# Review of Convolution and LTI System Properties

6.097 Problem Set 1 Solution

Compiled from Various Sources

Issued: 1/12/09, Due: 1/16/09

1. **Check Your Knowledge of System Properties (15 pts)**

In this problem, we assume that the following input-output relations for each system are known:

$$\begin{aligned} x_1 &\longrightarrow y_1 \\ x_2 &\longrightarrow y_2 \end{aligned}$$

Also, let $a$ and $b$ be constants. Let $A$ and $\tau$ be positive real numbers. Let $N$ be some positive integer.

(a) $y(t) = x(4t + 3)$

- **Time invariance - NO** As explained in Ex. 1.16 (p.52) of O&W, time scaled systems are not time-invariant. For input $x_1(t)$, the corresponding output $y_1(t)$ is:

$$y_1(t) = x_1(4t + 3)$$

Suppose we delay the input $x_1(t)$ by $\tau$, and call the delayed signal $x_2(t) = x_1(t - \tau)$. Then, the corresponding output $y_2(t)$ to the input $x_2(t)$ is:

$$\begin{aligned} y_2(t) &= x_2(4t + 3) \\ &= x_1(4t + 3 - \tau) \end{aligned}$$

Now, we need to check if $y_2(t)$ is equal to $y_1(t)$ delayed by $\tau$, $y_1(t - \tau)$:

$$\begin{aligned} y_1(t - \tau) &= x_1(4(t - \tau) + 3) \\ &= x_1(4t - 4\tau + 3) \\ &\neq y_2(t) \end{aligned}$$

Thus, the system is NOT time invariant. This problem shows that time shifting and time scaling are not commutative.

- **Linearity - YES** Let $x(t) = ax_1(t) + bx_2(t)$, then the corresponding output $y(t)$ is:

$$\begin{aligned} y(t) &= ax_1(4t + 3) + bx_2(4t + 3) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Thus, the system is linear.

- **Causality - NO** Consider $y(t)$ at $t = 0$, then

$$y(0) = x(3),$$

so, $y(0)$ relies on the future value of $x(t)$, i.e., $x(3)$. Thus, the system is NOT causal.

- **Stability - YES** Suppose $x(t)$ is bounded, i.e., $|x(t)| < A$ for all $t$. Then,

$$\begin{aligned} |y(t)| &= |x(4t + 3)| \\ &< A \end{aligned}$$

Thus, $y(t)$ is bounded. Therefore, the system is stable. This problem shows that time transformations do not affect stability.

(b) $y[n] = \cos(x(t)u(t))$

- **Time invariance - NO** We show this by a counterexample. Let $x_1(t) = u(t)$. Then the corresponding output $y_1(t)$ is:

$$\begin{aligned} y_1(t) &= \cos(u(t)u(t)) = \cos(u(t)) \\ &= \begin{cases} \cos(1) & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases} \end{aligned}$$

Now, consider an input $x_2(t)$ which is $x_1(t)$ advanced by 2, i.e., $x_2(t) = x_1(t+2) = u(t+2)$. The corresponding output $y_2(t)$ is:

$$\begin{aligned} y_2(t) &= \cos(u(t+2)u(t)) = \cos(u(t)) \\ &= \begin{cases} \cos(1) & \text{if } n \geq 0 \\ 1 & \text{if } n < 0 \end{cases} \\ &\neq y_1(t+2) \end{aligned}$$

Thus, the system is NOT time invariant.

- **Linearity - NO** Let $x(t) = ax_1(t) + bx_2(t)$, then the corresponding output $y(t)$ is:

$$\begin{aligned} y(t) &= \cos((ax_1(t) + bx_2(t))u(t)) \\ &= \cos(ax_1(t)u(t) + bx_2(t)u(t)) \\ &= \cos(ax_1(t)u(t))\cos(bx_2(t)u(t)) - \sin(ax_1(t)u(t))\sin(bx_2(t)u(t)) \\ &\neq a\cos(x_1(t)u(t)) + b\cos(x_2(t)u(t)) \end{aligned}$$

Therefore, the system is NOT linear.

