notion of logical subtraction defined on arbitrary  $A$  and arbitrary consequences  $B$  of  $A$ ?<sup>28</sup> Is there always an  $R$  deserving of the title of what is “left over” when  $B$  is subtracted from  $A$ ?

Yes, if we enlarge the field of propositions to allow *gappy* directed propositions (not defined on all worlds).<sup>29</sup> This section shows how to obtain the intension  $\mathbf{R}$  of  $R$ , on the assumption that  $A$  implies  $B$ . (Subject matters are left for later.) It suffices to specify in which worlds  $R$  is true/false.  $A$ - $B$ 's intensional content is the  $\mathbf{R}$  such that

- $\mathbf{R}$  is true in  $w$  iff  $A$  adds truth and no falsity to  $B$  in  $w$
- $\mathbf{R}$  is false in  $w$  iff  $A$  adds falsity and no truth to  $B$  in  $w$
- $\mathbf{R}$  is otherwise undefined in  $w$

A slightly more demanding remainder  $A \ominus B$  will be defined too.  $A$ - $B$  and  $A \ominus B$  are false in the same cases. But  $A \ominus B$ 's truth requires a bit more than  $A$ - $B$ 's.  $A$  should add, not just truth to  $B$ , but “its own” truth to  $B$ , in the following sense.  $A$  adds truth, as we know, iff  $B \supset A$  has a  $B$ -compatible truthmaker; it adds *its own* truth if some such truthmaker is implied by a truthmaker for  $A$ . The only difference, then, is that  $A \ominus B$  is sometimes undefined when  $A$ - $B$  is true. It will be undefined if  $A$  adds truth and no falsity, *but* the truth it adds is not its own. (This is not a common occurrence; an example is given below.)  $A \ominus B$ 's intensional content is the  $\mathbf{R}$  such that

- $\mathbf{R}$  is true in  $w$  iff  $A$  adds *its own* truth to  $B$  in  $w$ , and adds no falsity to  $B$  in  $w$ .
- $\mathbf{R}$  is false in  $w$  iff  $A$  adds falsity to  $B$  in  $w$  and adds no truth to  $B$  in  $w$ .
- $\mathbf{R}$  is otherwise undefined in  $w$ .

$A \ominus B$  is called the *partialized* remainder, for short the p-remainder. “Partialized” because it can be undefined where  $A$ - $B$  is defined, and because (see below) it is always *part* of  $A$ .

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<sup>28</sup>Here as above I assume that sentences  $S$  correspond one-one to directed contents  $\mathbf{S}$ .

<sup>29</sup>Think of a field extension aimed at providing solutions to all equations of a certain type.

## 22.1 Subtraction in PC

Remainders are officially semantic objects, but in a propositional logic setting we can let them be sentences. One bit of new notation is required to accommodate gappy remainders—ones that are not assigned truth-values by all models. Suppose  $X$  and  $Y$  are PC sentences. Then  $X \wedge \partial Y$  is a further sentence, whose semantics is as follows: it is defined iff  $Y$  is true, and where defined it has the same truth-value as  $X$ .<sup>30</sup> Examples of remainders.  $\dot{\vee}$  is exclusive disjunction.<sup>31</sup>

$X$	$Y$	$X - Y$	$X \ominus Y$
$p \wedge q$	$q$	$p$	$p$
$p \wedge q$	$p$	$q$	$q$
$p$	$p \vee q$	$p \wedge \partial(p \vee q)$	$p \wedge \partial(p \vee q)$
$q$	$p \vee q$	$q \wedge \partial(p \vee q)$	$q \wedge \partial(p \vee q)$
$p \leftrightarrow q$	$p \supset q$	$q \supset p$	$q \supset p$
$p \leftrightarrow q$	$q \supset p$	$p \supset q$	$p \supset q$
$p \dot{\vee} q$	$p \vee q$	$\neg(p \wedge q)$	$\neg(p \wedge q)$
$p \wedge q$	$p \leftrightarrow q$	$(p \vee q) \wedge \partial(p \leftrightarrow q)$	$(p \vee q) \wedge \partial(p \leftrightarrow q)$
$p \wedge q$	$p \vee q$	$p \wedge q$	$p \wedge q$
$pq \vee r$	$p \vee r$	$q \vee r$	$q \vee r$
$pq \vee rs$	$p \vee r$	$pq \vee rs \vee qs$	$(pq \vee rs) \wedge \partial(qs \supset (p \vee r))$

## 23 Implication and Equivalence

The theorems in this section assume that  $A$  and  $B$  are bivalent.  $\Rightarrow$  is implication, the relation  $X$  bears to  $Y$  iff  $Y$  is true in every world where  $X$  is true.  $\Leftarrow$  is converse implication.  $\Leftrightarrow$  is necessary equivalence, or truth in the same worlds.

**Theorem 10** *If  $A \Rightarrow B$ , then  $A \Rightarrow A - B$ .*

<sup>30</sup>I get the  $\partial$ -notation from Beaver, *Presupposition and Assertion in Dynamic Semantics*.

<sup>31</sup>The one unattractive result here is  $(p \wedge q) - (p \vee q) = p \wedge q$ . One might, in response to this, make it harder for  $A$  to add falsity to  $B$ , requiring a truthmaker for  $B \supset \neg A$  that not only (i) doesn't imply  $\neg B$  (that's our existing condition of  $B$ -compatibility), but also (ii) is not implied by  $\neg B$ . (Suppose  $p$  and  $q$  are both false.  $(p \vee q) \supset \neg(p \wedge q)$  is true for the  $p \vee q$ -compatible reason that  $p$  is false; but this is a reason implied by the falsity of  $B = p \vee q$ , so in violation of (ii).) This stronger requirement has a certain appeal; for again,  $B$ 's falsity seems of no possible relevance to whether falsity is added to  $B$  by  $A$ . But the stronger requirement has a cost:  $A - B$  is no longer guaranteed to agree with  $A$  on  $B$ -worlds.  $p - (p \vee q)$  will come out undefined, for instance, when  $p$  is false and  $p \vee q$  is true, for the reason already noted ( $p$ 's falsemaker  $\bar{p}$  is implied by  $\neg(p \vee q)$ ). This goes against the AGREEMENT condition, which does a great deal of good work below. Better to hold onto AGREEMENT than rejig everything for the sake of  $(p \wedge q) - (p \vee q)$ .

