Liquidity and Market Crashes

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Abstract

In this paper, we develop an equilibrium model for stock market liquidity and its impact on asset prices when participation in the market is costly. We show that, even when agents' trading needs are perfectly matched, costly participation prevents them from synchronizing their trades, which gives rise to the endogenous need for liquidity. Moreover, the endogenous liquidity need, when it occurs, is dominated by excessive selling of significant magnitude. Such a liquidity-driven selling leads to market crashes in the absence of any aggregate shocks. It also gives rise to negative skewness and fat-tails in stock returns.

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1 Introduction

Market crashes refer to large, sudden drops in asset prices in the absence of big news on the fundamentals—such as future asset payoffs. They exhibit some distinct features. Crashes are one-sided—there are no sudden market surges. They are typically accompanied by large selling pressures in the market. Moreover, the drop in prices occurs quickly but the recovery is much slower. Even though there is little consensus on what causes a crash, the lack of liquidity has always been identified as its symptom and has been blamed for exacerbating its consequences.[1] In particular, crashes are commonly portrayed as market conditions in which the excessive selling pressure drives prices below the fundamental value, while little new capital rushes into the market to take advantage of the price deviations.

This view is supported by the cumulating evidence that despite the profitable opportunities after a crash—at least as perceived by some observers—new capital flows in only after long lags. For example, following the 1987 stock market crash, many companies announced repurchases of their own shares, reflecting the belief that their stocks were undervalued. However, these announcements spread over several months and took even longer to be substantially implemented.[2] The LTCM episode in 1998 was followed by substantial capital outflows from hedge funds operating in the same markets as LTCM (e.g., fixed income arbitrage and global macro strategies). The trend only started to reverse several quarters after, despite the opportunities in these markets (see, e.g., Mitchell, Pedersen, and Pulvino (2006) and Tremont (2006)).[3] These evidence suggests that capital movement is costly. The costs range from informational costs to institutional rigidities, as discussed extensively in Merton (1987) among other.[4] When the abnormal trading pressure hits, only limited supply of liquidity is available to accommodate the trades and prices have to shift drastically.

This perspective focuses on the lack of liquidity supply during severe market conditions. However, it does not explain what gives rise to the initial need for liquidity, why it is always

[1]For example, the report by the Committee on the Global Financial System (CGFS (1999)) provides an overview of the "⋯ deterioration in liquidity and elevation of risk spreads ⋯" in many international financial markets in autumn 1998.

[2]Earlier analysis of the share repurchases after 1987 crash include Gammill and Marsh (1988) and Netter and Mitchell (1989). More recent studies of firms' share repurchase behavior include Ikenberry, Lakonishok, and Vermaelen (1995), Stephens and Weisbach (1998), and Dittmar (2000).

[3]We thank Cristian-Ioan Tiu and Mila Getmansky for bringing to our attention the Tremont Asset Flows Report for data on hedge fund flows.

[4]In Section 2.3, we discuss in more detail the nature of various costs to market participation and capital mobility and the empirical evidence.

in the form of excessive selling, and why it comes in large magnitudes. In this paper, we show that the same cost that hinders the ex post supply of liquidity also generates the need for liquidity in the first place. Despite the symmetric nature of market participants' idiosyncratic trading needs, the aggregate need for liquidity—when it arises—is asymmetric, usually on the selling side, and of large sizes. With limited supply of liquidity in the market, these sudden surges of endogenous liquidity needs will lead to large price drops, as in market crashes.

We start with a model that captures two important aspects of liquidity, the need to trade and the cost to trade. The trading needs arise from idiosyncratic shocks to agents' wealth, which they want to unload in the market by adjusting their asset holdings. By definition, idiosyncratic shocks sum to zero at the aggregate level. Thus, agents' trading needs are always symmetric and perfectly matched—for each potential seller there is a potential buyer with offsetting trading needs. If trading is costless, all potential buyers and sellers will be in the market at all times. Their trades will be perfectly synchronized and matched, and there will be no need for liquidity. The market-clearing price always reflects the "fundamental value" of the asset, such as asset payoffs and investor preferences. Idiosyncratic shocks only generate trading but have no impact on prices.

When trading or participation in the market is costly, the need for liquidity arises endogenously and idiosyncratic shocks can affect prices. The cost to participate has two important effects. First, it prevents potential traders from being in the market constantly. They will enter the market only when they are far away from their desired positions and the expected gains from trading outweigh the cost. Infrequent participation implies that traders who are hit by idiosyncratic risks will not always be able to unload them in the market, which makes them more risk averse. Second, potential traders with offsetting trading needs perceive different gains from trading. In particular, the gains from trading for potential sellers—those who receive a positive shock in their idiosyncratic risk—are always larger than the gains from trading for potential buyers. The reason is that, as both positive and negative shocks push them away from their optimal positions, traders become more risk averse and less willing to hold the asset. This exacerbates the selling needs for traders with negative shocks and dampens the buying demand for traders with positive shocks. The asymmetry between the desire to trade between potential buyers and sellers leads to order imbalances in the form of excess supply and a need for liquidity. The price has to decrease in response.

Moreover, the endogenous liquidity need is highly nonlinear in the idiosyncratic shocks that drive agents' trading needs. When the magnitude of idiosyncratic shocks is moderate, gains from trading are relatively small. As a result, all traders will stay out of the market and there is no need for liquidity. Only when the idiosyncratic shocks are sufficiently large, gains from trading exceed the participation cost and some potential traders start to enter the market. They enter with large trading needs and more on the selling side. Thus, the order imbalance and the need for liquidity, when they occur, are large in magnitude, causing the price to drop discretely in the absence of any aggregate shocks. Such a behavior of the market—namely, infrequent but large price drops accompanied by large selling pressure with no big news on fundamentals—clearly resembles the features of market crashes. It also leads to skewness and fat-tails in return distributions. As a result, higher-order moments of the price, Value-at-Risk measures, and trading volumes provide valuable information regarding market liquidity.

There is a large theoretical literature on explaining market crashes. Most of the existing models rely on both information asymmetry and market frictions.[5] The general mechanism of these models is similar. Frictions prevent prices from reflecting the private information agents have and severely more so in some states than others. Those states are typically not related to asset payoffs, often referred to as noise. But they affect how information gets into prices. When the market shifts from less informative states to more informative states, the price experiences large changes as private information is impounded in.

A challenge for these models is that they lead to both market crashes and surges. In other words, if a price change is caused by an infusion of private information, it can be either positive or negative as the information can be positive or negative.[6] Another challenge is that the price change is permanent as it reflects more information about fundamentals. It does not capture the possibility that prices may actual move away from the fundamental value during a crash, giving rise to transitory deviations.

The mechanism we have identified—the endogenous need of liquidity—has the unique feature that it only leads to large excessive selling pressure and market crashes. Moreover, the price drop is purely a liquidity effect, independent of the information on future payoffs. Thus, it represents a temporary dip in the price below the fundamental value. As the participation

[5]For example, Grossman (1988), Gennotte and Leland (1990), and Romer (1993) consider models with information asymmetry with incomplete markets. Yuan (2005) examines the interaction between information asymmetry and borrowing constraints. Hong and Stein (2003) and Bai, Chang, and Wang (2006) analyze the impact of information asymmetry under short-sale constraints.

[6]Models with short-sale constraints, such as Hong and Stein (2003), Yuan (2005), and Bai, Chang, and Wang (2006), can generate negative skewness in returns. But the skewness arises from the asymmetric distribution of small price changes, not discrete price drops.

cost also limits the flow of capital into the market to provide liquidity, the price deviation only recovers slowly. Although information may well play an important role in exacerbating the initial selling demand and amplifying market crashes, we identify a unique mechanism that can explain the sudden rise of excessive selling demand in the first place. Moreover, it can explain the slow movement of capital into the market after crashes, the prolonged price depression, and the eventual recovery.

The literature on the influence of liquidity on asset prices is extensive.[7] In studying the impact of liquidity, much of the attention has focused on the supply of liquidity, taking the demand of liquidity as given.[8] Our work extends the existing literature on liquidity by modelling how the need for liquidity arises endogenously, how it behaves and how it influences asset prices. For example, Grossman and Miller (1988) consider the role of market makers in providing liquidity and reducing price volatility and, taking as given the non-synchronization in trades, how participation costs limit their ability to do so. Our analysis shows that it is the participation costs that generate the non-synchronization in trades. Moreover, the endogenous liquidity need exhibits distinctive features, which lead to unique predictions on its price impact.

In this regard, our paper is similar in spirit to that of Allen and Gale (1994), who consider the ex-ante participation decisions of agents with different future liquidity needs. They show that the ex-ante optimal level of participation can be inadequate ex-post when the realized liquidity need is much larger than expected, causing additional volatility in prices. We focus more on the dynamic aspect of liquidity by allowing traders to make their participation decisions after observing new shocks to their trading needs over time. Thus, we are able to study how the need for liquidity occurs in response to new idiosyncratic shocks to the traders. As we show, the properties of the endogenous liquidity need (e.g., one-sided and fat-tailed) can be quite different from those assumed for exogenous liquidity shocks.

[7]The theoretical work includes Grossman and Miller (1988), Hirshleifer (1988), Pagano (1989), Allen and Gale (1994), Orosel (1998), Huang (2003), Lo, Mamaysky, and Wang (2004), and Vayanos (2004). Recent empirical work on the impact of liquidity on asset prices includes Brennan and Subrahmanyam (1996), Chordia, Roll, and Subrahmanyam (2000), Amihud (2002), Pastor and Stambaugh (2003), and Acharya and Pedersen (2005). Cochrane (2005) provides a review of the recent literature.

[8]See, for example, Amihud and Mendelson (1980), Ho and Stoll (1981), and Huang (2003). In the market micro-structure literature, which has liquidity as a central focus, the need for liquidity, as described by the order flow process, is often taken as given. See, for example, Glosten and Milgrom (1985), Kyle (1985), and Stoll (1985). Admati and Pfleiderer (1988) and Spiegel and Subrahmanyam (1995), however, do allow the order flow process to be influenced by equilibrium. Several recent papers, e.g., Kyle and Xiong (2001), Gromb and Vayanos (2002), and Brunnermeier and Pedersen (2006), allow liquidity needs to be partially endogenous due to the impact of market prices on agents' wealth.

Another distinctive feature of our model is that liquidity need arises purely from idiosyncratic shocks, which would have no pricing implications in absence of the liquidity effect. Most of the existing models on the impact of liquidity including Allen and Gale (1994) rely on aggregate shifts in demand.[9] The presence of aggregate shocks makes market crashes and surges equally likely, as they can be either positive or negative. Moreover, it blurs the distinction between the effects of liquidity and risk (and/or preferences). In these models, liquidity merely plays the role of exacerbating the impact of exogenous aggregate shocks.[10] In our model, it is the idiosyncratic shocks that generates endogenous selling demand at the aggregate level.

Our model is closely related to the model of Lo, Mamaysky, and Wang (2004), who consider the impact of fixed transactions costs on trading volume and the level of asset prices. In a continuous-time stationary setting, they show that gains from trading are in general asymmetric between traders with offsetting shocks when trading is infrequent. Such an asymmetry naturally leads to non-synchronization in trades, the need for liquidity, and price deviations. In order to focus on the impact of trading costs on price levels, Lo, Mamaysky, and Wang (2004) avoid order imbalances by allowing the participation cost to be allocated endogenously so that the trades of different market participants are always synchronized in equilibrium. As we show in this paper, it is the order imbalance that leads to changes in liquidity needs and the instability in asset prices.

The paper proceeds as follows. Section 2 describes the basic model. Section 3 solves for the intertemporal equilibrium of the economy. In Section 4, we analyze how individual traders' participation decisions give rise to the need for liquidity and the properties of equilibrium liquidity needs. In Section 5, we examine how the endogenous need for liquidity affects asset prices. Section 6 concludes. The appendix provides the proofs.

2 The Model

We construct a parsimonious model that captures two important factors in analyzing liquidity, the need to trade and the cost to constantly participate in the market. We describe the econ-

[9]See also Campbell, Grossman, and Wang (1993), Campbell and Kyle (1993), and Kyle and Xiong (2001), among others.

[10]Effectively, these models generate large fluctuations in the marginal utility of marginal investors in response to aggregate shocks. In their reduced form (i.e., if we ignore how aggregate shocks lead to changes in marginal utility), these models become similar to representative agent models with a state-dependent marginal utility. For example, Campbell and Hentschel (1992) model the arrival of news as an increase in the aggregate risk which depresses asset prices. This overall negative impact on prices offsets the effect of good news and amplifies the effect of bad news, leading to skewness in returns.

omy and the notion of equilibrium in Sections 2.1 and 2.2. We then provide more discussion on the key features of the model in Sections 2.3 and 2.4.

2.1 Economy

For simplicity, we consider a discrete-time, infinite-horizon setting.

A Asset Market

A stock is traded in a competitive asset market. It yields a risky dividend D_t at time t, where $t = 0, 1, 2, \ldots$ Dividends are i.i.d. normally distributed with a mean of $\bar{D}$ and volatility of σ_D. Let P_t denote the ex-dividend stock price at time t. In addition, there is a short-term riskless bond, which yields a constant interest rate of $r > 0$ per period.

B Agents

At $t = 0, 1, 2, \ldots$, a set of agents are born who live for one period. Agents born at t are referred to as generation t. They are born with initial wealth W_t, which they invest in the stock and the bond. They sell all their assets for consumption at time $t + 1$.

Each generation consists of two types of agents who face different endowments and trading costs. As described below, agents' heterogeneity in endowments will gives rise to their trading needs in our model. The first type of agents, denoted by m, are "market makers." They have no inherent trading needs, but are present in the market at all times, ready to trade with others. The second type of agents are "traders" who have inherent trading needs. Traders are split between two equal subgroups, denoted by a and b, respectively. The population weight of the market makers and the traders are μ and 2ν, respectively.