- **Causality - YES** The output depends only on the current input. Thus, the system is causal.
- **Stability - YES** If we restrict ourself to deal with only real-valued signals, then it is fairly easy to conclude that $|\cos(x(t)u(t))| \leq 1$ for any real-valued $x(t)$. However, if we allow $x(t)$ to be a complex-valued signal, then we cannot say that $|\cos(x(t)u(t))| \leq 1$ any more. Suppose $x(t)$ is a complex-valued signal, but its magnitude is bounded, i.e., $|x(t)| < A$ for all $n$. Then, we can express $x(t)$ as

$$x(t) = x_r(t) + jx_i(t), \quad \text{where } x_r(t), x_i(t) \text{ real }, \quad \sqrt{x_r^2(t) + x_i^2(t)} < A \quad \text{for all } n.$$

With this expression of $x(t)$, $y(t) = \cos(x(t)u(t))$ can be expressed as follows:

$$\begin{aligned} y(t) &= \cos(x(t)u(t)) = \frac{e^{jx(t)u(t)} + e^{-jx(t)u(t)}}{2} \\ &= \frac{e^{j(x_r(t)+jx_i(t))u(t)} + e^{-j(x_r(t)+jx_i(t))u(t)}}{2} \\ &= \frac{e^{jx_r(t)u(t)}e^{-x_i(t)u(t)}}{2} + \frac{e^{-jx_r(t)u(t)}e^{x_i(t)u(t)}}{2} \end{aligned}$$

Thus,

$$\begin{aligned}
|y(t)| &= \left|\frac{e^{jx_r(t)u(t)}e^{-x_i(t)u(t)}}{2} + \frac{e^{-jx_r(t)u(t)}e^{x_i(t)u(t)}}{2}\right| \\
&\leq \left|\frac{e^{jx_r(t)u(t)}e^{-x_i(t)u(t)}}{2}\right| + \left|\frac{e^{-jx_r(t)u(t)}e^{x_i(t)u(t)}}{2}\right| \\
&\leq \left|\frac{e^{-x_i(t)u(t)}}{2}\right| + \left|\frac{e^{x_i(t)u(t)}}{2}\right|, \quad |e^{jB}| = 1 \text{ for real } B \\
&\leq \frac{e^A}{2} + \frac{e^A}{2} \\
&= e^A \\
\Rightarrow |y(t)| &\leq e^A
\end{aligned}$$

Hence, $y(t)$ is bounded even if $x(t)$ is complex-valued as long as $x(t)$ is bounded. Therefore, the system is stable.

(c) $y(t) = \int_{-\infty}^{t} x(-\tau)d\tau$
Define a change of variable, $\xi = -\tau$, which gives $d\xi = -d\tau$. Then $y(t)$ can be rewritten as:

$$\begin{aligned}
y(t) &= -\int_{\infty}^{-t} x(\xi)d\xi \\
&= \int_{-t}^{\infty} x(\xi)d\xi
\end{aligned}$$

- **Time invariance - NO** Let $y_1(t)$ be the output corresponding to input $x_1(t)$:

$$y_1(t) = \int_{-t}^{\infty} x_1(\xi)d\xi$$

Supposed we shift $x_1(t)$ by $\tau$ and call the shifted signal $x_2(t) = x_1(t-\tau)$, then the corresponding output $y_2(t)$ to $x_2(t)$ is:

$$\begin{aligned}
y_2(t) &= \int_{-t}^{\infty} x_2(\xi)d\xi = \int_{-t}^{\infty} x_1(\xi - \tau)d\xi \\
&= \int_{-t-\tau}^{\infty} x_1(\xi^{'})d\xi^{'}, \quad \xi^{'} = \xi - \tau \\
y_1(t-\tau) &= \int_{-(t-\tau)}^{\infty} x_1(\xi)d\xi \\
&= \int_{-t+\tau}^{\infty} x_1(\xi)d\xi \\
\Rightarrow y_2(t) &\neq y_1(t-\tau)
\end{aligned}$$

Thus, the system is NOT time invariant.

- **Linearity - YES** Let $x(t) = ax_1(t) + bx_2(t)$, the corresponding output $y(t)$ is:

$$\begin{aligned}
y(t) &= \int_{-t}^{\infty} x(\xi)d\xi = \int_{-t}^{\infty} (ax_1(\xi) + bx_2(\xi))dt \\
&= \int_{-t}^{\infty} ax_1(\xi)d\xi + \int_{-t}^{\infty} bx_2(\xi)d\xi \\
&= a\int_{-t}^{\infty} x_1(\xi)d\xi + b\int_{-t}^{\infty} x_2(\xi)d\xi \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Thus, the system is linear.