*Proof* Suppose  $A$  is true in  $w$  (so that  $B$  is also true). To show  $A-B$  is true in  $w$ , we must show that (i)  $A$  adds truth to  $B$  there, and (ii)  $A$  does not add falsity to  $B$  in  $w$ . (i) holds because  $B \supset A$  is true in  $w$  for what can only (given that  $B$  is true in  $w$ ) be a  $B$ -compatible reason. (ii) holds because  $B \supset \neg A$  is false in  $w$ .  $\square$

**Theorem 11** *If  $A \Rightarrow B$ , then  $A \Rightarrow A \ominus B$ .*

*Proof* Suppose  $A$  is true in  $w$ . Then  $A$  has a truthmaker  $\mathbf{T}^A$  there. By the 2nd Law of Truthmakers,  $\mathbf{T}^A$  implies a truthmaker  $\mathbf{T}^{B \supset A}$  for  $B \supset A$ .  $\mathbf{T}^{B \supset A}$  is  $B$ -compatible since  $B$  is true in  $w$ . So  $B \supset A$  has a  $B$ -compatible truthmaker in  $w$  that is implied by a truthmaker for  $A$ , which is what it means for  $A$  to add its own truth to  $B$  in  $w$ . The proof of Theorem 10 shows that  $A$  doesn't add falsity. So  $A \ominus B$  is true in  $w$ .  $\square$

**Theorem 12** *If  $A \Rightarrow B$ , then  $(A-B) \wedge B \Leftrightarrow A$ .*

*Proof* [ $\Leftarrow$ ]  $A$  implies  $B$  by assumption, and it implies  $A-B$  by the previous result. [ $\Rightarrow$ ] Suppose  $(A-B) \wedge B$  is true in  $w$ . If  $A$  were false in  $w$ , then (i)  $A$  would add falsity to  $B$  in  $w$  (since  $B$  is true there), and (ii)  $A$  would not add truth (since  $B \supset A$  would be false). Hence  $A-B$  would be false in  $w$ , contrary to assumption.  $\square$

**Theorem 13** *If  $A \Rightarrow B$ , then  $(A \ominus B) \wedge B \Leftrightarrow A$ .*

*Proof* [ $\Leftarrow$ ]  $A$  implies  $B$  by assumption, and it implies  $A \ominus B$  by Theorem 11. [ $\Rightarrow$ ] Follows from the previous result, given that  $A \ominus B$  implies  $A-B$ .  $\square$

## 24 Aboutness for Remainders

Now we equip  $A-B$  and  $A \ominus B$  with subject matters, for each  $A$  and  $B$  such that  $A$  implies  $B$ .

Suppose  $A-B$  is true in  $w$ . Then  $A$  adds truth to  $B$  in  $w$ , which means  $B \supset A$  has a targeted truthmaker  $\mathbf{T}_w^{B \supset A}$  in  $w$ .  $A-B$ 's truthmakers in  $w$  are (stipulated to be) all and only these targeted truthmakers  $\mathbf{T}_w^{B \supset A}$ . The set of them as  $w$  varies is  $A-B$ 's subject matter.<sup>32</sup>

Suppose  $A \ominus B$  is true in  $w$ . Then  $A$  "adds its own truth" to  $B$  in  $w$ , which means  $B \supset A$  has targeted truthmakers  $\mathbf{T}_w^{B \supset A}$  in  $w$  that are implied by truthmakers for  $A$ .  $A \ominus B$ 's truthmakers in  $w$  are (stipulated to be) all and only these  $\mathbf{T}_w^{B \supset A}$ s.<sup>33</sup>

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<sup>32</sup> $A-B$  is true iff it adds truth and no falsity.  $\mathbf{T}_w^{B \supset A}$  testifies only to the first part of this, the fact that  $A$  adds truth. Thus  $\mathbf{T}_w^{B \supset A}$  can hold even when  $A-B$  lacks truth-value, because  $A$  also adds falsity to  $B$ . This forces us to relax our conception of truthmaker-hood a bit;  $X$ 's truthmakers imply, not that  $X$  is true, but that it is true if it has a truth-value at all.

<sup>33</sup> $A \ominus B$ 's truthmakers are thus the same as  $A-B$ 's, except we drop any not implied by truthmakers  $\mathbf{T}^A$  for  $A$ .

Suppose  $A-B$  is false in  $w$ . Then  $A$  adds falsity to  $B$ . So  $B \supset \neg A$  has at least one targeted truthmaker in  $w$ —which, given that  $A$  implies  $B$ , is just to say that  $A$  has a  $B$ -compatible falsemaker  $\mathbf{F}^A$  in  $w$ .  $A-B$ 's falseness in  $w$  are (stipulated to be) all and only these  $B$ -compatible  $\mathbf{F}^A$ s.  $A \odot B$ 's falseness in  $w$  are the same as  $A-B$ 's falseness. The set of these across all possible  $w$  is  $A-B$ 's subject anti-matter =  $A \odot B$ 's subject anti-matter.

$A-B$ 's overall subject matter = its subject matter and subject anti-matter combined = the set of its truthmakers  $\mathbf{T}_w^{B \supset A}$  and falseness  $\mathbf{F}^A$ .  $A \odot B$ 's overall subject matter = its subject matter and subject anti-matter combined = the set of its truthmakers  $\mathbf{T}_w^{B \supset A}$  and falseness  $\mathbf{F}^A$ .

## 25 The Part of $A$ Not About BLAH

Given a sentence  $A$  and a lewisian subject matter  $\mathbf{m}$ , we now define  $A_{(\mathbf{m})}$  = the part of  $A$  not about  $\mathbf{m}$ .<sup>34</sup> (The brackets around  $\mathbf{m}$  are to suggest that we're looking for a version of  $A$  in which  $\mathbf{m}$  is "bracketed.")