The per capital supply of the stock is $\bar{\theta}$, which is positive (i.e., $\bar{\theta} > 0$). In addition, each agent i of generation t receives a non-traded payoff N^i_{t+1} at the end of his life-span, given by

$$N^i_{t+1} = \lambda^i Z\, n_{t+1}, \quad i = m, a, b \tag{1}$$

where Z and n_{t+1} are mutually independent, normal random variables with a mean of zero and a volatility of σ_z and σ_n, respectively, λ^i is a binomial random variable drawn independently for each agent within his group, where

$$\lambda^m = 0, \quad \lambda^a = -\lambda^b = \begin{cases} 1, & \text{with probability } \lambda \\ 0, & \text{with probability } 1 - \lambda. \end{cases} \tag{2}$$

Here we have suppressed the time subscript for λ^i and Z for brevity. Thus, market makers receive no non-traded payoff, while a fraction λ of traders within each trader group receives non-traded payoffs. Since $\lambda^a = -\lambda^b$, the two groups of traders receive perfectly offsetting non-traded payoffs. By construction, we have

$$\sum_{i=a,b,m} N^i_{t+1} = 0. \tag{3}$$

The non-traded payoff is assumed to be correlated with the stock dividend D_{t+1}. In particular, we let $n_{t+1} = D_{t+1} - \bar{D}$.[11]

In the absence of risks from non-traded payoffs, all agents are identical and there will be no trading needs among them. However, in the presence of non-traded risks, traders who receive them want to trade in order to share these risks. In particular, given the correlation between the non-traded payoff and the stock payoff, they want to adjust their stock positions in order to hedge their non-traded risks. Thus, traders' idiosyncratic risk exposures give rise to their inherent trading needs.

Since the non-traded risks sum to zero as in (3), the traders' underlying trading needs are perfectly matched. If all traders are present in the market at all times, a seller is always matched with a buyer and there is perfect synchronization in their trades. If, however, only some traders are present at a given time, trades may not be always synchronized and the need for liquidity arises.

For tractability, we assume that all agents have a utility function of constant absolute risk aversion over their terminal wealth. The utility function for generation-t agents is

$$\mathrm{E}\left[-e^{-\alpha W^i_{t+1}}\right], \qquad i = a, b, m \tag{4}$$

where W^i_{t+1} denotes agent i's terminal wealth.

C Trading Costs

All agents can trade in the market at no cost at the beginning and the end of their life-span. That is, agents of generation t can trade in the market at t and $t+1$ without cost. In addition, market makers can also trade at no cost at any time between t and $t+1$. The traders, however, face a fixed cost $c \geq 0$ if they want to trade between t and $t+1$.

[11] We only need the correlation between n_{t+1} and D_{t+1} to be non-zero. The qualitative nature of our results are independent of the sign and the magnitude of the correlation. To fix ideas, we set it to 1.

D Time Line

We now describe in detail the timing of events and actions. At t, agents of generation t are born. They purchase shares of the stock from the old generation and construct their optimal portfolio θ_t^i, $i = a, b, m$. Market equilibrium at t determines P_t.

After t, traders learn if they will be exposed to any idiosyncratic risks (i.e., their draws of λ^i). Those who will ($\lambda^i \neq 0$) also observe a signal S about its potential magnitude Z:

$$S = Z + u \tag{5}$$

where u is the noise in the signal, normally distributed with a mean of zero and a variance of $\sigma_u^2 > 0$. For future convenience, we denote by X the expectation of Z conditional on signal S and $\sigma_{\hat{z}}^2$ the conditional variance. We then have

$$X \equiv \mathrm{E}[Z|S] = \frac{\sigma_z^2}{\sigma_z^2 + \sigma_u^2}\, S, \quad \sigma_{\hat{z}}^2 \equiv \mathrm{Var}[Z|S] = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_z^2}\, \sigma_z^2. \tag{6}$$

Under normality, X is a sufficient statistic for signal S. Thus, we will use X to denote these traders' information about the magnitude of their idiosyncratic risks.

After learning about their idiosyncratic risks, traders face the choice of staying out of the market (until their terminal date) or paying a cost c to enter the market. Those who choose to enter the market will then trade among themselves as well as with market makers. To fix ideas, we assume that signal X and entry decisions occur at $t + 1/2$ and trading occurs right after. This specification has the advantage of carrying forward interests more easily.

A trader's choice to enter the market depends on his draw of λ^i and the signal X on the magnitude of the idiosyncratic risk if $\lambda^i \neq 0$. Let η^i be the discrete choice variable of trader i ($i = a, b$) for whether to enter the market, where $\eta^i = 1$ denotes entry and $\eta^i = 0$ denotes no entry. Among group i traders ($i = a, b$) who receive idiosyncratic shocks (i.e., $\lambda^i \neq 0$), we use $\omega^{i,L}$ to denote the fraction that choose to enter the market. Similarly, $\omega^{i,NL}$ denotes the fraction of traders without idiosyncratic shocks that choose to enter. We also use $\theta_{t+1/2}^i(\eta^i)$ to denote the number of stock shares agents i ($i = m, a, b$) holds after trading at date $t + 1/2$. Of course, $\theta_{t+1/2}^i(\eta^i = 0) = \theta_t^i$. Summarizing the description above, Figure 1 illustrates the time-line of the economy.

For agent i, his terminal financial wealth, denoted by F_{t+1}^i, is

$$F_{t+1}^i = R^2 W_t - R\,\eta^i c^i + \theta_t^i R\,(P_{t+1/2} - RP_t) + \theta_{t+1/2}^i(\eta^i)\,(D_{t+1} + P_{t+1} - RP_{t+1/2}) \tag{7}$$

where $R = (1+r)^{1/2}$ is the gross interest rate for each $1/2$ period, $c^i = c$ for $i = a, b$ and $c^i = 0$

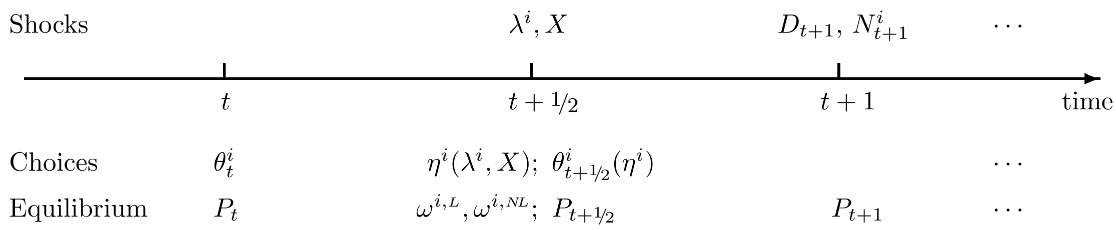

Figure 1: The time line of the economy.

for $i = m$. His total wealth at date $t+1$ is then given by

$$W^i_{t+1} = F^i_{t+1} + N^i_{t+1} \tag{8}$$

where N^i_{t+1} is the income from the non-traded asset in (1).

2.2 Definition of Equilibrium

The equilibrium of the economy requires three conditions. First, taking prices as given, all agents optimize with respect to their participation and trading decisions. Second, agents' participation reaches an equilibrium. Third, the stock market clears. Given the repetitive nature of each generation, we only need to focus on the equilibrium over the life-span of one generation, say, generation t.

At time t, generation-t is born and they purchase the shares from the previous generation. Since the total supply of the stock is $(\mu + 2\nu)\,\bar{\theta}$, the equilibrium stock price P_t is determined by the market clearing condition

$$\mu\,\theta^m_t + \nu\,(\theta^a_t + \theta^b_t) \;=\; (\mu + 2\nu)\,\bar{\theta}. \tag{9}$$

At time $t + 1/2$, after learning about whether or not they are exposed to idiosyncratic risks, traders decide whether to pay a cost to enter the market. A participation equilibrium is reached if all traders within the same group make identical participation decisions, i.e., $\omega^{i,j} = 0$ or 1, $i = a, b$ and $j = L,\ NL$, or if they are indifferent about whether to participate or not at an interior value $\omega^{i,j} \in (0, 1)$.

Given the participation decision, market makers and participating traders determine the market equilibrium. Let $\theta^{i,j}_{t+1/2} \equiv \theta^i_{t+1/2}\big(\eta^i(\lambda^i)\!=\!1\big)$ $(i = a, b)$ denote the stock holding at $t + 1/2$ of participating traders from each group, with and without idiosyncratic shock $(\lambda^i = 0, 1)$, respectively. The clearing of the stock market at $t + 1/2$ requires

$$\mu\,\theta^m_{t+1/2} + \nu \sum_{i=a,b} \left[\lambda\,\omega^{i,L}\theta^{i,L}_{t+1/2} + (1-\lambda)\omega^{i,NL}\theta^{i,NL}_{t+1/2}\right] = \mu\,\theta^m_t + \nu \sum_{i=a,b} \left[\lambda\,\omega^{i,L} + (1-\lambda)\omega^{i,NL}\right]\theta^i_t \quad (10)$$

which determines the equilibrium stock price at $t + 1/2$.

At time $t+1$, the economy repeats itself: A new generation is born; the current generation sells their assets and consume their total wealth as given in (8); and the stock market clears as the new generation purchases all the stock shares. Given the stationary nature of the economy, we consider a stationary equilibrium in which

$$P_{t+1} = P_t. \quad (11)$$

It is worth noting that we assumed a constant interest rate and thus do not require the bond market to clear.

2.3 Discussions and Simplifications

In this subsection, we provide additional discussions and motivations on some of the key features of the model. The two key ingredients of the model are the need to trade and the cost to participate in the market. Little justification is needed for introducing trading needs, given the large trading volume we observe in the market, its variations over time, and its abnormal level during market crashes.[12] In order to model trading needs, we must capture certain forms of heterogeneity among agents.[13] We use heterogeneity in agents' non-traded payoffs as a device to introduce trading needs for risk sharing. This modelling choice is mainly for tractability.

In general, the two trader groups can receive arbitrary non-traded risks, i.e., $N^i_{t+1} = Y^i n_{t+1}$, $i = a, b$ with arbitrary Y^a and Y^b. Let $Y = (Y^a + Y^b)/2$ and $Z = (Y^a - Y^b)/2$. We have $Y^i = Y + Z$ and $Y^b = Y - Z$. Hence, Y captures the aggregate exposure to non-traded risk and Z captures the idiosyncratic exposure. Since it is the idiosyncratic risk that drives trading (and liquidity), we will focus on it and simply assume $Y = 0$.[14] Thus, there is no loss

[12]For example, Gammill and Marsh (1988) provides a partial picture of trading activity by different types of market participants during the 1987 crash.

[13]Trading for risk sharing can arise from heterogeneity in endowments (e.g., Wang (1994) and Lo, Mamaysky, and Wang (2004)), preferences (e.g., Dumas (1992) and Wang (1996)), or beliefs (e.g., Harris and Raviv (1993) and Detemple and Murthy (1994)). By the No-Trade Theorem of Milgrom and Stokey (1982), in the absence of other heterogeneity, information alone does not necessarily generate trading. In the presence of other trading needs, information asymmetry can lead to additional trading (e.g., Wang (1994) and He and Wang (1995)).

[14]This is not to say that aggregate risk does not matter. Fluctuations in aggregate risk can interact with

of generality in our specification of idiosyncratic risks in (1) and (2).

In the presence of non-traded payoffs, a trader's utility is influenced by the risk they bring. For example, a trader with only non-traded assets has an expected utility of

$$\mathrm{E}\left[-e^{-\alpha Z n_{t+1}}\right] = -\,\mathrm{E}\left[e^{\frac{1}{2}\alpha^2\sigma_z^2 n_{t+1}^2}\right].$$

For it to be well defined (i.e., finite), we need

$$\alpha^2\sigma_n^2\sigma_z^2 < 1 \tag{12}$$

where σ_n^2 denotes the variance of n_{t+1}. For $n_{t+1} = D_{t+1} - \bar{D}$, $\sigma_n^2 = \sigma_D^2$. For technical convenience, we assume that (12) holds throughout the paper.

Another key ingredient of our model is the cost to participate in the market. It is intended to capture the frictions preventing the full participation of all potential players in a given market at any given moment. Although these frictions are mostly absent in the neoclassical asset pricing theory, they are nonetheless part of market reality. As pointed out in Grossman and Miller (1988), financial markets do not operate like "a gigantic town meeting in which all potential buyers and sellers participate directly." Instead, participation is always limited. Information costs and institutional rigidities are abundant in preventing full participation or instant capital flow. Gathering and processing information, devising trading strategies and their support systems in response to new information, raising capital and making changes in business practice to implement these strategies all involve costs and take time. After providing an extensive discussion on the importance of these costs, Merton (1987) observes that "On the time scale of trading opportunities, the capital stock of dealers, market makers and traders is essentially fixed. Entry into the dealer business is neither costless nor instantaneous."[15] Commenting on the 1987 crash, Leland and Rubinstein (1988) suggest that "... at any one time, there is a limited pool of investors available with the ability to evaluate stocks and take appropriate action in the market."[16]

Merely recognizing the costs and limits to participation is not sufficient. Their magnitude also matters. Although direct measurements of these costs are hard to come by, there is cu-

the demand and supply of liquidity. We leave the impact of aggregate risk for future work.

[15] It is worth noting that the information costs Merton (1987) refers to are much broader than the simple cost of gathering information. They also include the costs and time spent to process these information, to devise strategies to take advantage of it, to raise capital, and to implement these strategies. Finally, the costs also involve deeper issues like the incentive structure of the institutions and the regulatory environment.

[16] See also Brennan (1975), Hirshleifer (1988), Chatterjee and Corbae (1992), and Gromb and Vayanos (2002) among others for more discussion of participation/fixed costs in financial markets.

mulating evidence demonstrating their significance. For example, Coval and Stafford (2006) find that widespread selling by financially distressed mutual funds leads to significantly depressed prices for stocks in their portfolio. The price effects can last over a couple of quarters and take several more quarters to reverse. Moreover, the effect occurs despite the fact that many of the stocks sold are widely held by other mutual funds who are not suffering outflows. Mitchell, Pedersen, and Pulvino (2006) examine several markets such as convertible bonds and mergers and acquisitions, in which hedge funds actively pursue pricing anomalies. They show that when hedge funds in a particular market (say convertible bonds) face large redemptions, prices deviate significantly from the fundamentals. However, funds are slow in flowing back into the market. As a result, the price deviations persist for long periods of time before they start to diminish. The persistence in large price deviations caused by liquidity events as documented in these studies implies substantial frictions that prevent instantaneous capital flow or participation.[17]

Two more comments are in order regarding participation costs. First, they may confront certain subset of agents, but not necessarily all, and different agents may face different kinds of participation costs. Other than the distinction between traders (who face costs) and market makers (who face no cost,) our model does not fully reflect the richness of the cost structure in the market place. Second, the participation cost in essence reflects the cost to capital flow. More capital in a market tends to reduces the risk aversion of marginal investors (e.g., Grossman and Vila (1992)). In our setting, all agents have constant risk aversion. But participation of more agents brings down the effect risk aversion of the market maker (see also Grossman and Miller (1988)).

