# 6.097 Review of Signals and Systems

6.097 Problem Set 2Solutions

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1. **DT Convolution**

(a)

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \\ &= \sum_{m=-\infty}^{\infty} \alpha^m u[m]\beta^{n-m}u[n-m] \\ &= \sum_{m=0}^{n} \alpha^m \beta^{n-m} \\ &= \beta^n \sum_{m=0}^{n} \left(\frac{\alpha}{\beta}\right)^m \\ &= \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}u[n] \end{aligned}$$

(b)

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \\ &= \sum_{m=-\infty}^{\infty} (u[m]-u[m-5])(0.25)2^{n-m}u[-1-n+m] \end{aligned}$$

Working out the different cases,

$$y[n] = \begin{cases} (0.25)2^n \sum_{m=0}^{4} \left(\frac{1}{2}\right)^m = \frac{1}{4}2^n \frac{1-(1/2)^5}{1/2} = \frac{31}{64}2^n, & n \leq -1 \\ (0.25)2^n \sum_{m=n+1}^{4} \left(\frac{1}{2}\right)^m = \frac{1}{4}2^n \frac{(1/2)^{n+1}-(1/2)^5}{1/2} = \frac{1}{4}\left(1-2^{n-4}\right), & 0 \leq n \leq 3 \\ 0, & n \geq 4 \end{cases}$$

(c) Working out the math graphically or using the convolution equation,

$$y[n] = [-1, -4, -9, -15, -12, 0, 12, 15, 9, 4, 1].$$

2. **DT Fourier Series**

(a) i. $x[n]\cos(\frac{6\pi n}{N})$.
Let $z[n] = x[n]\cos(\frac{6\pi n}{N})$. Then noting that $\cos(6\pi n/N) = \frac{1}{2}(e^{j(6\pi/N)n} + e^{-j(6\pi/N)n})$, we have $b_3 = b_{-3} = \frac{1}{2}$ Thus,

$$c_k = \sum_{k=\langle N\rangle} b_l a_{k-l} = \frac{1}{2}a_{k+3} + \frac{1}{2}a_{k-3}.$$

ii. $x[n]\sum_{r=-\infty}^{\infty}\delta[n-rN]$.
Let $y[n] = \sum_{r=-\infty}^{\infty}\delta[n-rN]$. Then verify that the FS coefficients are $b_k = 1/N$ for all $k$.

$$c_k = \sum_{k=\langle N\rangle} b_l a_{k-l} = \frac{1}{N}\sum_{l=0}^{N-1} a_l$$

iii. $x[n]\left(\sum_{r=-\infty}^{\infty}\delta[n-\frac{rN}{3}]\right)$. Assume $N$ is divisible by 3. (*Optional*)
Let $y[n] = \sum_{r=-\infty}^{\infty}\delta[n-\frac{rN}{3}]$. The FS coefficients of $y[n]$ are

$$\begin{aligned} b_k &= \frac{1}{N}\sum_{n=0}^{N-1} y[n]e^{-jk(2\pi/N)n} \\ &= \frac{1}{N}(1 + e^{-jk(2\pi/N)(N/3)} + e^{-jk(2\pi/N)(2N/3)}) \\ &= \frac{1}{N}(1 + e^{-jk(2\pi/3)} + e^{-jk(4\pi/3)}) \end{aligned}$$

Thus gives

$$\begin{aligned} c_k &= \sum_{l=\langle N\rangle} a_{k-l}\frac{1}{N}(1 + e^{-jl(2\pi/3)} + e^{-jl(4\pi/3)}) \\ &= \frac{1}{N}\sum_{l=0}^{N-1}(1 + e^{-jl(2\pi/3)} + e^{-jl(4\pi/3)})a_{k-l} \end{aligned}$$

(b) First we need to find the DTFS coefficients of $y[n]$.

$$\begin{aligned} b_k &= \frac{1}{12}\sum_{n=-5}^{6} y[n]e^{-jkn(2\pi/12)} \\ &= \frac{1}{12}\sum_{n=-3}^{3} e^{-jkn(2\pi/12)} \\ &= \frac{1}{12}\frac{\sin((7/12)k\pi)}{\sin((1/12)k\pi)} \end{aligned}$$

Also $a_k = 1/2$ for $k = 2$ and $k = 10$ and zero otherwise. This gives

$$\begin{aligned} c_k &= \sum_{k=\langle N\rangle} a_l b_{k-l} \\ &= \frac{1}{2}\left[\frac{1}{12}\frac{\sin((7/12)(k-2)\pi)}{\sin((1/12)k\pi)} + \frac{1}{12}\frac{\sin((7/12)(k-10)\pi)}{\sin((1/12)k\pi)}\right], \quad 0 \le k \le 11 \end{aligned}$$