Let's first define another proposition  $A_{|\mathbf{m}|}$ , which stands to  $A_{(\mathbf{m})}$  roughly as  $A-B$  stands to  $A \odot B$ .  $A_{(\mathbf{m})}$ , like  $A \odot B$ , is sure to be part of  $A$ ;  $A_{|\mathbf{m}|}$ , like  $A-B$ , may or may not be.

$A_{|\mathbf{m}|}$  is defined by specifying first its intensional content (= the worlds where it's true/false), and then its subject matter (truthmakers) and subject anti-matter (falseness).

A world's  $\mathbf{m}$ -condition may help  $A$  to be true; but it should not help  $A_{|\mathbf{m}|}$  to be true.  $A_{|\mathbf{m}|}$ 's truth should be sensitive only to the *further* features worlds must have for  $A$  to be true in them.  $A_{|\mathbf{m}|}$  will be true in  $A$ -worlds AND in worlds that fail to be  $A$ -worlds only because of what they are like  $\mathbf{m}$ -wise—that are  $A$ -worlds "apart from  $\mathbf{m}$ ."

To each of  $\mathbf{m}$ 's cells  $\mathbf{m}_k$  corresponds an  $M_k$  such that  $M_k$  is true in  $w$  for the reason(s)  $w$  belongs to  $\mathbf{m}_k$ . Then the facts about  $w$ , beyond its  $\mathbf{m}$ -condition  $M_k$ , by virtue of which  $w$  is  $A$ , are the targeted truthmakers for  $M_k \supset A$  that obtain in  $w$ .

$A_{|\mathbf{m}|}$  is true in  $u$  iff

1.  $A$  adds truth to some  $M_k$  in  $u$ , and
2.  $A$  does not add falsity to any  $M_k$  in  $u$ .

$A_{|\mathbf{m}|}$  is false in  $u$  iff

1.  $A$  adds falsity to some  $M_k$  in  $u$ , and
2.  $A$  does not add truth to any  $M_k$  in  $u$ .

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<sup>34</sup>Strictly we are defining a directed content  $\mathbf{A}_{(\mathbf{m})}$  from a directed content  $\mathbf{A}$  and a subject matter  $\mathbf{m}$ , not a sentence as the notation  $A_{(\mathbf{m})}$  suggests, but it is easier to think in terms of sentences.

Now to subject matter, that is, truthmakers and falsmakers.  $A_{|\mathbf{m}|}$ 's truthmakers in worlds  $u$  where it is true are the targeted truthmakers for  $M_k \supset A$  that obtain in  $u$ , and its falsmakers in worlds  $w$  where it is false are the targeted truthmakers for  $M_k \supset \neg A$  that obtain in  $w$ .

The part  $A_{(\mathbf{m})}$  of  $A$  not about  $\mathbf{m}$  is obtained by limiting  $A_{|\mathbf{m}|}$  to worlds where  $A_{|\mathbf{m}|}$  is true/false for reasons implied by truthmakers/falsmakers for  $A$ . Or, using the notion of adding "its own" truth/falsity from above, limiting  $A_{|\mathbf{m}|}$  to worlds where  $A$  adds *its own* truth/falsity to  $M_k$ .

$A_{(\mathbf{m})}$  (the part of  $A$  not about  $\mathbf{m}$ ) is true in  $u$  iff

1.  $A$  adds its own truth to some  $M_k$  in  $u$ , and
2.  $A$  does not add falsity to any  $M_k$  in  $u$ .

$A_{(\mathbf{m})}$  is false in  $u$  iff

1.  $A$  adds its own falsity to some  $M_k$  in  $u$ , and
2.  $A$  does not add truth to any  $M_k$  in  $u$ .

$A_{(\mathbf{m})}$ 's truthmakers are those of  $A_{|\mathbf{m}|}$ 's truthmakers that are implied by truthmakers for  $A$ . Its falsmakers are those of  $A_{|\mathbf{m}|}$ 's falsmakers that are implied by falsmakers for  $A$ .

**Theorem 14** *If  $A$  implies  $B$ , then  $A-B = A_{|\mathbf{whetherB}|}$*

*Proof* Let  $\mathbf{m}$  be **whether B**. [Intensional content]  $A_{|\mathbf{m}|}$  is true in worlds where  $A$  (i) adds truth to  $B$ , or (ii) adds truth to  $\neg B$ , but  $A$  (iii) does not add falsity to  $B$ , and (iv) does not add falsity to  $\neg B$ .  $A$  adds truth to  $\neg B$  in  $w$  iff  $\neg B \supset A$  has a targeted, hence  $\neg B$ -compatible, truthmaker in  $w$ . But  $\neg B \supset A$  is equivalent to  $B$ , since  $A$  implies  $B$ . Clearly  $B$  does not have a  $\neg B$  compatible truthmaker in any world. So (ii) is impossible.  $A$  adds falsity to  $\neg B$  iff  $\neg B \supset \neg A$  has a targeted, hence  $\neg B$  compatible, truthmaker in  $w$ .  $\neg B \supset \neg A$  is a necessary truth since  $A$  implies  $B$ , so no non-trivial truthmaker exists. Thus (iv) is irrelevant, too, and we may focus on (i) and (iii). According to them,  $A_{|\mathbf{m}|}$  is true in worlds where  $A$  adds truth to  $B$  without adding any falsity. That's the truth-condition for  $A-B$  as well. So  $A-B$  and  $A_{|\mathbf{m}|}$  are true in the same worlds. A similar argument shows they are false in the same worlds. [Subject matter]  $A_{|\mathbf{m}|}$ 's truthmakers are all and only (i) targeted truthmakers for  $B \supset A$  and (ii) targeted truthmakers for  $\neg B \supset A$ . There are no truthmakers of type (ii) since  $\neg B \supset A$  is equivalent to  $B$ . So  $A_{|\mathbf{m}|}$ 's truthmakers are all targeted truthmakers for  $B \supset A$ . These are also  $A-B$ 's truthmakers.  $A_{|\mathbf{m}|}$ 's falsmakers are all (i) targeted truthmakers for  $B \supset \neg A$  and (ii) targeted truthmakers for  $\neg B \supset \neg A$ . There are no non-trivial truthmakers of type (ii) since  $\neg B \supset \neg A$  is a necessary truth. So  $A_{|\mathbf{m}|}$ 's falsmakers are  $B \supset \neg A$ 's targeted truthmakers. These are also, by definition,  $A-B$ 's falsmakers.  $\square$