Our model also makes an important technical assumption that, at the time of making their participation decisions, traders only partially learn about their future idiosyncratic risk. That is, they receive only a noisy signal S about the true magnitude of the total exposure Z. Note that, if Z is fully known at the time of the participation decision, a single trade is expected to remove all future non-traded risks; the model degenerates into a static one. By assuming a partial observation of Z, we capture the intertemporal effect that future risks cannot be fully eliminated by one trade. In other words, even if a trader chooses to enter the market now, he still expects to bear some future idiosyncratic risk since he may not be in the

[17]The evidence on the limited mobility of capital is quite extensive. See also Harris and Gurel (1986), Shleifer (1986), Lynch and Mendenhall (1997), Wurgler and Zhuravskaya (2002), and Chen, Noronha, and Singal (2004) on the price effect of stock deletions from the S&P index, Frazzini and Lamont (2006) on the price impact of capital flows to mutual funds, and Tremont (2006) on market behavior and hedge fund flows.

market in the future. Such an expectation influences his current participation decision. As we will see in the next section, this additional uncertainty leads to asymmetric participation decisions for traders with matching trading needs. This result arises from the intertemporal nature of the model. In a fully intertemporal setting, Lo, Mamaysky, and Wang (2004) show that when participation costs force traders to trade infrequently, they always expect to bear some idiosyncratic risks and the asymmetry in their trading is a general outcome. Our setup provides a simple way to capture the same effect.

As long as it occurs after the participation decision, the exact timing of the full revelation of Z is not critical. For simplicity, we assume that by the time of trading all traders who receive idiosyncratic risks also observe the realization of Z.[18]

Another assumption we make is that traders without idiosyncratic risk ($\lambda^i = 0$) do not observe the magnitude of the shocks that others receive. This helps to simplify the model as these traders will typically stay out of the market. We should emphasize that our main results remain the same if they also observe the shocks (e.g., by observing prices). This assumption only helps to simplify our analysis.

2.4 Equilibrium with Costless Participation

Before solving for the equilibrium, we describe the special case of participation costs being zero for all agents. This case serves as a benchmark when we examine the impact of participation costs on liquidity and stock prices. If $c^i = 0 \ \forall \ i = a, b, m$, all traders and market makers will be in the market at all times, and $\omega^{i,L} = \omega^{i,NL} = 1 \ \forall \ i = a, b$. The equilibrium price and agents' equilibrium stock holdings are:

$$\begin{aligned} P_t &= \bar{P} \equiv \tfrac{1}{r}\left(\bar{D} - \alpha\sigma_D^2\,\bar{\theta}\right), & \theta_t^i &= \bar{\theta} \\ P_{t+1/2} &= RP_t, & \theta_{t+1/2}^i &= \bar{\theta} - \lambda^i Z \end{aligned} \tag{13}$$

where $t = 0, 1, 2, \ldots$ and $i = a, b, m$.

In this case, stock prices, P_t and $P_{t+1/2}$, are determined by the stock's expected future dividends $\bar{D}$, the dividend risk σ_D, and the aggregate (per capita) risk exposure $\bar{\theta}$. We call these the "fundamentals" of the stock. Prices do not depend on the idiosyncratic risk exposure $\lambda^i Z$. For traders exposed to non-traded risks, their stock holding equal the per capita shock share $\bar{\theta}$ plus an additional component $\lambda^i Z$, which reflects their hedging demand to offset the

[18] We have also solved the model under the assumption that Z is revealed at $t+1$. The equilibrium price and participation decisions are qualitatively the same.

exposure to the non-traded risk. It is important to note that because these traders' underlying trading needs are perfectly matched ($\lambda^a = -\lambda^b$), so are their trades when they are all in the market. In this case, the market is perfectly liquid in the sense that order flows have no price impact. There is no need for liquidity and market makers perform no role (their holdings stay at $\theta^m = \bar{\theta}$).

3 Equilibrium

We now solve for the equilibrium when participation is costly in three steps. First, taking the stock price at time $t+1$ and agents' initial stock holdings and participation decisions at t as given, we solve for the stock market equilibrium at $t + 1/2$. Next, we solve for individual agents' participation decisions and the participation equilibrium, given the market equilibrium at $t + 1/2$ and their initial stock holdings at t. Then, we solve for the market equilibrium at time t. Using the condition $P_{t+1} = P_t$, we finally obtain the full stationary equilibrium of the economy.

In the first two steps (Sections 3.1-3.3), we assume that traders who receive no idiosyncratic shocks ($\lambda^i = 0$) stay out of the market until the end of their horizon, that is, $\omega^{i,NL} = 0$, $i = a, b$. We then only consider those traders who do receive shocks and solve for their participation decisions, the participation equilibrium, and the market equilibrium at $t + 1/2$. In these subsections, unless stated otherwise, traders refer only to those with $\lambda^i \neq 0$ and participation weights, $\omega^a = \omega^{a,L}$ and $\omega^b = \omega^{b,L}$, refer to fractions of them who choose to participate. In the last step (Section 3.4), we include all traders and confirm that in equilibrium those who receive no idiosyncratic shocks indeed choose not to participate in the market.

3.1 Market Equilibrium at $t + 1/2$

At $t + 1/2$, we first take agents' initial stock holdings and their participation decisions as given and solve for the market equilibrium. Let $\theta \equiv (\theta^a_t, \theta^b_t, \theta^m_t)$ denote agents' stock holdings at t and $\omega \equiv (\omega^a, \omega^b)$. Since our analysis focuses on generation t, we have omitted the time subscript t of θ and ω for brevity. Together with Z, $\{\theta, \omega\}$ defines the state of the economy at $t + 1/2$. Two variables are of particular importance in describing the market condition:

$$\hat{\theta} \equiv \frac{\mu\theta^m + \lambda\,\nu(\omega^a\theta^a + \omega^b\theta^b)}{\mu + \lambda\,\nu(\omega^a + \omega^b)}, \qquad \delta \equiv \frac{\lambda\,\nu}{\mu + \lambda\,\nu(\omega^a + \omega^b)}\,(\omega^a - \omega^b) \tag{14}$$

where $\hat{\theta}$ gives the per capita stock supply in the market (brought in by participating agents) and δ measures the difference in participation between the two trader groups. Since the participation equilibrium at t depends on the information X about the non-traded risk, ω^a and ω^b and thus $\hat{\theta}$ and δ are in fact functions of X.

The following proposition solves the market equilibrium at $t + 1/2$, taking as a given the market condition $\hat{\theta}$ and δ and the magnitude of idiosyncratic risks Z.

Proposition 1. *Let P_{t+1} be the equilibrium price at time $t+1$. Given $\{\theta, \omega\}$, the equilibrium stock price at $t + 1/2$ is*

$$P_{t+1/2} = \frac{1}{R}\left(\bar{D} + P_{t+1} - \alpha\sigma_D^2\,\hat{\theta} - \alpha\sigma_D^2\,\delta Z\right) \tag{15}$$

and the equilibrium stock holding of participating agent i is

$$\theta^i_{t+1/2} = \hat{\theta} + \delta Z - \lambda^i Z, \qquad i = a, b, m. \tag{16}$$

When $\delta = 0$, the participation of the two groups of traders is symmetric. The participating agents' holdings are equal to the per capita holding $\hat{\theta}$ minus the hedging demand $\lambda^i Z$. Since $\lambda^a = -\lambda^b$, there is a perfect match between the buy and sell orders among traders, and the equilibrium price is not affected by the idiosyncratic shock Z. This situation is reminiscent of the benchmark case when participation is costless.

When $\delta \neq 0$, the participation of the two groups of traders is asymmetric. The quantity δZ measures the excess exposure (per capita) to the non-traded risk due to the asymmetric participation of traders. In this case, the optimal holding in (16) has an extra term δZ for all participating agents since they equally share this additional source of risk. The idiosyncratic shock Z now affects the equilibrium price. Thus in our model, even though traders face offsetting shocks, asymmetry in their participation can give rise to a mismatch in their trades and cause the price to change in response to these shocks.

Here, we have taken traders' participation and the resulting δ and $\hat{\theta}$ as given. In the next subsection, we show that when individual participation decisions are made endogenously, asymmetric participation occurs as an equilibrium outcome.

3.2 Optimal Participation Decision

Given the market equilibrium at $t + 1/2$ and the signal X for future idiosyncratic shocks, we now solve the participation equilibrium. First, taking as given the participation decision of others and price $P_{t+1/2}$, we derive the optimal participation policy of an individual trader. Next, we find the competitive equilibrium for traders' participation decisions.

For trader i, let J_P and J_{NP} denote his utility after he chooses to participate or not to participate, respectively. We have

$$J_P(\theta^i; \lambda^i, X; \hat{\theta}, \delta) \;=\; \mathrm{E}\left[\max_{\theta^i_{t+1/2}} \mathrm{E}_{t+1/2}\left[-e^{-\alpha W^i_{t+1}}\right] \,\middle|\, \lambda^i, X;\; \eta^i = 1\right] \tag{17a}$$

$$J_{NP}(\theta^i; \lambda^i, X; \hat{\theta}, \delta) \;=\; \mathrm{E}\left[-\,e^{-\alpha W^i_{t+1}} \,\middle|\, \lambda^i, X;\; \eta^i = 0,\; \theta^i_{t+1/2} = \theta^i\right] \tag{17b}$$

where θ^i denotes his initial stock holding, λ^i and X define his exposure to the non-traded risk, $\hat{\theta}$ and δ define the market condition, and $\mathrm{E}[\cdot|X]$ denotes the expectation conditional on X. His net gain from participation can be defined as the certainty equivalence gain in wealth:

$$g(\theta^i; \lambda^i, X; \hat{\theta}, \delta) = -\frac{1}{\alpha} \ln \frac{J_P(\cdot)}{J_{NP}(\cdot)}. \tag{18}$$

The minus sign on the right-hand side adjusts for the fact that $J_P(\cdot)$ and $J_{NP}(\cdot)$ are negative. The optimal decision for trader i is to participate if and only if the net gain from participating is positive. The following proposition describes the optimal participation policy for an individual trader.

Proposition 2. *For trader i with initial stock holding θ^i, idiosyncratic shock $\lambda^i \neq 0$ and X, and under market condition $\hat{\theta}$ and δ, his net gain from participation is*

$$g(\theta^i; \lambda^i, X; \hat{\theta}, \delta) = g_1(\theta^i; \lambda^i, X; \hat{\theta}, \delta) + g_2(\lambda^i; \delta) - Rc^i \tag{19}$$

where

$$g_1(\cdot) = \frac{\alpha\sigma_D^2(1-k\,\lambda^i\delta)^2}{2(1-k)[1-k+k\,(1-\lambda^i\delta)^2]}\left(\theta^i - \hat{\theta}^i\right)^2 \tag{20a}$$

$$g_2(\cdot) = \frac{1}{2\alpha}\ln\left[1 + (1-\lambda^i\delta)^2 k/(1-k)\right] \tag{20b}$$

and

$$\hat{\theta}^i \equiv \frac{1-k}{1-k\,\lambda^i\delta}\,\hat{\theta} - \frac{1-\lambda^i\delta}{1-k\,\lambda^i\delta}\,\lambda^i X, \quad k \equiv \alpha^2\sigma_D^2\sigma_{\hat{z}}^2 \tag{21}$$

He chooses to participate if and only if $g(\cdot) > 0$.

The net gain from participation consists of three terms, $g_1(\cdot)$, $g_2(\cdot)$ and $-Rc^i$. The first term, $g_1(\cdot)$, represents the expected gain from trading given the current signal X on non-traded risks. This term depends on trader i's initial holding θ^i, the per capita stock supply of all participating agents $\hat{\theta}$, and the expected idiosyncratic risk, $\lambda^i X$. It is important to note that as long as the initial holding θ^i is different from $\frac{1-k}{1-\lambda^i k\,\delta}\hat{\theta}$, this term is not symmetric between the two trader groups. As we will see below, the asymmetry in trading gains gives rise to asymmetric participation decisions. The second term, $g_2(\cdot)$, captures the expected gain from trading to offset future shocks to non-traded risks. This term depends on the market condition δ and the variation in future trading needs, which is captured by k. The last term, $-Rc^i$, simply reflects the cost of participation.

Given the gain from participation, the optimal participation decision becomes very intuitive. Since both $g_1(\cdot)$ and $g_2(\cdot)$ are always positive, the gain $g(\cdot)$ is always positive when the participation cost is smaller than the gain from offsetting future shocks, i.e., when $c \le R^{-1}g_2(\cdot)$. Trader i always participates in this case, independent of signal X. The more interesting case is when $c > R^{-1}g_2(\cdot)$ and trader i chooses to participate only if the expected gain $g_1(\cdot)$ from trading against his current expected exposure is sufficiently large. Note that $g_1(\cdot)$ is zero when his current holding θ^i is equal to $\hat{\theta}^i$. Thus, we can interpret $\hat{\theta}^i$ as trader i's desired stock holding after observing his idiosyncratic risk, given by λ^i and X. In this case, a trader chooses to participate when his holding θ^i is sufficiently far away from the desired position $\hat{\theta}^i$.

3.3 Participation Equilibrium

Given traders' optimal participation decisions, we now solve for the participation equilibrium. In the absence of participation costs, traders with positive idiosyncratic shocks ($\lambda^i X > 0$) expect to sell the stock while traders with negative shocks ($\lambda^i X < 0$) expect to buy. Depending on the realization of $\lambda^i X$, each group of traders become either potential sellers or buyers. For example, when $X > 0$, group-a traders become potential sellers ($\lambda^a X > 0$) while group-b traders become potential buyers ($\lambda^b X < 0$). When $X < 0$, the opposite is true.

In order to solve for ω^a and ω^b in equilibrium, we substitute the expression of $\hat{\theta}$ and δ in (14) into the definition of $g(\cdot)$ and define a function of participation gain for group-a and b traders respectively,

$$g^a(\omega^a, \omega^b) \equiv g(\theta^a; \lambda^a, X; \hat{\theta}, \delta), \qquad g^b(\omega^a, \omega^b) \equiv g(\theta^b; \lambda^b, X; \hat{\theta}, \delta). \tag{22}$$

In general, participation gain depends on agents' initial positions. We only consider the situation in which their initial holdings satisfy the following condition:

$$|\theta^i - \theta^m| \le \min\left\{\frac{\mu\,\sigma_{\hat{z}}}{\mu + \lambda\nu},\ k\,\theta^m\right\}, \qquad i = a, b. \tag{23}$$

We verify later that this condition is satisfied in equilibrium (see Theorem 1).