- **Causality - NO** The output $y(t)$ requires future values of $x(t)$, including the value at $t = \infty$. Thus, the system is NOT causal.
- **Stability - NO** Suppose $x(t) = 1$ for all $t$. Clearly $x(t)$ is bounded. Then the corresponding output $y(t)$, when $t$ is positive, is:

$$\begin{aligned} y(t) &= \int_{-t}^{\infty} x(\xi)d\xi \\ &= \int_{-t}^{\infty} 1d\xi \\ &= \infty \end{aligned}$$

  Thus, $y(t)$ is not bounded for all $t$. Therefore, the system is NOT stable.

(d) $y[n] = n^2 \left(\frac{1}{4}\right)^n x[n]$.

- **Time invariance - NO** Suppose we delay $x_1[n]$ by $N$ and call the delayed signal $x_2[n] = x_1[n-N]$. The output $y_2[n]$ corresponding to $x_2[n]$ is:

$$y_2[n] = n^2 \left(\frac{1}{4}\right)^n x_2[n] = n^2 \left(\frac{1}{4}\right)^n x_1[n-N]$$

  On the other hand, if we delay $y_1[n] = n^2 \left(\frac{1}{4}\right)^n x_1[n]$ by $N$, we get:

$$y_1[n-N] = (n-N)^2 \left(\frac{1}{4}\right)^{n-N} x_1[n-N]$$

  which is clearly not the same as $y_2[n]$. Thus, the system is NOT time invariant.
- **Linearity - YES** Let $x[n] = ax_1[n] + bx_2[n]$, then the corresponding output $y[n]$ is:

$$\begin{aligned} y[n] &= n^2 \left(\frac{1}{4}\right)^n (ax_1[n] + bx_2[n]) \\ &= n^2 \left(\frac{1}{4}\right)^n (ax_1[n]) + n^2 \left(\frac{1}{4}\right)^n (bx_2[n]) \\ &= a\left(n^2 \left(\frac{1}{4}\right)^n x_1[n]\right) + b\left(n^2 \left(\frac{1}{4}\right)^n x_2[n]\right) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

  Therefore, the system is linear.
- **Causality - YES** The output only depends on the current input. Thus, the system is memoryless, and is causal.
- **Stability - NO** Suppose $x[n]$ is bounded such that $|x[n]| < A$ for all $n$. Then,

$$\begin{aligned} |y[n]| &= \left|n^2 \left(\frac{1}{4}\right)^n x[n]\right| = |n^2| \left|\left(\frac{1}{4}\right)^n\right| |x[n]| \\ &\leq |n^2| \left(\frac{1}{4}\right)^n A. \end{aligned}$$

  If we take the limit of $|y[n]|$ as $n \to -\infty$, then both $|n|$ and $\left(\frac{1}{2}\right)^n$ blow up to $\infty$ and $A$ is a finite constant.

$$\lim_{n=-\infty} |y[n]| = \infty.$$

  Therefore, the system is NOT stable.

(e) $y[n] = x[n+1] + 7e^{-j\pi}$
Because $e^{-j\pi} = e^{j\pi} = -1$, $y[n]$ can be simplified to:

$$y[n] = x[n+1] - 7$$

- **Time invariance - YES** The output corresponding to input $x_1[n]$ is:

$$y_1[n] = x_1[n+1] - 7$$

So if $x_2[n] = x_1[n+N]$,

$$\begin{aligned} y_2[n] &= x_2[n+1] - 7 \\ &= x_1[n+N+1] - 7 \end{aligned}$$

Similarly,

$$y_1[n+N] = x_1[n+N+1] - 7$$

which is identical to $y_2[t]$. Thus, the system is time invariant.

- **Linearity - NO** Let $x[n] = ax_1[n] + bx_2[n]$, the corresponding output $y[n]$ is:

$$\begin{aligned} y[n] &= x[n+1] - 7 \\ &= ax_1[n+1] + bx_2[n+1] - 7 \\ &\neq (ax_1[n+1] + 7) + (bx_2[n+1] + 7) \\ &= ay_1[n] + by_2[n] \\ \Rightarrow y[n] &\neq ay_1[n] + by_2[n]. \end{aligned}$$

Thus, the system is NOT Linear.