**Theorem 15** *If  $A$  implies  $B$ , then  $A \ominus B = A_{(\mathbf{whetherB})} =$  the part of  $A$  not about **whether B**.*



*Proof sketch:*  $A\ominus B$  is obtained from  $A-B$  by restricting the latter to worlds where the truth/falsity that  $A$  adds to  $B$  is  $A$ 's own truth/falsity, meaning that the witnessing truthmakers/falsemakers are implied by truthmakers/falsemakers for  $A$ . The part of  $A$  not about **whether B** is obtained from  $A|_{\text{whetherB}}$  in the same way.  $\square$

## 26 Part-Whole for Remainders

**Theorem 16** *If  $B$  is part of  $A$ , then  $A-B$  and  $A\ominus B$  are disjoint from  $B$ .*

*Proof* Suppose for contradiction that  $C$  is a non-trivial common part of  $B$  and  $A-B$ . That  $C$  is non-trivial means it has at least one falsemaker  $\mathbf{F}^C$ .  $\mathbf{F}^C$  is a falsemaker for  $B$  (by the 3rd Truthmaker Law) since  $C$  is part of  $B$ , hence  $\mathbf{F}^C$  is  $B$ -incompatible.  $\mathbf{F}^C$  is also a falsemaker for  $A-B$  since  $C$  is part of  $A-B$ . But a falsemaker for  $A-B$  is a  $B$ -compatible truthmaker for  $B\supset\neg A$ . Hence  $\mathbf{F}^C$  is both compatible with  $B$  and incompatible with it. Contradiction. So  $C$  is not a non-trivial common part of  $C$  and  $A-B$  after all. The proof for  $A\ominus B$  is similar.  $\square$

**Theorem 17** *Suppose  $B$  is part of  $A$ . Then  $A\ominus B$  is part of  $A$ , but  $A-B$  may not be part of  $A$ .*

*Proof* Suppose  $B\leq A$ . We saw in section 23 that  $A\ominus B$  is implied by  $B$   $A$ . It remains to show that  $A\ominus B$ 's subject matter is part of  $A$ 's subject matter and  $A\ominus B$ 's subject anti-matter is part of  $A$ 's subject anti-matter. *Subject Matter* Any truthmaker for  $A\ominus B$  is a  $\mathbf{T}^{B\supset A}$  implied by some  $\mathbf{T}^A$ . Thus each of  $A-B$ 's truthmakers is implied by one of  $A$ 's truthmakers. *Subject Anti-Matter* A falsemaker for  $A\ominus B$  is a targeted truthmaker for  $B\supset\neg A$ , which since  $A$  implies  $B$  is the same as a  $B$ -compatible falsemaker for  $A$ . Thus each of  $A-B$ 's falsemakers is implied by one of  $A$ 's falsemakers. Finally, an example to show that  $A-B$  may not be part of  $A$ . Let  $A$  and  $B$  be  $pq\vee rs$  and  $p\vee r$ .  $A-B$  is  $pq\vee rs\vee qs$  (see section 22.1).  $\{\mathbf{qt}, \mathbf{st}\}$  is a minimal model of  $A-B$  that is not contained in any minimal model of  $A$ .  $\square$

## 27 Extrapolation, when $A$ implies $B$

If  $A$  implies  $B$ , then the  $A$ -region of logical space is included in the  $B$ -region. Suppose we wanted to extrapolate the  $A$ -region beyond the  $B$ -region into the rest of logical space. How would we go about it? How do we find the  $R$  that picks up where  $A$  leaves off and continues “in the same way” into the  $\neg B$ -region? If possible we would like to specify  $R$  up to directed content, that is, we would like to specify  $R$ 's intensional content—the set of worlds where it is true (false)—and its subject matter—the reasons it is true (false) in those worlds. A four-stage procedure seems reasonable.

**S1** Fix  $R$ 's i-content in the  $B$ -region; specify in which  $B$ -worlds  $R$  is to be true (false).

**S2** Fix  $R$ 's subject matter in the  $B$ -region; specify *why*  $R$  is true (false) in a given  $B$ -world.

**S3** Fix  $R$ 's subject matter outside the  $B$ -region; specify why  $R$  is true (false) in  $\neg B$ -worlds.

**S4** Fix  $R$ 's i-content outside the  $B$ -region; specify in which  $\neg B$ -worlds  $R$  is true (false).

Here is how we might implement the procedure.

**S1**  $R$  is true (false) in a  $B$ -world  $w$  iff  $A$  is true (false) in  $w$ .

**S2**  $R$  is true (false) in a  $B$ -world  $w$  for the reasons  $w$  is  $A$  ( $\neg A$ ), *given* that it is  $B$ .

**S3**  $R$ 's reasons for being true (false) in  $\neg B$ -worlds are the same as in  $B$ -worlds.

**S4**  $R$  is true (false) outside  $B$  where it has reason to be and no reason not to be.<sup>35</sup>

Now we implement the four stage strategy formally by imposing four conditions on  $R$ .

*$R$  extrapolates  $A$  beyond  $B$  iff*

AGREEMENT  $R$  is true (false) in a  $B$ -world  $w$  iff  $A$  is true (false) in  $w$ .

BASIS  $R$ 's t-makers (f-...) in  $B$ -worlds  $w$  are  $B \supset A$ 's ( $B \supset \neg A$ 's) targeted t-makers in  $w$ .

FIDELITY  $R$ 's t-makers (f-...) in  $\neg B$ -worlds  $w$  are those from  $B$ -worlds that obtain in  $w$ .