The following proposition describes the participation equilibrium.

Proposition 3. *When agents' initial stock holdings satisfy (23) and $X > 0$, there exists a unique participation equilibrium. Let*

$$\hat{s}^a = \begin{cases} 0, & \text{if } g^a(0,0) \leq 0 \\ 1, & \text{if } g^a(1,0) \geq 0 \\ s^a, & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{s}^b = \begin{cases} 0, & \text{if } g^b(1,0) \leq 0 \\ 1, & \text{if } g^b(1,1) \geq 0 \\ s^b, & \text{otherwise} \end{cases}$$

where s^a and s^b are the solutions to $g^a(s^a, 0) = 0$ and $g^b(1, s^b) = 0$, respectively. The equilibrium is fully specified as follows:

A. If $g^a(1, \hat{s}^b) \geq 0$, then $\omega^a = 1$ and $\omega^b = \hat{s}^b$.

B. If $g^a(1, \hat{s}^b) < 0$ and $g^b(\hat{s}^a, 0) \leq 0$, then $\omega^a = \hat{s}^a$ and $\omega^b = 0$.

C. Otherwise, $\omega^a, \omega^b \in (0,\, 1)$ and solve both $g^a(\omega^a, \omega^b) = 0$ and $g^b(\omega^a, \omega^b) = 0$.

Moreover, $\omega^a \geq \omega^b$. For $X < 0$, all the results hold by exchanging subscripts a and b.

Cases A and B describe two polar cases when we have corner solutions, either all potential sellers participate (case A) or no buyers do (case B). Case A corresponds to the situation in which trading gains for sellers are overwhelming so that they will all enter the market, irrespective of what buyers do. The presence of a large number of sellers increases the trading gain for buyers. Thus, in this case some buyers may also choose to participate. Case B corresponds to the situation in which not all sellers will participate but independent of what they do the net trading gains for buyers remains negative. In this case, some sellers choose to participate but no buyers do. Case C corresponds to the intermediate case when we have an interior solution. In this case, participation of each group depends on the degree of participation of the other group.

Proposition 3 confirms that there are always more sellers entering the market than buyers in equilibrium. Additional sellers bring excess sell orders to the market and the need for liquidity, which is provided by market makers.

3.4 Full Equilibrium of the Economy

We now turn to the solution to the full equilibrium of the economy. We start by computing the value function for all agents at time t, including traders who receive no idiosyncratic risks.

First, we observe that the indirect utility function, J_P or J_{NP}, defined in (17) is valid for each agent i ($i = a, b, m$) conditional on his own λ^i, X, given his initial stock holding θ_t^i. Next, for a trader with $\lambda^i \neq 0$, his unconditional value function becomes

$$J^L(\theta_t^i;\theta_t) = \mathrm{E}\left[\max\{J_P(\theta_t^i;\lambda^i,X;\hat{\theta},\delta),\ J_{NP}(\theta_t^i;\lambda^i,X;\hat{\theta},\delta)\}\ \middle|\ \lambda^i \neq 0\right]. \tag{24}$$

and for a trader with $\lambda^i = 0$, who does not observe X, his value function is

$$J^{NL}(\theta_t^i;\theta_t) = \max\left\{\mathrm{E}\left[J_P(\theta_t^i;\lambda^i,X;\hat{\theta},\delta)\ \middle|\ \lambda^i = 0\right],\ \mathrm{E}\left[J_{NP}(\theta_t^i;\lambda^i,X;\hat{\theta},\delta)\ \middle|\ \lambda^i = 0\right]\right\} \tag{25}$$

where $\hat{\theta}$ and δ are defined in (14), which depend on the equilibrium participation ratio ω^a and ω^b in Proposition 3 and thus are are functions of X (and θ_t), and $\mathrm{E}[\cdot]$ denotes expectation over X.[19] The ex-ante utility of any trader before any information regarding idiosyncratic shocks can then be defined as a weighted average of J^L and J^{NL}:

$$J^i(\theta_t^i;\theta_t) = \lambda\, J^L(\theta_t^i;\theta_t) + (1-\lambda)\, J^{NL}(\theta_t^i;\theta_t), \quad i = a, b. \tag{26}$$

Finally, for market makers, the ex-ante utility is simply

$$J^m(\theta_t^m;\theta_t) = \mathrm{E}\left[J_P(\theta_t^m;\lambda^m,X;\hat{\theta},\delta)\ \middle|\ \lambda^m = 0,\ c^i = 0\right]. \tag{27}$$

To solve for the full equilibrium of the economy, we first take P_{t+1} as given to derive the equilibrium price P_t and stock holding θ_t, then we impose the stationarity condition (11) (i.e., $P_{t+1} = P_t$) to derive the full equilibrium. In addition, we need to confirm that in equilibrium, traders receiving no idiosyncratic shocks optimally choose to stay out of the market, i.e.,

$$\mathrm{E}\left[J_P(\theta_t^i;\lambda^i,X;\hat{\theta},\delta)\ \middle|\ \lambda^i = 0\right] \leq \mathrm{E}\left[J_{NP}(\theta_t^i;\lambda^i,X;\hat{\theta},\delta)\ \middle|\ \lambda^i = 0\right]. \tag{28}$$

The following proposition describes the condition that defines the equilibrium.

Proposition 4. *A stationary equilibrium of the economy is determined by the set of prices and stock holdings* $\{P_t, \theta_t\}$ *that solves the agents' optimality condition at* t

$$0 = \frac{\partial}{\partial \theta_t^i}\, J^i(\theta_t^i;\theta_t), \quad i = a, b, m, \tag{29}$$

the market clearing condition (9), the stationarity condition (11), and satisfies conditions (23) and (28).

Equation (29) is agents' first order condition for optimal portfolio choice at t before they

[19] For J^{NL}, the expectation is taken over X before the maximization because a trader with $\lambda^i = 0$ does not observe X and makes his participation decisions independent of the realization of X.

receive any idiosyncratic shocks.

We can solve the equilibrium explicitly when the probability of idiosyncratic shock λ is small as shown in the appendix, which leads to the following theorem:

Theorem 1. *When the probability of idiosyncratic shock λ is small, there exists a stationary equilibrium as described by Proposition 4.*

For arbitrary λ, we have to solve the equilibrium numerically.

4 Limited Participation and the Need for Liquidity

The equilibrium under costly participation shows two striking features. First, despite the fact that the two groups of traders have perfectly offsetting trading needs, their actual trades are not synchronized when participation in the market is costly. The non-synchronization in their trades gives rise to the need for liquidity in the market. A group of traders may bring their orders to the market while traders with offsetting trading needs are absent, creating an imbalance of orders. The stock price adjusts in response to the order imbalance to induce market makers to provide liquidity and to accommodate the orders. As a result, the price of the stock not only depends on the fundamentals (i.e., its expected future payoffs and total risks), but also depends on idiosyncratic shocks that market participants face. Second, despite the symmetry between shocks to potential buyers and sellers, the order imbalance observed in the market tends to be asymmetric and is on average dominated by sell orders. Thus, the endogenous liquidity need typically takes the form of excessive selling, which causes the price to tank. In the next two sections, we examine in more detail these results and the economic intuition behind them.

4.1 Gains from Trading and Individual Participation Decisions

We start with the individual participation decision, taking as given the initial holding, the idiosyncratic shocks, and the equilibrium participation in the market. A trader bases his participation decision on the trade-off between the cost to be in the market and the gain from trading. Our key result regarding the trading gain is stated below.

Result 1. *The gain from trading is in general different between buyers and sellers even when their trading needs are perfectly matched.*

We start with the simple situation in which the market participation rate is symmetric, i.e., $\omega^a = \omega^b$ and $\delta = 0$. The gain from participating in the market for trader i is $g_1(\theta^i; \lambda^i, X; \hat{\theta}, 0) + g_2(\lambda^i; 0) - Rc^i$. Since $g_2(\cdot)$ and c^i are identical for both trader groups, we only need to focus on $g_1(\cdot) = g_1(\theta^i; \lambda^i, X; \hat{\theta}, 0)$, which is fully determined by the distance between current stock holding θ^i and the desired holding $\hat{\theta}^i$. In particular, in this case we have $\hat{\theta}^i = (1-k)\hat{\theta} - \lambda^i X$ and

$$g_1(\theta^i; \lambda^i, X; \hat{\theta}, 0) = \frac{\alpha\sigma_D^2}{2(1-k)} \left[\theta^i - (1-k)\hat{\theta} + \lambda^i X\right]^2, \quad i = a, b. \tag{30}$$

The trading gain is symmetric between the two groups of traders, who have opposite $\lambda^i X$, only when $\theta^i = (1-k)\hat{\theta}$. When λ is small, the equilibrium θ^i is close to $\bar{\theta}$, and hence $\theta^i > (1-k)\hat{\theta}$. Thus, the trading gain is different for the two trader groups.

It is important to recognize that Result 1 is a general phenomenon when trading is costly. When the traders can trade without cost, they will constantly maintain the optimal position, and the gains from trading are always symmetric for small deviations from the optimal position. Let $u(\theta)$ denote the utility from holding θ, and θ^* be the optimal holding. Then, $u'(\theta^*) = 0$. For a small deviation $x = \theta - \theta^*$ from the optimum, the gain from trading is given by $u(\theta^*) - u(\theta^* + x) \simeq -u''(\theta^*)\, x^2/2$, which is the same for an opposite deviation $-x$. Thus, at the margin, traders with offsetting shocks or trading needs have the same gain from trading. This is no longer the case when trading is costly. Facing a cost, traders no longer trade constantly. They only trade when the deviation from the optimal is sufficiently large. As Figure 2 illustrates, the trading gain is no longer symmetric for finite deviations from the optimum since in general $u(\theta^*) - u(\theta^* + x) \neq u(\theta^*) - u(\theta^* - x)$. Hence, the gains from trading become different between traders with perfectly offsetting trading needs.

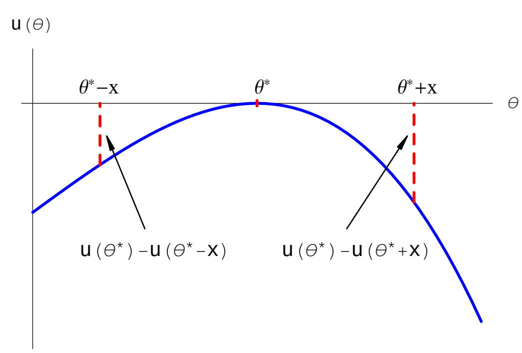

Figure 2: Utility gain from trading is asymmetric when trading is costly.

In our specific setting, we can go beyond simply confirming the above general result. In particular, we can make directional predictions regarding the relative size of trading gains for buyers and sellers.

Result 2. *When the probability of idiosyncratic shock λ is small, in equilibrium sellers always enjoy larger gains from trading than buyers.*

To understand this result, we continue our previous example with $\delta = 0$. From (30), it is apparent that $g_1(\cdot)$ is higher for potential sellers (who have $(\lambda^i X > 0)$ than buyers (who have $\lambda^i X < 0)$. Figure 3 illustrates the changes in the desired stock holding before and after traders observe whether or not they receive an idiosyncratic shock (λ^i) and its expected magnitude (X). The solid lines represent the desired stock holdings. A trader i starts with an initial holding θ_t^i $(i = a, b)$ before receiving any information on his idiosyncratic risk. Then, he learns whether or not he is exposed to the idiosyncratic risk, i.e., his draw of λ^i. If he is not exposed, i.e., $\lambda^i = 0$, he stays out of the market and keeps his initial holding.[20] If instead he is exposed, i.e., $\lambda^i \neq 0$, even without learning the actual sign or the magnitude of the shock X, the trader's preferred stock holding changes to $(1-k)\,\hat{\theta}$, given by the dashed line in Figure 3, which is lower than θ^i, his initial holding. It is worth pointing out that this new preferred holding level is independent of the sign of his idiosyncratic shock, that is, whether $\lambda^i = 1$ or -1. Moreover, even if the expected idiosyncratic shock is zero, $X = 0$, the desired holding changes to $(1-k)\,\hat{\theta}$ and the gain from trading is not zero, as (30) shows.

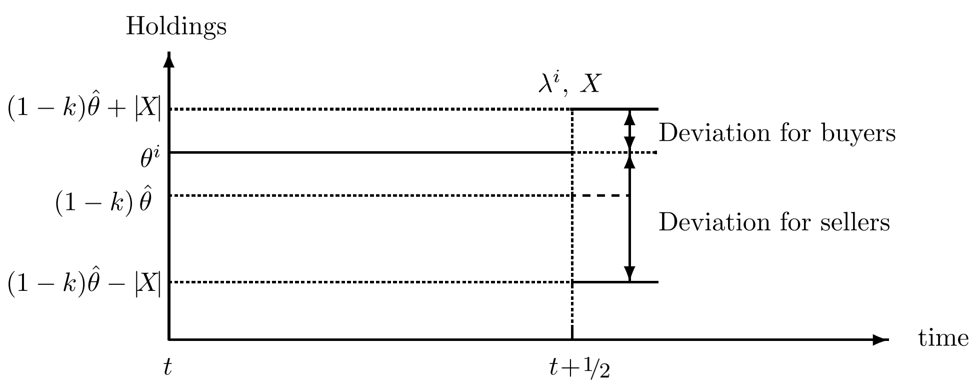

Figure 3: Traders' desired stock holdings before and after observing liquidity signals.

The desired holding for the trader has a straightforward interpretation: He chooses to hold $(1-k)$ times the per-capita stock supply in the market. Since $k > 0$, the trader prefers an overall stock exposure that is lower than the per-capita supply even though he has the same utility function as the rest of agents in the market. The reason is the following. The

[20] In this case, his preferred stock holding will actually increase. Without observing X, as stated in Theorem 1, the expected gain from participation remains negative and he will not enter the market.

cost of participation prevents the trader from trading in the market at all times. As a result, the trader has to bear some idiosyncratic risk, at least sometimes. This extra risk effectively reduces the risk tolerance of the trader and lowers his desired stock exposure relative to market makers, who face no cost and can always trade.[21] The percentage reduction in the trader's desired position, captured by k, is proportional to the level of the remaining uncertainty in his idiosyncratic risk exposure.