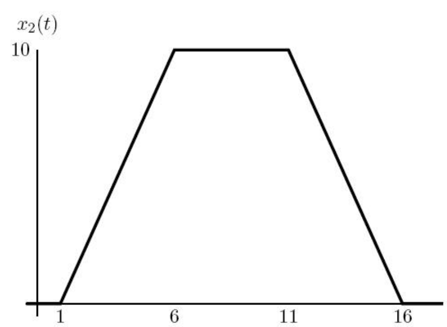
- **Causality - NO** The output $y[n]$ requires future input $x[n+1]$. Thus, the system is NOT causal.
- **Stability - YES** Suppose $x[n]$ is bounded, i.e., $|x[n]| < A$ for all $n$. Then,

$$\begin{aligned} |y[n]| &= |x[n+1] - 7| \\ &\leq |x[n+1]| + |-7| \\ &< A + 7 \\ \Rightarrow |y[n]| &< A + 7. \end{aligned}$$

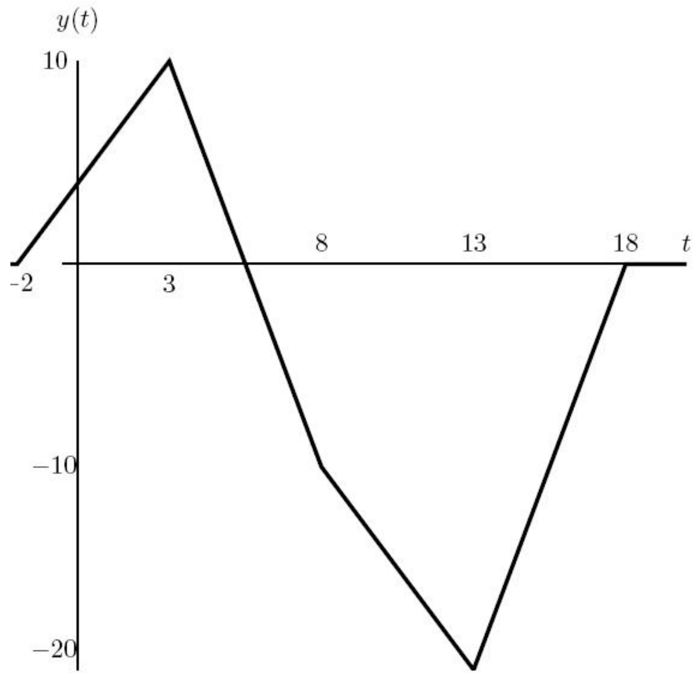
Thus, $y[n]$ is bounded. Therefore, the system is stable.

## 2. Practice with Convolution

We plot the output (denoted by $x_2(t)$) of $h_1(t)$ driven by the input $x(t)$, calculated by the flip and drag convolution method




The output of $h_2(t)$ driven by the input $x_2(t)$ is a superposition of two shifted and scaled versions of $x_2(t)$, namely $x_2(t+3)$ and $-2x_2(t-2)$




3. **Practice with System Functions in Time and Frequency**

(a) First, we have

$$x(t) = e^{-3t}u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = \frac{1}{j\omega + 3}$$

and

$$y(t) = 2te^{-3t}u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(j\omega) = 2j\frac{d}{d\omega}\frac{1}{j\omega + 3} = \frac{2}{(j\omega + 3)^2}.$$

Since $Y(j\omega) = H(j\omega) \cdot X(j\omega)$,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{j\omega + 3},$$

and thus

$$h(t) = 2e^{-3t}u(t).$$

(b) From (b), we have already had

$$v(t) = Y(j\omega)|_{\omega=2t} = \frac{2}{(j2t + 3)^2} = \frac{1}{2(jt + 3/2)^2}$$

and use the duality property of Fourier transform, we can obtain

$$V(j\omega) = 2\pi \cdot \frac{1}{2}(-\omega)e^{-\frac{3}{2}(-\omega)}u(-\frac{\omega}{2}) = -\pi\omega e^{\frac{3}{2}\omega}u(-\frac{\omega}{2}).$$

(c) Since $r(t)$ is a real and even signal, its Fourier transform $R(j\omega)$ is real and even, too, and thus

$$\begin{aligned}\int_0^{\infty} R(j\omega)d\omega &= \frac{1}{2}\int_{-\infty}^{\infty} R(j\omega)d\omega \\ &= \frac{1}{2} \cdot 2\pi \cdot r(0) = 2\pi e^{-9}\end{aligned}$$