EXTENT  $R$  is true (...) in  $\neg B$ -world  $w$  iff  $R$  has t-makers (...) but no f-makers (...) in  $w$ .<sup>36,37</sup>

**Theorem 18** *If  $A$  implies  $B$ , there is a unique  $R$  extrapolating  $A$  beyond  $B$ .*<sup>38</sup>

*Proof* AGREEMENT fixes  $R$ 's intension within the  $B$ -region. BASIS fixes the reasons for  $R$ 's truth (falsity) inside the  $B$ -region. FIDELITY fixes the reasons for  $R$ 's truth (falsity) outside the  $B$ -region. EXTENT fixes  $R$ 's truth-value outside the  $B$ -region. AGREEMENT and EXTENSION thus determine  $R$ 's intensional content; BASIS and FIDELITY determine its overall subject matter.  $R$ 's intensional content and overall subject matter determine  $R$  up to directed content.  $\square$ <sup>39</sup>

We allow ourselves to speak of “the  $R$  that extrapolates (p-extrapolates)  $A$  beyond  $B$ ,” on the understanding that  $R_1$  and  $R_2$  are the same for these purposes if they agree in directed content.

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<sup>35</sup>Meaning, a reason  $R$  had for being true (false) in some  $B$ -world also obtains in  $w$ , while there does not obtain in  $w$  any reason  $R$  had for being false (true) in a  $B$ -world.

<sup>36</sup>Spelled out more fully, this means that  $R$  is true in  $w$  if  $B \supset A$  has  $B$ -compatible,  $B$ -efficient t-makers in  $w$ , while  $B \supset \neg A$  lacks them; and  $R$  is false in  $w$  if it's the other way around.

<sup>37</sup>It will help to have a notion that stands to extrapolation as p-remainder stands to remainder.  $R$  *p-extrapolates*  $A$  beyond  $B$  under the same four conditions, except that “ $B \supset A$ 's truthmakers” are restricted to those that are implied by truthmakers for  $A$  and “ $B \supset \neg A$ 's truthmakers” are restricted to those that are implied by falsmakers for  $A$ .

<sup>38</sup>Unique up to directed content.

<sup>39</sup>A similar argument shows there is a unique  $R$  p-extrapolating  $A$  beyond  $B$ .

## 28 Remainders = Extrapolators

**Theorem 19** *If  $A$  implies  $B$ , then  $A-B$  (as defined in 22) is the  $R$  extrapolating  $A$  beyond  $B$ .*

*Proof* Uniqueness was argued above, so it's enough to show  $A-B$  extrapolates  $A$  beyond  $B$ . In section 23 we saw that  $A-B$  is true (false) in the same  $B$ -worlds as  $A$  (that's AGREEMENT). The definition of subject matter for remainders (section 24) specifies that  $A-B$  is true (false) in a  $B$ -world  $w$  for whatever reason(s)  $B \supset A$  is true (false) in  $w$  (that's BASIS). The definition of subject matter for remainders further specifies that  $A-B$  is made true (false) in non- $B$ -worlds by  $B$ -compatible,  $B$ -efficient, truthmakers for  $B \supset A$  ( $B \supset \neg A$ ) that obtain in  $w$ . That these truthmakers are  $B$ -compatible means that they also obtain in the  $B$ -region; thus  $A-B$ 's truthmakers (falsmakers) outside the  $B$ -region are also truthmakers (falsmakers) within the  $B$ -region (that's FIDELITY). The definition of  $A-B$ 's intensional content in 22 says that  $A-B$  is true (false) iff  $A$  adds truth and no falsity (falsity and no truth) to  $B$ .  $A$  adds truth (falsity) to  $B$  iff  $B \supset A$  ( $B \supset \neg A$ ) has a  $B$ -compatible,  $B$ -efficient truthmaker in  $w$ . So  $A-B$  is true (false) in non- $B$ -worlds iff  $B \supset A$  ( $B \supset \neg A$ ) has a  $B$ -compatible,  $B$ -efficient truthmaker in  $w$  (that's EXTENT).  $\square$ <sup>40</sup>

## 29 Interpolation, when A implies B

This section compares  $A-B$  with the premise  $R$  that should be plugged in (“interpolated”) between  $B$  and  $A$  to make  $B, ??? \therefore A$  a valid argument.  $???$  stands in for the missing premise. We restrict ourselves in this section to the case where  $A$  implies  $B$ , leaving the more general problem for later.

Not any old  $R$  that might be put in for  $???$  strikes us as “completing the argument.” It often happens that there is just one choice of  $R$  that seems right. The propositional calculus inference  $pq, ??? \therefore pqr$  is completed by putting  $r$  in for  $???$ . Judgments about the appropriate completer seem principled. What are the principles?

The obvious strategy is to (i) take a gappy argument (an enthymeme) that *can* be completed, (ii) lay out some manifestly incorrect choices of  $???$ , and (iii) ask what is wrong with them. Consider again  $pq, ??? \therefore pqr$ . The following seem like *bad* things to put in for  $???$ , at any rate worse things to put in than  $q$ .

1.  $s$
2.  $rs$
3.  $qr$
4.  $pqr$
5.  $pq \supset pqr$

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<sup>40</sup>A similar argument shows that  $A \ominus B$  (as defined in 22) is the  $R$  p-extrapolating  $A$  beyond  $B$ .

The problem with  $s$  is of course that  $pq, s \therefore pqr$  is not a valid argument. The first thing we want is

SUFFICIENCY Given  $B$ ,  $???$  should be sufficient for  $A$ .

$s$  is not sufficient for  $pqr$  given  $pq$ , as there are  $pq$ -worlds where  $s$  is true without  $pqr$  being true. Another way to put the condition is that  $B$  should imply the material condition  $???\supset A$ .

What about  $rs$ ?  $rs$  does bridge the gap between the (initial) premise and the conclusion; combined with  $pq$ , it does imply  $pqr$ . However the bridge is a bridge too far. The second conjunct  $s$  is unnecessary and takes us beyond where we wanted to go. Our second desideratum is

NECESSITY Given  $B$ ,  $???$  should be necessary for  $A$ .

$rs$  is not necessary for  $pqr$  given  $pq$ , as there are  $pq$ -worlds where  $pqr$  is true without  $rs$  being true. Another way to put the condition is that  $B$  should imply the material conditional  $A\supset???$ .