Finally, taking into account the sign and the magnitude of his non-traded risk $\lambda^i X$ further increases or decreases his desired stock holding to $\hat{\theta}^i = (1{-}k)\,\hat{\theta}{-}\lambda^i X$. In particular, a potential buyer, who receives a negative shock to his risk exposure, has $\lambda^i X < 0$ and a desired holding of $(1-k)\,\hat{\theta}{+}|X|$, while a potential seller, who receives a positive shock, has $\lambda^i X > 0$ and a desired holding of $(1-k)\,\hat{\theta}{-}|X|$. Figure 3 shows that the desired holdings (given by the top and bottom solid lines, respectively) deviate further from the initial position (given by the dotted line) for a potential seller than for a potential buyer. Obviously, the gain from trading is higher for the seller.

The main intuition behind the above result is as follows. Since traders choose their initial holdings before they learn whether or not they will receive idiosyncratic shocks, they rationally choose a high initial holding if they expect a low probability of ever receiving a shock. However, once they are hit with shocks, their initial holding level becomes too high given the possibility of bearing some un-hedged risk. Irrespective of the sign of his idiosyncratic shock, he prefers to decrease his stock exposure. Obviously, potential sellers who have received additional positive exposure are further away from the desired holding level than are the potential buyers. As a result, sellers enjoy larger gains from trading.[22]

Intuitively, one expects that the asymmetry in gains from trading can lead to asymmetric participation between the traders. In particular, since potential sellers always have higher gains from trading than potential buyers in our setting, we further expect that sellers are more likely to participate in the market than buyers.[23] We verify this intuition in the next

[21]The result that traders become effectively more risk-averse with un-hedged idiosyncratic risks is clearly preference dependent. Kimball (1993) shows that it is true for "standard risk aversion," which is defined as a class of utility function that exhibits both DARA and decreasing absolute prudence.

[22]In a setting similar to ours, Lo, Mamaysky, and Wang (2004) show that even in continuous time the gain from trading is asymmetric around the optimal holding due to the fact that traders only trade infrequently.

[23]It is worth noting that our results are for $\bar{\theta} > 0$, i.e., when the market as a whole has a positive risk exposure. If $\bar{\theta} < 0$, i.e., when the market as a whole has a short position in some aggregate risk, the results would be reversed. That is, the liquidity need is in the form of excess buying pressure, causing prices to surge. Since this paper is about the behavior of aggregate markets such as stocks and bonds, it is natural to assume $\bar{\theta} > 0$. In markets for security borrowing such as repo or shorting (see, for example, Duffie (1996), Duffie,

subsection.

4.2 Non-Synchronized Trading and the Need for Liquidity

Given the individual trader's entry policy, we now examine the participation equilibrium, which is stated in Proposition 3. From the proposition, it is clear that in general, $\omega^a \neq \omega^b$. That is, even with perfectly offsetting trading needs, the traders fail to synchronize their trades under costly participation. Whenever the participation is asymmetric, there is a mismatch in the buy and sell orders in the market. This order imbalance then creates the need for liquidity. Thus, we have the following result.

Result 3. *In equilibrium, participation can be asymmetric among traders even when their trading needs are perfectly matched, giving rise to non-synchronization in their trades and the endogenous need for liquidity.*

Figure 4 shows the equilibrium participation decisions as functions of the idiosyncratic shock X. Panel (a) reports the fraction ω^i of traders within group i who choose to participate. The dotted line plots ω^a and the dashed line plots ω^b. Panel (b) reports the difference in participation ratio between the two groups of traders δ, defined in equation (14), as a function of X. When $X > 0$, group-a traders are potential sellers and group-b traders are potential buyers. Consistent with our earlier intuition, more sellers are participating than buyers as ω^a is always above ω^b in this region. In particular, when X is not too far from zero, $\omega^a > 0$ and $\omega^b = 0$, that is, no group-b traders choose to participate because the benefit from trading is too small, and only a fraction of group-a traders participates. This corresponds to Case B in Proposition 3. As X increases, the gains from trading increases for both groups and both ω^a and ω^b increases. In particular, for medium levels of X, ω^b becomes positive and ω^a reaches 1. When X is sufficiently large, $\omega^a = \omega^b = 1$. That is, the gain from trading dominates the cost for both groups of traders and they all choose to participate. This corresponds to Case A in Proposition 3. When $X < 0$, group-a traders become potential buyers and group-b traders become potential sellers. All the above results flip. In fact, ω^b is simply the mirror image of ω^a around the vertical axis, reflecting the fact that traders a and b face opposite idiosyncratic shocks. Interestingly, neither ω^a nor ω^b is symmetric around 0, consistent with the fact that a trader's gain from trading is asymmetric between positive and negative idiosyncratic shocks.

Gârleanu, and Pedersen (2002), and Vayanos and Weill (2007),) the opposite situation can arise, at least for a segment of the market. This is beyond the scope of the current paper and we leave it for future research.

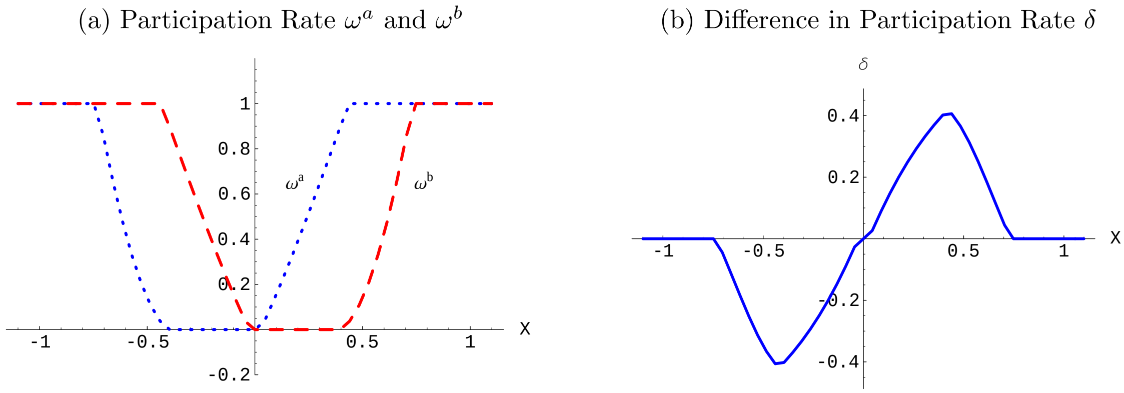

Figure 4: Equilibrium Participation. The figure plots the equilibrium participation rate for the two trader groups for different values of idiosyncratic shock X. Panel (a) reports the equilibrium fraction of group i traders who choose to participate, where the dotted and the dashed lines refer to group a and b traders respectively. Panel (b) reports the difference in participation decisions, $\delta = \lambda\nu(\omega^a - \omega^b)/[\mu + \lambda\nu(\omega^a + \omega^b)]$. Other parameters are set at the following values: $\bar{\theta} = 1$, $\alpha = 4$, $r = 0.05$, $\bar{D} = 0.36$, $c = 0.09$, $\sigma_D = 0.3$, $\sigma_z = 0.7$, $\sigma_u = 0.7$, $\mu = 1$, $\nu = 5$, and $\lambda = 0.15$.

A trader's decision to participate in the market generates both positive and negative externalities. On one hand, the participation of one trader group has positive externality on the other group as they offset each other's idiosyncratic risks. On the other hand, both groups of traders are competing for market makers' risk sharing capacity. Their participation generates a negative externality on each other, especially for traders within the same group. In general, these externalities are not efficiently internalized through the market mechanism, and the participation equilibrium can be suboptimal. The asymmetric participation is also a reflection of such an inefficiency given that traders' trading needs are perfectly matched.[24]

The externalities from traders' participation decisions are also the reason that the interior solution described in Case C of Proposition 3 is rare—we do not observe it in Figure 4. More participation of potential sellers enhances the participation gains of potential buyers. As a result, more buyers will be lured into the market, which in turn encourages more sellers to participate. Hence, starting with an interior participation level, increasing participation for both groups generally improves utility for both groups of traders. An equilibrium is reached when all sellers and enough buyers are in the market so that the gain from further entry by buyers diminishes to zero. The interior solution in Case C is only possible when traders start with large enough initial stock holdings and the competition for market makers' risk sharing capacity overwhelms their gains from offsetting each other's idiosyncratic risks.

Panel (b) of Figure 4 shows that the normalized difference between ω^a and ω^b is always

[24] A general discussion on the welfare impact of participation costs is beyond the scope of this paper and left for future work. See Huang and Wang (2006).

positive when $X > 0$, indicating that more group-a traders are participating. Since they are potential sellers when $X > 0$, the aggregate order imbalance is skewed towards sell orders. Similarly, when $X < 0$, δ is always negative, indicating more group-b traders are participating. Since group-b traders are potential sellers when $X < 0$, the order imbalance is again skewed towards sell orders.[25] In summary,

Result 4. *When the probability of idiosyncratic shock λ is small, potential sellers always participate more in the market than potential buyers in equilibrium. The aggregate order imbalance always takes the form of an excess supply.*

5 Liquidity and Stock Price

Our analysis above suggests that self-interest fails to coordinate agents' participating decisions and synchronize their trades even when their trading needs perfectly match. This non-synchronization in trades gives rise to imbalances in asset demand and the need for liquidity. Such exogenous order imbalances are the starting point for Grossman and Miller (1988) and market microstructure models such as those in Ho and Stoll (1981) and Glosten and Milgrom (1985). In our model, by explicitly modelling the motives and the costs of being in the market, we endogenously derive the order imbalance. In particular, we show that, in the absence of any aggregate shocks, the order imbalance often takes the form of an excess supply. This directional prediction leads to interesting implications for equilibrium prices, to which we now turn our attention.

5.1 Liquidity and Crashes

By construction, the equilibrium stock price is stationary over time at the beginning of each generation, $P_{t+1} = P_t = P$. And it fluctuates in-between each generation as a function of the idiosyncratic shocks. As (15) indicates, the intermediate price consists of two components:

[25] In the model we solve, we have assumed that traders receiving no idiosyncratic shocks $\lambda^i = 0$ make their participation decisions without observing X. The analysis above shows that the order imbalance occurs for relatively large values of X (but not extremely large). Moreover, this situation occurs when some potential buyers choose to stay out of the market as the participation cost dominates gains from trading. From Proposition 2, we also see that for large X, gains from trading is higher for traders with idiosyncratic shocks ($\lambda^i \neq 0$) than for those without ($\lambda^i = 0$). From this, we conclude that even if traders with $\lambda^i = 0$ also observe X at the time of their participation decision, they won't participate for those states with large order imbalances.

the risk-adjusted fundamental value, $R^{-1}\big(\bar{D}+P-\alpha\sigma_{D}^{2}\,\hat{\theta}\big)$, and the liquidity component,

$$\tilde{p} \equiv -(\alpha\sigma_{D}^{2}/R)\,\delta Z. \tag{31}$$

We call the first term "the fundamental value" since it determines the stock price when the expected future idiosyncratic risk (signal X) is zero. It is simply equal to the expected future payoffs (dividend plus resale price) minus a risk premium. The liquidity component $\tilde{p}$, on the other hand, captures the premium related to market illiquidity. It is non-zero only when agents anticipate future idiosyncratic shocks since the symmetry between trader groups implies that $\delta = 0$ when $X = 0$. Moreover, it is proportional to the per-capita order imbalance, driven by the asymmetric participation between buyers and sellers. Since our purpose here is to understand the endogenous nature of order imbalances and its impact on asset prices, we will focus our discussion on the liquidity component.

To understand the properties of the liquidity component $\tilde{p}$, we first solve the stationary equilibrium described in Theorem 1. Then we derive the equilibrium prices at time $t+1/2$ by substituting in the stationary equilibrium price and optimal θ_t^i and θ_t^m. The $\tilde{p}$ in (31) depends on the difference in market participation rate δ, which is determined after observing the signal X, and the realized idiosyncratic shock Z, which is equal to the signal X plus some noise, $Z-X \sim N(0,\sigma_{\hat{z}}^2)$. The noise term makes the liquidity component more dispersed without qualitatively changing the liquidity effect. In Figure 5, we average out the noise term and report the expected liquidity component conditional on the signal X,

$$\hat{p} = \mathrm{E}[\,\tilde{p}\,|\,X\,] = -(\alpha\sigma_{D}^{2}/R)\,\delta X. \tag{32}$$

We plot $\hat{p}$ as a function of X in panel (a), and plot its probability distribution in panel (b).

When traders face no participation costs, they all participate in the market and $\delta = 0$. There is no need for liquidity. By (31), the liquidity component $\tilde{p}=0$. The stock price equals the fundamental value and does not depend on the idiosyncratic shocks individual traders face. In the presence of participation costs, partial participation leads to non-synchronized trades among traders, and $\delta \neq 0$. There is a need for liquidity. The stock price has to adjust to attract the market makers to provide the liquidity and to accommodate the trades. In general, $\hat{p} \neq 0$ and the stock price becomes dependent on the idiosyncratic shocks of individual traders.

As Result 4 states, potential sellers are more willing to enter the market to sell the stock. Thus, the order imbalance, as captured by $-\delta X$, is negative, which leads to a negative $\hat{p}$. It is important to note that the sign of $\hat{p}$ is independent of the sign of X. In other words, it does

(a) Conditional Liquidity Component (b) Probability Distribution of $\hat{p}$

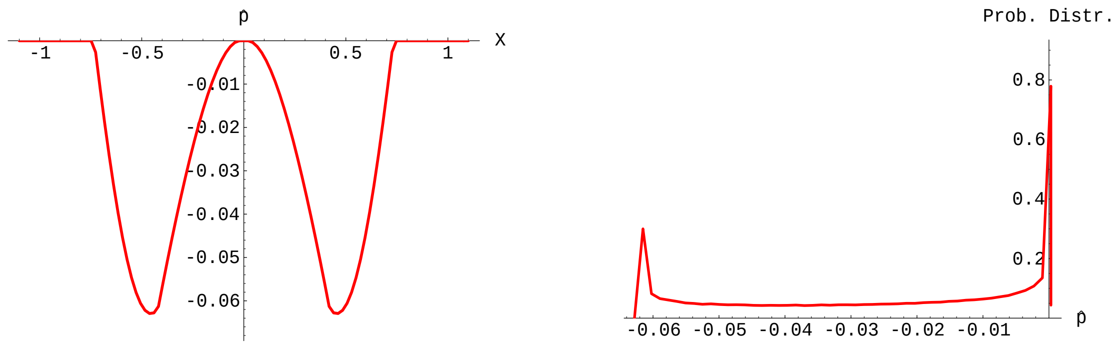

Figure 5: The conditional liquidity component in price $\hat{p}$. We define $\hat{p}$ in (32) to capture the expected price movement in excess of the "fundamentals." Panel (a) plots $\hat{p}$ as a function of the signal X. Panel (b) plots the probability density function of $\hat{p}$, except at the point 0 where the value corresponds to the total probability mass at the point (since the density function should be infinity at the point). Other parameters are set at the following values: $\bar{\theta} = 1$, $\alpha = 4$, $r = 0.05$, $\bar{D} = 0.36$, $c = 0.09$, $\sigma_D = 0.3$, $\sigma_z = 0.7$, $\sigma_u = 0.7$, $\mu = 1$, $\nu = 5$, and $\lambda = 0.15$.

not depend on the distribution of idiosyncratic shocks among the traders. Therefore, we have the following result:

Result 5. *The impact of liquidity needs always decreases asset prices.*

The magnitude of the liquidity effect on price depends on the signal X. Figure 5(a) plots $\hat{p}$ against X. First, we note that the liquidity effect on the price is always negative, as mentioned before. Second, the impact of liquidity on the stock price is highly non-linear in X, the idiosyncratic shocks to the traders. In particular, for small values of X, gains from trading are small for all traders and they do not enter the market. As a result, there is no need for liquidity and the price is equal to the fundamental value. For large values of X, gains from trading are sufficiently large for all traders and they all enter the market. As a result, their traders are synchronized and there is no need for liquidity. The stock price also equals the fundamental. For intermediate ranges of X, the gains from trading are large enough for some traders to enter the market, but not for all traders to do so. It is in this case that trades are non-synchronized and liquidity is needed in the market, which will in turn affect the stock price. As Figure 5 shows, the price impact of liquidity reaches the maximum for a certain magnitude of the idiosyncratic shock.