3. **DT Fourier Transform**

(a)

$$\begin{aligned}
\tilde{c}_n &= \frac{1}{T}\int_T \tilde{f}_N(t)e^{-j\omega_0 nt}dt \\
&= \frac{1}{T}\int_T \left[\sum_{m=-N}^{N} c_m e^{j\omega_0 mt}\right] e^{-j\omega_0 nt}dt \\
&= \sum_{m=-N}^{N} c_m \frac{1}{T}\int_T e^{j\omega_0 (m-n)t}dt \\
&= \sum_{m=-N}^{N} c_m \delta_{m,n} \\
&= \begin{cases} c_n, & \text{if} -N \le n \le N \\ 0, & \text{else} \end{cases}
\end{aligned}$$

The sequence $d_n$ such that $\{\tilde{c}_n\} = \{d_n c_n\}$ is therefore

$$d_n = \begin{cases} 1, & \text{if} -N \le n \le N \\ 0, & \text{else} \end{cases}$$

(b) Convolution in the time domain corresponds to multiplication in the frequency domain. If we let $k_n$ represent the Fourier coefficients of $g_N(t)$,

$$\tilde{f}_N(t) = f(t) * g_N(t) \leftrightarrow \tilde{c}_n = c_n k_n$$

Since from part (a) $\tilde{c_n} = c_n d_n$, then $k_N = d_N$.

$$\begin{aligned}
g_N(t) &= \sum_{n=-\infty}^{\infty} d_n e^{j\omega_0 nt} \\
&= \sum_{n=-N}^{N} e^{j\omega_0 nt} \\
&= \frac{e^{-j\omega_0 Nt} - e^{j\omega_0 (N+1)t}}{1 - e^{j\omega_0 t}} \\
&= \frac{\sin\left(\omega_0\left(N+\frac{1}{2}\right)t\right)}{\sin\left(\frac{\omega_0}{2}t\right)}
\end{aligned}$$

Because the Fourier coefficients are discrete, $g_N(t)$ is periodic.

4. **DT Fourier Transform**

(a) The system function $H(e^{j\omega})$ is the product of $H_1$ and $H_2$.

$$\begin{aligned}
H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\
&= \frac{2 - e^{-j\omega}}{1 - \frac{1}{8}e^{-3j\omega}}
\end{aligned}$$

Thus

$$Y(e^{j\omega})(1 - \frac{1}{8}e^{-3j\omega}) = X(e^{j\omega})(2 - e^{-j\omega})$$

Converting to the time domain,

$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1]$$

(b) (*Optional, Tedious*) Using a tedious partial-fraction expansion $H(e^{j\omega})$ can be expressed as

$$H(e^{j\omega}) = \frac{4/3}{1+\frac{1}{2}e^{-j\omega}} + \left(\frac{2}{3}\right)\frac{1-e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-2j\omega}}$$

Using the inverse DTFT

$$h[n] = \frac{4}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1+j\sqrt{3}}{3}\left(\frac{1}{2}e^{j(2\pi/3)}\right)^n u[n] + \frac{1-j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j(2\pi/3)}\right)^n u[n]$$

## 5. The $z$-Transform

(a) For $H(z)$ to be causal the ROC must extend outward from the outermost pole and include $z=\infty$. The ROC is thus $|z| > \frac{1}{2}$.

(b) Yes, the system is stable since the ROC includes the unit circle.

(c)

$$y[n] = -\tfrac{1}{3}\left(-\tfrac{1}{4}\right)^n u[n] - \tfrac{4}{3}(2)^n u[-n-1]$$
$$Y(z) = -\frac{\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1+z^{-1}}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-2z^{-1}\right)}$$

Since the first term of $y[n]$ is right-sided, the corresponding ROC constraint is $|z| > \frac{1}{4}$; likewise, the second term being left-sided leads to the ROC constraint $|z| < 2$. Therefore, the ROC for $Y(z)$ is $\frac{1}{4} < |z| < 2$.

$$X(z) = \frac{Y(z)}{H(z)} = \frac{\left(1+z^{-1}\right)}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-2z^{-1}\right)}\frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}{\left(1+z^{-1}\right)} = \frac{1-\frac{1}{2}z^{-1}}{1-2z^{-1}}$$

Only the pole at $z=2$ remains, so the ROC for $X(z)$ is $|z| < 2$.

(d)

$$H(z) + \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1+\frac{1}{4}z^{-1}}, |z| > \frac{1}{2}$$
$$h[n] = 2\left(\tfrac{1}{2}\right)^n u[n] - \left(-\tfrac{1}{4}\right)^n u[n]$$

(e)

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$
$$Y(z) - \tfrac{1}{4}z^{-1}Y(z) - \tfrac{1}{8}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$
$$y[n] - \tfrac{1}{4}y[n-1] - \tfrac{1}{8}y[n-2] = x[n] + x[n-1]$$