Next we come to  $qr$ .  $qr$  is necessary for  $pqr$ , obviously. One cannot accuse  $qr$  of being a bridge too far. A complaint one can make, however, is that it's a bridge whose starting point is too near;  $qr$  repeats something is already there in the initial premise  $pq$ . This shows up formally as follows: should  $qr$  be false *due to the falsity of the repeated element*, then  $pq$  must be false as well. Our third desideratum is

NON-REDUNDANCY No falsemaker for  $???$  precludes the truth of  $B$ .

Once again,  $qr$  fails this test. If it is false because  $r$  is false, no problem. But if it is false because  $q$  is false, it brings the initial premise down with it. The reason for calling this a non-redundancy requirement is that redundancy entails a repeated element, in our terms, a part in common; a falsemaker for  $???$  that targets the shared part is going to force  $B$  to be false as well.

Consider now  $pqr$ . This already violates non-redundancy but we use it to motivate a further condition. The problem with  $pqr$  is that it makes the initial premise  $pq$  irrelevant, or less relevant than it could have been.<sup>41</sup> What we seem to want is that  $???$  should be true for the reasons  $A$  is true *given*  $B$ , which we can understand as the  $B$ -efficient reasons  $B\supset A$  has for being true in  $B$ -worlds.

RELEVANCE  $???$  is true for the reasons  $A$  is true (false) *given*  $B$ ; formally,  $???$  is made true (false) only by  $B$ -efficient truthmakers for  $B\supset A$  ( $B\supset\neg A$ ).

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<sup>41</sup>This is not quite the same as redundancy, because the initial premise can fail to pull its weight even if  $???$  does not repeat anything in it. Imagine we had suggested  $pq$  as the missing premise in  $p\leftrightarrow q, ???, \therefore pq$ .  $pq$  is not redundant in the sense just defined—its false-makers do not force  $p\leftrightarrow q$  to be false. But it doesn't make good use of the first premise either.

$pqr$  fails this condition because it is true for the reasons the argument’s conclusion is true (it *is* the conclusion), as opposed to the reasons that conclusion is true given the premise.

Consider finally  $pq \supset pqr$ . This does not make the initial premise irrelevant. Neither is it redundant. (It has one falsemaker ( $\{pt, qt, rf\}$ ), and that falsemaker, far from implying that  $B$  is false, implies that  $B$  is true.)  $pq \supset pqr$  is necessary for  $pqr$  given  $pq$ ;  $pqr$  can’t be true in a  $pq$ -world unless  $pq \supset pqr$  is true there. And it is clearly sufficient, too. The problem is that  $???$  should do more than imply  $B \supset A$ . That is just a matter of ruling out  $B$ -worlds in which  $A$  is false. And we want a  $???$  that “rules in”  $B$ -worlds where  $A$  is true—a  $???$  that combines with  $B$  to ensure the truth of  $A$  in these worlds.  $???$  cannot do that unless it obtains, when it does, for  $B$ -compatible reasons. This gives us our fifth condition on enthymeme-completers:

ADDITIVITY  $B$  should be freely combinable with  $???$ ’s truthmakers.

The problem with  $pq \supset pqr$  is that its truthmakers are not always compatible with with  $B = pq$ ; for the conditional can be made true by making its antecedent  $pq$  false. That it can be true via  $pq$ ’s falsity means that it does not pick out the characteristic of certain  $pq$ -worlds that makes them moreover  $pqr$ .

The claim is that  $R$  completes the argument  $B, ??? \therefore A$  iff

1. SUFFICIENCY  $R$  is sufficient for  $A$  in  $B$ -worlds.
2. NECESSITY  $R$  is necessary for  $A$  in  $B$ -worlds.
3. NON-REDUNDANCY No falsemaker for  $R$  precludes the truth of  $B$ .
4. RELEVANCE  $R$  is true (false) for the reasons  $A$  is true (false) *given*  $B$ .
5. ADDITIVITY  $B$  is freely combinable with each of  $R$ ’s truthmakers.

## 30 Remainders as Interpolants

Extrapolation as defined above turns out to have a great deal in common with interpolation as we are defining it now. An extrapolator, recall, is an  $R$  that satisfies AGREEMENT, BASIS, FIDELITY, and EXTENSION. SUFFICIENCY and NECESSITY are together equivalent to AGREEMENT, while NON-REDUNDANCY and ADDITIVITY are together the FIDELITY condition. EXTENSION is a general background assumption about the relation between truth and falsity, on the one hand, and truth- and falsity-makers, on the other.<sup>42</sup> BASIS is RELEVANCE restricted to  $B$ -worlds; conversely RELEVANCE follows from BASIS in the context of FIDELITY.

So,  $R$  completes the enthymeme  $B, ??? \therefore A$  iff  $R$  is  $A$ - $B$ . Turning this around, if you want to find  $A$ - $B$ , look for the  $R$  that fits most naturally into the  $???$ -slot of  $B, ??? \therefore A$ .

<sup>42</sup>EXTENSION says that (\*)  $R$  is true in  $w$  iff it has a truthmaker and no falsemaker in  $w$ .

Example: What is *Jones is a bachelor again* minus *Jones is male*? Now we have a new way of approaching this question. *Jones is a bachelor again* minus *Jones is male* = whatever it takes to complete the argument that begins with *Jones is male* and concludes with *Jones is a bachelor again*. What it takes is *Jones is single again*. So the remainder when *Jones is male* is subtracted from *Jones is a bachelor again* is *Jones is single again*.

## 31 Taking Stock

We have described three operations on  $A$  and  $B$ . The first is *subtraction*; it results in a remainder proposition  $A-B$  that is true (false) in  $w$  when  $A$  adds truth and no falsity (falsity and no truth) to  $B$  in  $w$ . The second is *extrapolation*; it results in the proposition, let's call it  $\chi(A,B)$ , that you get by continuing the proposition that  $A$  past the boundaries imposed by  $B$ . The third is *interpolation*; it takes  $A$  and  $B$  to the proposition, let's call it  $\iota(A,B)$ , that plugs the gap in the enthymeme  $B, ??? \therefore A$ .