The result that the price impact of liquidity need is one-sided and highly non-linear arises from the fact that liquidity needs are endogenous in our model. In most of the existing models of liquidity, such as that in Grossman and Miller (1988), liquidity needs are exogenously specified; consequently, its price impact is linear in the exogenous liquidity needs and symmetrically distributed. Our analysis shows that modelling the liquidity needs endogenously is important for understanding the impact of liquidity on prices. After all, it is the same economic factor, namely, the cost to participate in the market, that drives both the liquidity needs of the traders and the liquidity provision of market makers.

The non-linearity in the price impact of liquidity leads to another interesting result: large and frequent price movements in the absence of any aggregate shocks. Figure 5(b) plots the probability distribution of $\hat{p}$. As a benchmark, when participation is costless, the idiosyncratic shock X does not affect stock prices and there is no liquidity effect. The distribution is simply a delta-function at zero. When traders face costs to participate in the market, however, the stock price also depends on the idiosyncratic shock X. Moreover, even though the underlying idiosyncratic shocks that drive the individual traders' trading needs are normally distributed, their price impact as measured by $\hat{p}$ is always negative and has a fat-tailed distribution. Aside from a non-zero probability mass at the origin, the distribution peaks at a finite and negative value, reflecting the fact that liquidity becomes important and affects the price for a range of finite shocks. Moreover, the impact of liquidity gives rise to the possibility of a large price movement in the absence of any shocks to the fundamentals of the stock. Since such a price movement is associated with a large imbalance in trades and a surge of liquidity needs, it is a market crash driven purely by liquidity needs. We call it a "liquidity crash." Summarizing the results above, we have the following:

Result 6. *The impact of liquidity can lead to "liquidity crashes" in which large price drops occur in the absence of any shocks to the fundamentals.*

5.2 Price Distribution

The above discussion focuses on $\hat{p}$, which gives the expected impact of liquidity need on the stock price conditional on X, the signal on future idiosyncratic shocks. The actual price at $t + 1/2$, as given in (31), will depend on Z, the actual realization of idiosyncratic shock. Although the behavior of $\tilde{p}$ is qualitatively similar to that of $\hat{p}$, its distribution is slightly different from that of $\hat{p}$ due to the additional noise in the realized idiosyncratic shock. Figure 6 plots the unconditional distribution of $\tilde{p}$.

From the figure, we observe that the distribution exhibits negative skewness and fat tails. To confirm this observation, we report in Table 1 the moments of the liquidity component $\tilde{p}$ (for the same parameters,) along with the values of $\tilde{p}$ at the 1st and 5th percentiles. We can interpret these values as the "Value-at-Risk" (VaR) for a portfolio invested only in the stock.

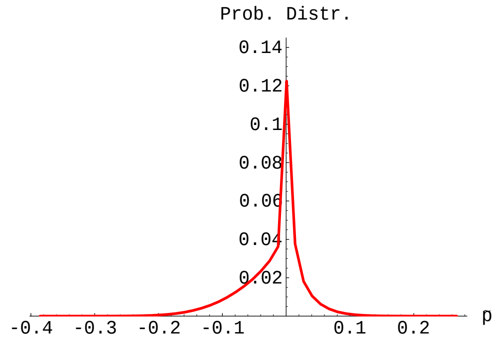

Figure 6: The unconditional liquidity component in price $\tilde{p}$. We define $\tilde{p}$ in (31) to capture the price movement in excess of the "fundamentals," and plot the probability density function of $\tilde{p}$. Other parameters are set at the following values: $\bar{\theta} = 1$, $\alpha = 4$, $r = 0.05$, $\bar{D} = .36$, $c = 0.09$, $\sigma_D = 0.3$, $\sigma_z = 0.7$, $\sigma_u = 0.7$, $\mu = 1$, $\nu = 5$, and $\lambda = 0.15$.

Table 1: **Characteristics of the liquidity component $\tilde{p}$ in the stock price.**

Mean	St. dev.	Skewness	Kurtosis	VaR (5%)	VaR (1%)
-0.024	0.047	-1.033	5.078	-0.119	-0.173

Note that in the absence of liquidity effect, the return distribution will simply be a delta function at zero. If we were to include any news on the fundamentals (i.e., future dividends) that is normally distributed, the return would be normal. Hence, we have the following result.

Result 7. *The impact of liquidity can significantly increase the downside risk and lead to negative skewness and fat tails in asset prices.*

5.3 Cross-Sectional Implications

The effect of liquidity depends on two factors, investors' trading needs, which is captured by σ_z, and the cost of participation c. Both of these factors can vary substantially across markets, so does the resulting market liquidity. Since the impact of σ_z is straightforward—it increases the influence of liquidity on prices, we now consider the impact of the cost c.

Figure 7(a) plots the first three central moments—the mean, standard deviation and skewness—of price impact of liquidity $\tilde{p}$ for different values of c. When the cost is close to zero, everyone participates and there is no liquidity effect. As the cost increases, the liquidity effect becomes more pronounced, as reflected by the larger moments of $\tilde{p}$. The fact that the mean and skewness are always negative reflects the one-sided nature of endogenous order imbalances. However, when the cost is sufficiently high, the dotted line in Panel (a) indicates that the average liquidity effect goes to zero.

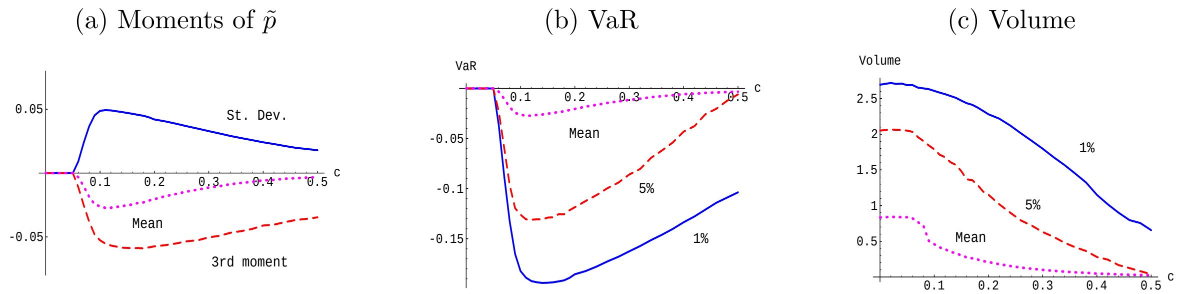

Figure 7: Participation cost and liquidity. Panel (a) plots the first three central moments of the liquidity component $\tilde{p}$, which is defined in (31) and captures the price movement in excess of the "fundamentals." Panel (b) plots the 1st- and 5th-percentile values of $\tilde{p}$, which correspond to the "Value-at-Risk" for a portfolio invested only in stocks. Panel (c) plots the average volume (the dotted line) along with the 1st- and 5th-percentile volumes. Other parameters are set at the following values: $\bar{\theta} = 1$, $\alpha = 4$, $r = 0.05$, $\bar{D} = 0.36$, $\sigma_D = 0.3$, $\sigma_z = 0.7$, $\sigma_u = 0.7$, $\mu = 1$, $\nu = 5$, and $\lambda = 0.15$.

However, it would be misleading to declare the high-cost market liquid based only on the average liquidity effect. In fact, such a market is very illiquid in the sense that most of the time they are too costly for any traders to participate. Liquidity events, although rare, may still occur and when they do, the magnitudes are large. In Figure 7(b) and (c), we show how $\tilde{p}$ and the trading volume, which is defined as

$$V \equiv \mu\,|\delta Z| \,+\, \lambda\,\nu \sum_{i=a,b} \omega^i |\delta Z - \lambda^i Z|, \tag{33}$$

change with c. The dotted lines in Panels (b) and (c) plot the average price impact of liquidity and volume for different values of c. Clearly, as c increases, trading volume decreases monotonically. When c reaches 0.4, the average price impact of liquidity becomes very small. However, such a market is far from being liquid. In fact, the volume is negligible, despite large trading needs from idiosyncratic shocks (σ_z stays at 0.7). Moreover, large liquidity events can still occur. The solid lines in Panels (b) and (c) indicate that at c greater than 0.4, with 1% chance, liquidity events can move the price drastically, accompanied by heavy trading volume.

Our analysis shows that in markets with relatively small participation costs, expected price impact provides a good measure of market liquidity. In markets with larger participation costs, however, expected price impact alone may provide a misleading signal of market liquidity. Therefore, we have the following result.

Result 8. *Higher-order moments of the price, Value-at-Risk measures, and trading volumes provide valuable information for market liquidity.*

6 Conclusion

In this paper, we show that frictions such as participation costs can induce non-synchronization in agents' trades even when their trading needs are perfectly matched. Each trader, when arriving at the market, faces only a partial demand/supply of the asset. The mismatch in the timing and the size of trades creates temporary order imbalances and the need for liquidity, causing asset prices to deviate from the fundamentals. Purely idiosyncratic shocks can affect prices, introducing additional price volatility. Moreover, the price deviations tend to be highly skewed and of large sizes. In particular, the shortage of liquidity always causes the price to decrease and when this happens, the price tends to drop significantly, resembling a crash due to a sudden surge in liquidity needs.

A few additional comments are in order. First, our analysis takes as given the population weight of market makers, which determines the amount of liquidity they can provide and thus the equilibrium impact of liquidity needs. As shown in Huang and Wang (2006), the population weight of market makers can be endogenized. In particular, they assume that all agents can either pay a low cost ex-ante to become a market maker or a high cost ex-post when trading needs arise. They show that typically only a small fraction of agents will choose to become market makes. In light of their analysis, we can interpret the relative population weight of market makers and traders as an equilibrium outcome. Second, in our model the idiosyncratic shocks are transitory. Thus, when a liquidity crash occurs, the stock price tanks but eventually recovers. The possibility of such a price pattern might seem puzzling since it seems to leave profitable opportunities. However, this is not so given the costs. With a small probability for such an event to happen, it is profitable for only a small number of market makers to enter the market ex-ante even if the cost for becoming a market maker is rather small. For others, the significant cost to jump in on the spot prevents them from taking advantage of the opportunities. Finally, in our setting, the cost to jump into the market on the spot does impose an upper bound on the potential impact of liquidity on prices. But, this is true only in the absence of aggregate shocks as we assumed in the model. In the presence of aggregate shocks, the potential impact of endogenous liquidity needs on prices becomes unbounded.

A Appendix

Proof of Proposition 1

Given P_{t+1}, participating agent i maximizes his expected utility over his terminal wealth W^i_{t+1} given in (8), which is obtained by integrating over the distribution of D_{t+1} given $\theta^i_{t+1/2}$:

$$\max_{\theta^i_{t+1/2}} -e^{-\alpha\left[R^2W_t - R\,c^i + R\,\theta^i_t\left(P_{t+1/2} - RP_t\right) + \theta^i_{t+1/2}\left(\bar{D} + P_{t+1} - RP_{t+1/2}\right) - \frac{1}{2}\alpha\sigma^2_D(\theta^i_{t+1/2} + \lambda^i Z)^2\right]}. \tag{A1}$$

His optimal holding is calculated by solving the first order condition with respect to $\theta^i_{t+1/2}$,

$$\theta^i_{t+1/2} = \frac{1}{\alpha\sigma^2_D}(\bar{D} + P_{t+1} - RP_{t+1/2}) - \lambda^i Z, \qquad i = a, b, m. \tag{A2}$$

Solving the market clearing condition (10) yields the equilibrium price $P_{t+1/2}$. Using δ and $\hat{\theta}$ defined in (14) yields the expression of $P_{t+1/2}$ in the proposition. The optimal holding in the proposition is obtained by substituting the equilibrium price $P_{t+1/2}$ back into (A2).

Proof of Proposition 2

First, we substitute the equilibrium price $P_{t+1/2}$ and holding $\theta^i_{t+1/2}$ back into (A1), and integrate over the distribution of Z conditional on X to derive the (indirect) utility function J_P in (17a) for the participating traders,

$$J_P(\cdot) = -\frac{1}{\sqrt{1-k+k\,(1-\lambda^i\delta)^2}}\, e^{-\alpha\left[R^2W_t - R\,c^i + \theta^i_t\left(\bar{D} + P_{t+1} - R^2P_t\right) - \frac{\alpha\sigma^2_D}{2(1-k)}(\theta^i_t + \lambda^i X)^2 + g_1(\cdot)\right]} \tag{A3}$$

where $g_1(\cdot)$ and k are defined in (20) and (21). Next, we calculate the value function for non-participating traders J_{NP} in (17b) by integrating over D_{t+1} and Z conditional on X,

$$J_{NP}(\cdot) = -\frac{1}{\sqrt{1-k}}\, e^{-\alpha\left[R^2W_t + \theta^i_t\left(\bar{D} + P_{t+1} - R^2P_t\right) - \frac{\alpha\sigma^2_D}{2(1-k)}(\theta^i_t + \lambda^i X)^2\right]}. \tag{A4}$$

Finally, we substitute J_P and J_{NP} into (18) to derive the gains from participation. Obviously, trader i chooses to participate in the market if and only if $g(\cdot) > 0$.