How are these three things—the remainder, the extrapolant (if I can call it that), and the interpolant—connected? Theorem 19 tells us that  $A-B$  agrees with  $\chi(A,B)$  when  $A$  implies  $B$ . And we have just seen that  $\chi(A,B)$  agrees with  $\iota(A,B)$  when  $A$  implies  $B$ . When  $A$  implies  $B$ , then,  $A-B = \chi(A,B) = \iota(A,B)$ .

The question that arises now is, is there any way of lifting the restriction on  $A$  that it has to imply  $B$ ?<sup>43</sup>

There are two strategies one might try. The reductive strategy tries to reduce the general case to the implicational case. Certainly  $A \wedge B$  implies  $B$ , so why not say that  $A-B$  necessarily has the same truth-value as  $A \wedge B - B$ ? The simple strategy takes our definitions as written and attempts to apply them across the board, whether  $A$  implies  $B$  or not.

## 32 Anti-Reductionism

The *reductionist* says that  $A-B$  is true in the same worlds as  $(A \wedge B) - B$ .<sup>44</sup> This has the advantage of taking us back to the previous case, since  $A \wedge B$  implies  $B$ .

But it also has a disadvantage. When  $A$  does not imply  $B$ ,  $A$  has a non-trivial truth-profile in worlds where  $B$  is false. This is potentially important information; the shape of  $A$ 's truth-profile outside  $B$  might be relevant to the proper interpretation of  $A-B$ .<sup>45</sup>

The problem with defining  $A-B$  as  $(A \wedge B) - B$  is that it just throws this information away. Suppose  $A_1$  and  $A_2$  agree in the  $B$ -region, but have a different truth-profile outside of that region.

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<sup>43</sup>I am thinking here just of truth-conditional or intensional content. Subject matter can be left out of it for now.

<sup>44</sup>And  $A \ominus B$  is true in the same worlds as  $(A \wedge B) \ominus B$ .

<sup>45</sup> $A \ominus B$

$A_1-B = (A_1 \wedge B)-B = (A_2 \wedge B)-B = A_2-B$ , so the contrasting behavior in  $\neg B$ -worlds has no influence on the result. Similar remarks apply to the interpolant and the extrapolant.

Of course, this is a problem only if the contrasting behavior in  $\neg B$ -worlds *should* affect the result. Here's an example to show that it indeed should. Consider two enthymemes (i) *The F = the G, ??? ∴ The F = the H*, and (ii) *The F = the G, ??? ∴ The G = the H*. (i)'s missing premise is *The G = the H* and (ii)'s is *The F = the H*. The reductive strategy cannot allow this, since  $A_1 \wedge B = A_2 \wedge B = \textit{The F, the G, and the H are identical}$ .

So let's try the simple strategy, defining  $A-B$  and  $A \ominus B$  the same way whether  $A$  implies  $B$  or not.

### 33 Subtraction, Generally

Whether  $A$  implies  $B$  or not,  $A-B$  is

- true in  $w$  iff  $A$  adds truth and no falsity to  $B$  in  $w$
- false in  $w$  iff  $A$  adds falsity and no truth to  $B$  in  $w$
- otherwise undefined in  $w$

Example: Let  $F$ ,  $G$  and  $H$  be predicates and let  $F = G$  be short for *The Fs = the Gs*. What is  $F=H - F=G$ , given that it is to be true just where  $F=H$  adds truth and no falsity to  $F=G$ .

$F=H$  adds truth to  $F=G$  in  $w$  iff  $F=G \supset F=H$  has a targeted truthmaker in  $w$ .  $F=H$  adds falsity to  $F=G$  in  $w$  iff  $F=G \supset F \neq H$  has a targeted truthmaker there.

So the hypothesis we are looking for is true in  $w$  iff  $F=G \supset F=H$  is true in  $w$  for an  $F=G$ -compatible and  $F=G$ -efficient reason, and the same cannot be said of  $F=G \supset F \neq H$

Here are the obvious candidates:

1.  $G=H$
2.  $F=H$
3.  $F=G=H$
4.  $F=G \supset G=H$
5.  $F=G \supset F=H$

Intuitively the answer is (a). The one and only  $F=G$ -compatible reason why a world would be  $F=H$ -if- $F=G$  is that that world's  $G$ s = its  $H$ s. The reason the primates are the renates if they're the cordates is not (b) that the primates are the renates, or (c) that all three groups are the same, or etc. It's (a) that the renates are the cordates.

$G=H$  is the right result. The question is whether our definitions predict it. Certainly  $G=H$  has the property we are looking for. It is true in a world just when  $F=G \supset F=H$  is true there for an  $F=G$ -compatible and -efficient reason; for the fact that (or in virtue of which)  $G=H$  is itself an  $F=G$ -compatible and -efficient truthmaker for  $F=G \supset F=H$ . But what is wrong with the other facts listed?

Against (b)— $F=H$ 's truthmakers are  $F=G$ -wasteful; because (unlike  $G=H$ 's truthmakers) they imply  $(F=G)^- \supset (F=H)$  for every part  $(F=G)^-$  of  $F=G$ —indeed they imply  $X \supset (F=H)$  for every  $X$  whatsoever.

Against (c)— $F=G=H$ 's truthmakers are  $F=G$ -wasteful; they imply  $X \supset (F=H)$  for every  $X$ .

Against (d)—in worlds where  $F=G$  and  $G=H$  are both false,  $F=G \supset G=H$  is true because of the fact that  $F \neq G$ , or the fact in virtue of which  $F \neq G$ . Hence  $F=G \supset G=H$ 's truthmakers are not (all)  $F=G$ -compatible.

Against (e)—in worlds where  $F=G$  and  $F=H$  are both false,  $F=G \supset F=H$  is true because of the fact that  $F \neq G$ , or the fact in virtue of which  $F \neq G$ . Hence  $F=G \supset G=H$ 's truthmakers are not (all)  $F=G$ -compatible.

We are thus led to conclude that  $F=H - F=G$  is  $G=H$ .

## 34 Extrapolation, Generally

Whether  $A$  implies  $B$  or not,  $R$  extrapolates  $A$  beyond  $B$  iff

AGREEMENT  $R$  is true (false) in a  $B$ -world  $w$  iff  $A$  is false in  $w$ .

BASIS  $R$ 's t-makers (f-makers) in a  $B$ -world  $w$  are  $B \supset A$ 's targeted t-makers there.

FIDELITY  $R$ 's t-makers (f-makers) in  $\neg B$ -worlds are its t-makers (f-makers) in  $B$ -worlds.

EXTENT  $R$  is true (false) in  $\neg B$ -worlds where  $R$  has only a t-maker (only an f-maker).<sup>46</sup>

Example: How do we extrapolate *The martini-drinker is a philosopher* beyond the region where *That guy is the martini-drinker*? AGREEMENT tells us to look for an **R** that agrees with *The martini-drinker is a philosopher* on worlds where *That guy is the martini-drinker*. The two obvious choices are *The martini-drinker is a philosopher* and *That guy is a philosopher*.

BASIS favors *That guy is a philosopher*. Let's first consider the question intuitively, then following the letter of the condition.

BASIS tells us to look for an  $R$  that is true, in worlds where that guy is the martini-drinker, for the same reasons as it is true in  $w$  that *The martini-drinker is a philosopher*, given that *That guy is the martini-drinker*. The question intuitively speaking is: why are some (but only some) worlds

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<sup>46</sup>Given FIDELITY and BASIS, this means that  $R$  is true (false) in a  $\neg B$ -world  $w$  iff  $B \supset A$  has a targeted truthmaker in  $w$  while  $B \supset \neg A$  lacks any such truthmaker (vice versa).



where that guy is the martini-drinker moreover such that the martini-drinker is a philosopher? The reason is that *That guy is a philosopher* in those worlds.

The question more precisely is this: do we have in the fact that **The martini-drinker is a philosopher** a targeted truthmaker for *That guy is the martini-drinker*  $\supset$  *The martini-drinker is a philosopher*. No, because that fact wastes the antecedent; it wastes *That guy is the martini-drinker*. The fact that **that guy is a philosopher** uses more of *That guy is the martini-drinker* than does *The martini-drinker is a philosopher*, in every relevant world.

## 35 Interpolation, Generally

Whether  $A$  implies  $B$  or not,  $R$  interpolates between  $B$  and  $A$  iff

SUFFICIENCY Given  $B$ ,  $R$  should be sufficient for  $A$ .

NECESSITY Given  $B$ ,  $R$  should be necessary for  $A$ .

NON-REDUNDANCY No falsemaker for  $R$  precludes the truth of  $B$ .

RELEVANCE  $R$  is true (false) for the reasons  $A$  is true (false) *given*  $B$ .<sup>47</sup>

ADDITIVITY  $B$  is freely combinable with each of  $R$ 's truthmakers.

Example: Let  $F$ ,  $G$  and  $H$  be predicates and let  $F = G$  be short for *The Fs = the Gs*. What is the  $R$  that completes the enthymeme,  $F=G$ ,  $???$ ,  $\therefore F=H$ ?

We are looking for an  $R$  that is necessary and sufficient, given  $F=G$ , for  $F=H$ ; whose falsemakers are compatible with  $F=G$  (that's NON-REDUNDANCY); whose truthmakers are compatible with  $F=G$  (that's ADDITIVITY); and the reasons for whose truth (falsity) are the reasons  $F=H$  is true (false) *given*  $F=G$  (that's RELEVANCE).

As before the candidates are

1.  $G=H$
2.  $F=H$
3.  $F=G=H$
4.  $F=G \supset G=H$
5.  $F=G \supset F=H$

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<sup>47</sup>These are plausibly taken to be  $B \supset A$ 's  $B$ -compatible,  $B$ -efficient truthmakers.

Each of these is necessary and sufficient for  $F=H$  given  $F=G$ .

Consider (a)  $G=H$ , the intuitive favorite. Are its falsemakers compatible with  $F=G$ ? Yes, for a  $G$  not to be  $H$ , or vice versa, is compatible with the  $F$ s being the  $G$ s. (That's NON-REDUNDANCY.) Are its truthmakers compatible with  $F=G$ ? Yes, that the  $G$ s are the  $H$ s leaves it open for them to be the  $F$ s as well. (That's ADDITIVITY.) By the same token,  $G=H$ 's truth- and falsemakers are also compatible with the  $F$ s *not* being the same as the  $G$ s; they are truthmakers (falsemakers) not for  $F=H$  simpliciter, but  $F=H$  *given that*  $F=G$  (that's RELEVANCE.)

$F=H$  and  $F=G=H$  violate RELEVANCE; their truthmakers explain not why  $F=H$  is true given  $F=G$ , but why it is true period.

$F=G \supset G=H$  and  $F=G \supset F=H$  violate ADDITIVITY. Both have as truthmakers the fact that (or in virtue of which)  $F \neq G$ , which is  $B$ 's negation and so not  $B$ -compatible.

We are thus led to to conclude that it is  $G=H$  rather than any of the other candidates that fills the gap between  $F=G$  and  $F=H$ .

Similar reasoning suggests that the gap between *That guy is the martini-drinker* and *The martini-drinker is a philosopher* is filled by *That guy is a philosopher*.

## 36 Assertive Content

Write  $S_{\langle \pi \rangle}$  for a sentence  $S$  presupposing that  $\pi$ , or such that  $\pi$  is presupposed in the utterance of  $S$  under discussion.

Hypothesis: What is asserted in that utterance of  $S_{\langle \pi \rangle}$  is  $S-\pi$ .

E.g., what is asserted in an utterance of *Jones is a bachelor again*, when it's presupposed that Jones is male, is *Jones is a bachelor again - Jones is male = Jones is single again*. Uttering *The martini guy is a philosopher* when it's presupposed that HE is the martini guy is a way of asserting *The martini guy is a philosopher - HE is the martini guy = HE is a philosopher*.