Proof of Proposition 3

We first introduce two lemmas.

Lemma 1. *When traders' initial stock holdings satisfy (23), the gain from participation* $g^a(\omega^a, \omega^b)$ *for group-a traders decreases with* ω^a *and increases with* ω^b*, while the opposite is true for group-b traders' gain* $g^b(\omega^a, \omega^b)$*.*

Here is the proof of Lemma 1. Given the definition of $g(\cdot)$ in (19), we compute its partial derivative with respect to ω^a and ω^b. Define $\delta^i \equiv \lambda^i \delta$ and $d^i \equiv 1-k+k(1-\delta^i)^2$. Following (22), let $g^i \equiv g(\theta^i_t; \lambda^i, X; \hat{\theta}, \delta)$. Then,

$$\frac{\partial g^i}{\partial \omega^j} = \left(\frac{\partial g_1}{\partial \delta^i} + \frac{\partial g_2}{\partial \delta^i}\right)\frac{\partial \delta^i}{\partial \omega^j} + \frac{\partial g_1}{\partial \hat{\theta}}\frac{\partial \hat{\theta}}{\partial \omega^j}, \quad j = a, b$$

where

$$\begin{aligned}
\frac{\partial g_1}{\partial \delta^i} &= -\frac{\alpha \sigma_D^2 (1-k\,\delta^i)}{(d^i)^2}\left[k\,\delta^i \theta^i_t + k\,(1-\delta^i)\hat{\theta} + \lambda^i X\right]\left(\theta^i_t - \frac{1-k}{1-k\,\delta^i}\,\hat{\theta} + \frac{1-\delta^i}{1-k\,\delta^i}\,\lambda^i X\right) \\
\frac{\partial g_2}{\partial \delta^i} &= -\frac{(1-\delta^i)\,k}{\alpha d^i} \\
\frac{\partial \delta^i}{\partial \omega^j} &= \lambda^i\,(\lambda^j - \delta)\hat{\lambda}, \qquad \hat{\lambda} \equiv \frac{\lambda \nu}{\mu + \lambda \nu(\omega^a + \omega^b)} \\
\frac{\partial g_1}{\partial \hat{\theta}} &= -\frac{\alpha \sigma_D^2 (1-k\,\delta^i)}{d^i}\left(\theta^i_t - \frac{1-k}{1-k\,\delta^i}\,\hat{\theta} + \frac{1-\delta^i}{1-k\,\delta^i}\,\lambda^i X\right) \\
\frac{\partial \hat{\theta}}{\partial \omega^j} &= \hat{\lambda}\,(\theta^i_t - \hat{\theta}).
\end{aligned}$$

We can further simplify the expression for $\partial g^i/\partial \omega^j$ by considering the cases of $j = i$ and $j \neq i$ separately. In particular, if $j = i$, then $\lambda^i \lambda^j = 1$ and

$$\frac{\partial g^i}{\partial \omega^i} = -\frac{\alpha \sigma_D^2 \hat{\lambda}\,(1-\delta^i)^2}{(d^i)^2}\left[\lambda^i X + k\,\hat{\theta} + \frac{1-k\,\delta^i}{1-\delta^i}(\theta^i_t - \hat{\theta})\right]^2 - \frac{k\hat{\lambda}\,(1-\delta^i)^2}{\alpha d^i}.$$

From (14), δ increases in ω^a and decreases in ω^b. Hence, $\delta \in [-\bar{\delta}, \bar{\delta}]$, where $\bar{\delta} = \frac{\lambda\nu}{\mu+\lambda\nu} < 1$, and so is δ^i. As a result, $\partial g^i/\partial \omega^i < 0$. If $j \neq i$, then $\lambda^i \lambda^j = -1$ and

$$\begin{aligned}
\frac{\partial g^i}{\partial \omega^j} &= \frac{\alpha \sigma_D^2 \hat{\lambda}\,(1-\delta^2)}{(d^i)^2}\left[\lambda^i X + k\,\hat{\theta} + \frac{(1+k)\,\delta^i - 2k\,\delta^2}{1-\delta^2}\,(\theta^i_t - \hat{\theta})\right]^2 \\
&\quad + \frac{\alpha \sigma_D^2 \hat{\lambda}}{1-\delta^2}\left[\frac{k\,(1-\delta^2)^2}{\alpha^2 \sigma_D^2 d^i} - (\theta^i_t - \hat{\theta})^2\right].
\end{aligned}$$

Since $\delta^2 \in [0, \bar{\delta}^2]$, $(1-\delta^i)^2 \in [0, 1+\bar{\delta}^2]$, and $k = \alpha^2 \sigma_D^2 \sigma_{\hat{z}}^2 \in [0, 1]$, we have

$$\frac{k\,(1-\delta^2)^2}{\alpha^2 \sigma_D^2 d^i} \geq \frac{\sigma_{\hat{z}}^2\,(1-\bar{\delta}^2)^2}{1-k+k\,(1+\bar{\delta})^2} > \sigma_{\hat{z}}^2\,(1-\bar{\delta})^2.$$

On the other hand, $\hat{\theta}$ in (14) is a weighted average of θ^i_t and θ^m_t with weights in-between 0 and 1. We have

$$(\theta^i_t - \hat{\theta})^2 \leq (\theta^i_t - \theta^m_t)^2 < \left(\frac{\mu\,\sigma_{\hat{z}}}{\mu + \lambda\nu}\right)^2 = \sigma_{\hat{z}}^2 (1-\bar{\delta})^2$$

where the second inequality is due to condition (23). Thus, $\partial g^i/\partial\omega^j > 0$ for $j \neq i$, proving the lemma.

Lemma 2. *When traders' initial stock holdings satisfy (23), under symmetric participation, sellers always enjoy larger gains from trading than buyers, i.e.,* $g^a(\omega,\omega) \geq g^b(\omega,\omega)\ \forall\, \omega \in [0,1]$.

Here is the proof of Lemma 2. When $\omega^a = \omega^b$, $\delta = 0$ and $g(\cdot)$ in (19) reduces to

$$g(\theta^i_t; \lambda^i, X; \hat{\theta}, 0) = \frac{\alpha\sigma^2_D}{2(1-k)} \left[\theta^i_t - (1-k)\,\hat{\theta} + \lambda^i X\right]^2 - \frac{1}{2\alpha}\ln(1-k) - Rc^i.$$

Hence,

$$g^a(\omega^a,\omega^b) - g^b(\omega^a,\omega^b) = \frac{\alpha\sigma^2_D}{1-k} \left[\theta^i_t - (1-k)\,\hat{\theta}\right](\lambda^a - \lambda^b)X.$$

If group-a traders are sellers and group-b are buyers, then $\lambda^a X \geq 0 \geq \lambda^b X$. Since $\hat{\theta}$ is weighted average of θ^i_t and θ^m_t, $\theta^i_t - (1-k)\,\hat{\theta} \geq \theta^i_t - (1-k)\,\theta^m_t > 0$, where the last inequality comes from (23). Hence, $g^a(\omega^a,\omega^b) \geq g^b(\omega^a,\omega^b)$ when $\omega^a = \omega^b$, which is the lemma.

Now we prove Proposition 3. First, from Lemma 1, we know that $g^a(\omega^a, 0)$ is a monotonically decreasing function of ω^a. If $g^a(0,0) > 0 > g^a(1,0)$, then there exists an $s^a \in (0,1)$ that solves $g^a(s^a,0) = 0$. Similarly, $g^b(1,\omega^b)$ is monotonically decreasing in ω^b and $g^b(1,0) > 0 > g^b(1,1)$ guarantees that the solution $s^b \in (0,1)$. Hence, $\hat{s}^a,\ \hat{s}^b \in [0,1]$.

We now consider the three possible cases, which are exhaustive. In case A, there are three subcases depending on the value of $\hat{s}^b$: First, if $\hat{s}^b = 0$, then $g^b(1,0) \leq 0 \leq g^a(1,0)$. The market condition $\omega^a = 1$ and $\omega^b = 0$ is the most favorable for buyers and the least favorable for sellers. Yet the gain from participation is still positive for potential sellers and negative for potential buyers. Hence, $\omega^a = 1$ and $\omega^b = 0$ is the solution. Second, if $\hat{s}^b = 1$, then $g^b(1,1) \geq 0$ and Lemma 2 implies $g^a(1,1) \geq 0$ as well. Hence, all traders participate and $\omega^a = \omega^b = 1$. And third, if $\hat{s}^b = s^b \in (0,\ 1)$, then $g^b(1,\hat{s}^b) = 0$. The condition $g^a(1,\hat{s}^b) \geq 0$ confirms that sellers enjoy positive gains in this case. Hence, at equilibrium participation $\omega^a = 1$ and $\omega^b = \hat{s}^b$, trader a enjoys positive gain and trader b is indifferent between participating or not.

In case B, there are only two subcases depending on the value of $\hat{s}^a$. Note that $\hat{s}^a = 1$ is not feasible under the condition $g^a(1,\hat{s}^b) < 0$, since $g^a(1,0) \leq g^a(1,\hat{s}^b) < 0$ according to Lemma 1, while $\hat{s}^a = 1$ requires $g^a(1,0) \geq 0$. The first subcase is when $\hat{s}^a = 0$. Then $g^a(0,0) \leq 0$. Since $g^b(0,0) < g^a(0,0)$ by Lemma 2, $\omega^a = \omega^b = 0$ is the only solution. The second subcase is when $\hat{s}^a = s^a \in (0,1)$ and solves $g^a(\hat{s}^a,0) = 0$. At $\hat{s}^a$, trader a is indifferent between participating or not. The condition $g^b(\hat{s}^a,0) \leq 0$ confirms that trader b does not want to participate when $\omega^a = \hat{s}^a$. Hence, $\omega^a = \hat{s}^a$ and $\omega^b = 0$ in equilibrium.

In case C, the condition is that $g^a(1, \hat{s}^b) < 0 < g^b(\hat{s}^a, 0)$. Similar to case B, the condition $g^a(1, \hat{s}^b) < 0$ still rules out the possibility that $\hat{s}^a = 1$. In addition, $g^b(\hat{s}^a, 0) > 0$ rules out the possibility that $\hat{s}^b = 0$, since $0 < g^b(\hat{s}^a, 0) < g^b(1, 0)$ by Lemma 1. Similarly, we can rule out $\hat{s}^a = 0$ and $\hat{s}^b = 1$. Hence, the condition in case C reduces to $g^a(1, s^b) < 0 < g^b(s^a, 0)$. Note that $g^a(s^a, 0) = 0$ and $g^b(s^a, 0) > 0$ implies that in equilibrium, $\omega^b > 0$. To prove this, assume by contradiction that $\omega^b = 0$. Then at the optimal $\omega^a = s^a$, trader a is indifferent while trader b can gain from participating. Thus $\omega^b = 0$ cannot be the equilibrium. Similarly, $g^a(1, s^b) < 0$ and $g^b(1, s^b) = 0$ implies $\omega^a < 1$ in equilibrium. Lemma 2 guarantees that both ω^a and ω^b are interior solutions. Both traders need to be indifferent between participating or not, i.e., $g^a(\omega^a, \omega^b) = 0$ and $g^b(\omega^a, \omega^b) = 0$. The monotonicity of g^a and g^b functions ensures the existence of a solution in this case. Finally, to prove that $\omega^a \geq \omega^b$, we assume by contradiction that $\omega^a < \omega^b$. Then $0 = g^a(\omega^a, \omega^b) > g^a(\omega^a, \omega^a) > g^b(\omega^a, \omega^a) > g^b(\omega^a, \omega^b) = 0$, yielding a contradiction. Note that the first inequality is because $g^a(\omega^a, \omega^b)$ increases in ω^b, and the last is because $g^b(\omega^a, \omega^b)$ decreases in ω^b. The middle inequality is because of Lemma 2.

Proof of Proposition 4

We first calculate the value function for traders with $\lambda^i = 0$. Conditional on the signal X, the utility if they choose to participate is:

$$J_P(\lambda^i = 0) = -\frac{1}{\sqrt{1+k\,\delta^2}} e^{-\alpha\left[R^2 W_t - R\,c^i + \theta_t^i\left(\bar{D} + P_{t+1} - R^2 P_t\right) - \frac{1}{2}\alpha\sigma_D^2 {\theta_t^i}^2 + \frac{1}{2}\frac{\alpha\sigma_D^2}{1+k\,\delta^2}(\theta_t^i - \hat{\theta} - \delta X)^2\right]}. \tag{A6}$$

If they choose not to participate, the utility is

$$J_{NP}(\lambda^i = 0) = -e^{-\alpha\left[R^2 W_t + \theta_t^i\left(\bar{D} + P_{t+1} - R^2 P_t\right) - \frac{1}{2}\alpha\sigma_D^2 {\theta_t^i}^2\right]}. \tag{A7}$$

By substituting in the participation and market equilibrium from Propositions 1 and 3, and integrating over X, we can derive the unconditional value function J^L and J^{NL} in (24) and (25). Hence, the ex-ante value function $J^i(\cdot)$ in (26) is well defined for all traders. Moreover, the utility conditional on X for the market maker is the same as $J_P(\lambda^i = 0)$ except for his initial holding θ_t^m and cost $c^m = 0$:

$$J_P^m(X) = -\frac{1}{\sqrt{1 + k\,\delta^2}} e^{-\alpha\left[R^2 W_t + \theta_t^m\left(\bar{D} + P_{t+1} - R^2 P_t\right) - \frac{1}{2}\alpha\sigma_D^2 {\theta_t^m}^2 + \frac{\alpha\sigma_D^2}{2(1+k\delta^2)}(\theta_t^m - \hat{\theta} - \delta X)^2\right]}. \tag{A8}$$

Integrating over X then yields the ex-ante utility $J^m(\cdot) = \mathrm{E}[J_P^m(X)]$.

With market clearing condition (9), stationarity condition (11), and three first order conditions in (29), we have five equations in total and five unknowns, i.e., $\{\theta_t^a, \theta_t^b, \theta_t^m, P_t, P_{t+1}\}$.

A solution to the system gives a full equilibrium of the economy.

Proof of Theorem 1

When traders face no idiosyncratic shocks, i.e., $\lambda = 0$, it is easy to show that traders never participate whenever $c^i > 0$, that is, $\omega^i = 0 \ \forall \ i = a, b$. The equilibrium prices are determined by market makers as representative agents, and are identical to those derived in (13). Equilibrium holdings of the stock for all agents are always equal to the per capita supply $\bar{\theta}$.

Proposition 4 describes conditions for an equilibrium. The ex-ante symmetry between the two groups of traders implies that $J^a = J^b$ and $\theta^a_t = \theta^b_t$. For simplicity, we use index i to denote traders a or b. Substituting in the stationarity condition ($P_{t+1} = P_t$) directly, we are left with three variables to solve for the equilibrium $\{P_t, \theta^i_t, \theta^m_t\}$ from three equilibrium conditions: two first order conditions (29) for agents i and m, respectively, and one market clearing condition (9).

For small λ, we expand the solution to equilibrium in λ to the first order:

$$P_t = \bar{P} + P_\lambda \lambda + o(\lambda) \tag{A9a}$$

$$\theta^i_t = \bar{\theta} + \theta^i_\lambda \lambda + o(\lambda) \tag{A9b}$$

$$\theta^m_t = \bar{\theta} + \theta^m_\lambda \lambda + o(\lambda) \tag{A9c}$$

where $\bar{P}$ is defined in (13) and $o(\lambda)$ denotes terms of higher order of λ. We then solve the equilibrium up to the first order of λ.

Given (A9), we first solve δ and $\hat{\theta}$ to the first order of λ. Note that both $\mu\, \sigma_{\hat{z}}/(\mu + \lambda\nu)$ and $k\, \theta^m$ in condition (23) are of order $O(1)$ while $\theta^i_t - \theta^m_t = \lambda(\theta^i_\lambda - \theta^m_\lambda)$ are of order $O(\lambda)$, where $O(\cdot)$ denotes terms of the same order. Condition (23) is satisfied when λ is small (i.e., to the first order of λ) and Proposition 3 holds. In particular, the trading gain in (19) can be simplified to

$$g^i(\cdot) = -\frac{1}{2\alpha} \ln(1-k) + \frac{\alpha\sigma_D^2}{2(1-k)} (k\bar{\theta} + \lambda^i X)^2 + O(\lambda)$$

Thus, trader i participates iff $g^i(\cdot) > 0$, which occurs iff $X > X^i_+$ or $X < X^i_-$, where

$$X^i_\pm = -\lambda^i\, k\, \bar{\theta} \pm h + O(\lambda)$$

and

$$h \equiv \begin{cases} \frac{1}{\alpha\sigma_D} \sqrt{(1-k)[2\alpha c^i R + \ln(1-k)]}, & \text{if} \quad 2\alpha c^i R + \ln(1-k) \geq 0 \\ 0, & \text{if} \quad 2\alpha c^i R + \ln(1-k) \geq 0 \end{cases}$$

Since δ and $\hat{\theta}$ depends on ω^i only through term $\lambda\omega^i$, we can ignore all $O(\lambda)$ terms for the calculation of ω^i. The equilibrium participation in Proposition 3 can be simplified to

$$\begin{cases} \omega^a = \omega^b = 1, & \delta = 0, & \text{if } X \le -k\,\bar{\theta} - h \\ \omega^a = 0,\ \omega^b = 1, & \delta = -\bar{\delta}, & \text{if } -k\,\bar{\theta} - h < X \le -|k\,\bar{\theta} - h| \\ \omega^a = \omega^b = 1, & \delta = 0, & \text{if } -|k\,\bar{\theta} - h| < X \le |k\,\bar{\theta} - h| \text{ and } k\,\bar{\theta} > h \\ \omega^a = \omega^b = 0, & \delta = 0, & \text{if } -|k\,\bar{\theta} - h| < X \le |k\,\bar{\theta} - h| \text{ and } k\,\bar{\theta} < h \\ \omega^a = 1,\ \omega^b = 0, & \delta = \bar{\delta}, & \text{if } |k\,\bar{\theta} - h| < X < k\,\bar{\theta} + h \\ \omega^a = \omega^b = 1, & \delta = 0, & \text{if } X \ge k\,\bar{\theta} + h. \end{cases} \tag{A10}$$

Since $\bar{\delta}$ is of order $O(\lambda)$, so is δ. The following equation linearizes δ and $\hat{\theta}$:

$$\delta(\theta^i_t, \theta^m_t, X) = \delta_\lambda(X)\lambda \tag{A11a}$$

$$\hat{\theta}(\theta^i_t, \theta^m_t, X) = \bar{\theta} + \frac{\mu\theta^m_\lambda + \lambda\,\nu(\omega^a + \omega^b)\theta^i_\lambda}{\mu + \lambda\,\nu(\omega^a + \omega^b)}\,\lambda + o(\lambda) = \bar{\theta} + \theta^m_\lambda\lambda + o(\lambda). \tag{A11b}$$

Using (A9) and (A11) and the definition of $\bar{P}$ in (13), the first order condition for market makers can be written as

$$\begin{aligned} 0 = \frac{\partial J^m}{\partial \theta^m_t} &= \mathrm{E}\left[-\alpha J^m_{P}(X)\left(\bar{D} + P_{t+1} - R^2 P_t - \alpha\sigma_D^2\theta^m_t - \frac{\alpha\sigma_D^2}{1 + k\delta^2}(\hat{\theta} - \theta^m_t + \delta X)\right)\right] \\ &= \mathrm{E}\left[-\alpha J_0\, e^{\alpha r\bar{\theta}P_\lambda\,\lambda + o(\lambda)}\left[-\left(rP_\lambda + \alpha\sigma_D^2\theta^m_\lambda + \alpha\sigma_D^2\delta_\lambda X\right)\lambda + o(\lambda)\right]\right] \\ &= \mathrm{E}\left[\alpha J_0\left(rP_\lambda + \alpha\sigma_D^2\theta^m_\lambda + \alpha\sigma_D^2\delta_\lambda X\right)\right]\lambda + o(\lambda) \end{aligned}$$

where $J^m_P(X)$ is defined in (A8), and $J_0 \equiv -e^{-\alpha\left(R^2 W_t + \frac{1}{2}\alpha\sigma_D^2\bar{\theta}^2\right)}$. Or equivalently,

$$rP_\lambda + \alpha\sigma_D^2\,\theta^m_\lambda + c_1 = 0 \tag{A12}$$

where

$$c_1 \equiv \frac{\nu}{\mu}\,\alpha\sigma_D^2\sigma_x\sqrt{\frac{2}{\pi}}\left(e^{-h_1^2} - e^{-h_2^2}\right), \qquad h_1 \equiv \frac{k\,\bar{\theta} - h}{\sqrt{2}\,\sigma_x}, \qquad h_2 \equiv \frac{k\,\bar{\theta} + h}{\sqrt{2}\,\sigma_x}$$

and $\sigma_x^2 = \frac{\sigma_z^4}{\sigma_z^2 + \sigma_u^2}$. Since $h_1 \le h_2$, we know that $c_1 \ge 0$.

We now consider the first order condition for trader i. First, we verify that traders with $\lambda^i = 0$ never participates. Combining (A6) and (A7), the gain from participation for these traders becomes:

$$g_{NL}(\theta^i_t; \theta) = -\frac{1}{\alpha}\ln\frac{\mathrm{E}[J_P(\lambda^i = 0)]}{J_{NP}(\lambda^i = 0)} = -\frac{1}{\alpha}\ln\mathrm{E}\left[\frac{e^{-\frac{1}{2}\frac{\alpha^2\sigma_D^2}{1 + k\delta^2}(\theta^i_t - \hat{\theta} - \delta X)^2}}{\sqrt{1 + k\delta^2}}\right] - R\,c^i \tag{A13}$$

where E[·] is taken with respect to X. Given (A9) and (A11) and the fact that $\theta_t^i - \hat{\theta}$ and δ are both of the order $O(\lambda)$, we have

$$g_{\scriptscriptstyle NL}(\theta_t^i;\theta) = O(\lambda) - Rc^i$$

which is negative as long as c^i is finite and λ is small enough. Thus, $J^{\scriptscriptstyle NL} = \mathrm{E}[J_{\scriptscriptstyle NP}(\lambda^i = 0)]$. When $\lambda^i \neq 0$, a trader's first order condition can be written as

$$\begin{aligned} 0 &= \lambda \frac{\partial J^{\scriptscriptstyle L}}{\partial \theta_t^i} + (1-\lambda)\frac{\partial J^{\scriptscriptstyle NL}}{\partial \theta_t^i} \\ &= \lambda \frac{\partial \mathrm{E}[J_{\scriptscriptstyle NP}]}{\partial \theta_t^i} + \lambda \frac{\partial \mathrm{E}[\mathbf{1}_{\{g(\cdot)>0\}}\,(J_{\scriptscriptstyle P} - J_{\scriptscriptstyle NP})]}{\partial \theta_t^i} + (1-\lambda)\frac{\partial J^{\scriptscriptstyle NL}}{\partial \theta_t^i} \end{aligned} \tag{A14}$$

where $J^{\scriptscriptstyle NL} = \mathrm{E}[J_{\scriptscriptstyle NP}(\lambda^i = 0)]$ is given in (A7), $J_{\scriptscriptstyle P}$ and $J_{\scriptscriptstyle NP}$ are defined in (A3) and (A4), and $g(\cdot)$ is the trading gain in (19).

$$\begin{aligned} \lambda \frac{\partial \mathrm{E}[J_{\scriptscriptstyle NP}]}{\partial \theta_t^i} &= \lambda \frac{\alpha\left(\bar{D} + P_{t+1} - R^2 P_t - \frac{\alpha\sigma_D^2\theta_t^i}{1-\bar{k}}\right)}{\sqrt{1-\bar{k}}}\, e^{-\alpha\left[R^2 W_t + \theta_t^i\left(\bar{D}+P_{t+1}-R^2P_t\right) - \frac{1}{2}\frac{\alpha\sigma_D^2}{1-\bar{k}}{\theta_t^i}^2\right]} \\ &= \lambda\, J_0\, \alpha\, c_2 + o(\lambda), \quad c_2 \equiv \frac{\bar{k}\,\alpha\sigma_D^2\,\bar{\theta}}{(1-\bar{k})^{3/2}}\, e^{\frac{1}{2}\frac{\bar{k}}{1-\bar{k}}\alpha^2\sigma_D^2\bar{\theta}^2} \end{aligned}$$

where $\bar{k} = \alpha^2\sigma_D^2\sigma_z^2$ captures the total uncertainty in idiosyncratic shocks. Since $J_{\scriptscriptstyle P} = J_{\scriptscriptstyle NP}$ when $g(\cdot) = 0$, for the second term in (A14), we have

$$\begin{aligned} \lambda \frac{\partial \mathrm{E}[\mathbf{1}_{\{g(\cdot)>0\}}\,(J_{\scriptscriptstyle P} - J_{\scriptscriptstyle NP})]}{\partial \theta_t^i} &= \lambda\, \mathrm{E}\left[\mathbf{1}_{\{g(\cdot)>0\}} \frac{\partial (J_{\scriptscriptstyle P} - J_{\scriptscriptstyle NP})}{\partial \theta_t^i}\right] \\ &= \mathrm{E}\left[-\mathbf{1}_{\{g(\cdot)>0\}}\, J_0\, \frac{\alpha^2\sigma_D^2\,(k\,\bar{\theta} + X)}{(1-k)^{3/2}}\, e^{\frac{\alpha^2\sigma_D^2}{2(1-k)}(k\,\bar{\theta}^2 + 2\bar{\theta}X + X^2)}\lambda + o(\lambda)\right] \\ &= -J_0\,\alpha\, c_2 c_3\,\lambda + o(\lambda) \end{aligned}$$

where

$$\begin{aligned} c_3 &\equiv \sqrt{\frac{1-\bar{k}}{2\pi(1-k)}}\frac{\sigma_x}{\bar{k}\,\bar{\theta}}\left(e^{-h_3^2} - e^{-h_4^2}\right) - \frac{1}{2}\left(\mathrm{Erf}(h_3) + \mathrm{Erf}(h_4) - 2\right) \\ h_3 &\equiv \frac{h(1-\bar{k}) - \bar{k}(1-k)\bar{\theta}}{\sqrt{2(1-k)(1-\bar{k})}\,\sigma_x}, \quad h_4 \equiv \frac{h(1-\bar{k}) + \bar{k}(1-k)\bar{\theta}}{\sqrt{2(1-k)(1-\bar{k})}\,\sigma_x}. \end{aligned}$$

Note that $\frac{\partial c_3}{\partial h} = -\frac{h}{\sqrt{2\pi}\,\bar{k}\,\sigma_x\bar{\theta}}\left(\frac{1-\bar{k}}{1-k}\right)^{3/2}(e^{-h_3^2} - e^{-h_4^2}) \leq 0$ (since $h_3 \leq h_4$.) Since $h \geq 0$, and $c_3 = 1$

when $h = 0$, we have $c_3 \leq 1$. For the third term in (A14), we have

$$\begin{aligned}(1-\lambda)\,\frac{\partial J^{NL}}{\partial \theta_t^i} &= (1-\lambda)\,\alpha\left(\bar{D}+P_{t+1}-R^2P_t-\alpha\sigma_D^2\theta_t^i\right)\; e^{-\alpha\left[R^2W_t+\theta_t^i\left(\bar{D}+P_{t+1}-R^2P_t\right)-\frac{1}{2}\alpha\sigma_D^2{\theta_t^i}^2\right]}\\ &= J_0\,\alpha(rP_\lambda+\alpha\sigma_D^2\,\theta_\lambda^i)\,\lambda+o(\lambda).\end{aligned}$$

Hence, to the first order of λ, the first order condition for trader i reduces to

$$rP_\lambda + \alpha\sigma_D^2\,\theta_\lambda^i + c_2(1-c_3) = 0. \tag{A15}$$

Finally, the market clearing condition (9) is reduces to

$$\mu\,\theta_\lambda^m + 2\nu\,\theta_\lambda^i = 0. \tag{A16}$$

Solving system (A12), (A15), and (A16), we derive the linear stationary equilibrium

$$P_\lambda = -\frac{\mu\,C_1 + 2\,\nu\,c_2(1-c_3)}{r\,(\mu+2\nu)} \tag{A17a}$$

$$\theta_\lambda^i = \frac{\mu\,[c_1 - c_2(1-c_3)]}{\alpha\sigma_D^2\,(\mu+2\nu)} \tag{A17b}$$

$$\theta_\lambda^m = -\frac{2\,\nu\,[c_1 - c_2(1-c_3)]}{\alpha\sigma_D^2\,(\mu+2\nu)} \tag{A17c}$$

Since $c_1 \geq 0$, $c_2 \geq 0$, and $c_3 \leq 1$, P_λ is always negative.

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