The z-Transform

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Outline

Motivation for the z-Transform

The response to a complex exponential z^n is

$$y[n] = H(z)z^n$$

Thus complex exponentials z^n are eigenfunctions.

Motivation for the z-Transform

The function $H(z)$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

is known as the z-transform of $h[n]$.
Don't forget to specify a region of convergence (ROC).

For $z = e^{j\omega}$, $H(z)$ reduces to the Fourier transform of $h[n]$.

An example

The z-transform of the signal $x[n] = a^n u[n]$ is given by

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Note that for $|a| > 1$, the ROC does not include the unit circle.
The DTFT does not converge.
How about for $x[n] = -a^n u[-n-1]$?

An example

Another example

$$x[n] = \left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right] u[n]$$

Note the ROC is the intersection of both $x_1[n]$ and $x_2[n]$.

$$|z| > \frac{1}{2}$$

Poles and Zeros

$X(z)$ can also be expressed as

$$X(z) = \frac{a_0}{b_0} \frac{\sum_{k=1}^{M} \left(1 - c_k z^{-k}\right)}{\sum_{k=1}^{N} \left(1 - d_k z^{-k}\right)}$$

c_k represent the non-zero zeros of $X(z)$: $X(z)|_{z=c_k} \to 0$
d_k represent the non-zero poles of $X(z)$: $X(z)|_{z=d_k} \to \infty$

Note that # zeros = # poles: Don't forget zeros and poles at $z = 0$ or $z = \infty$.

Properties of the ROC

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.
2. ROC does not contain any poles.
3. If $x[n]$ is left (right) sided, then ROC is inward (outward).
4. If $X(z)$ is rational and $x[n]$ is right-sided, then ROC is the region in the z-plane outside the outermost pole. Furthermore, if $x[n]$ is causal, the ROC also includes $z = \infty$.

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ROC Example

What are the possible ROCs for the following z-transform?

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}$$

The Inverse z-transform

The Inverse z-transform can be obtained via contour integration

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1}\, dz$$

There is often an easier way.

The Inverse z-transform

What's the signal $x[n]$ with z-transform

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

Use partial fraction expansion.

The Inverse z-transform

What's the signal $x[n]$ with z-transform

$$X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty$$

Use the power-series expansion method.

Properties of the z-transform

Analagous to FT and FS properties.

Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z), \quad ROC = R_1 \cup R_2$$

Watch how ROC changes.

Characterization of LTI systems using z-transform

1. A LTI system is causal iff (a) the ROC is the exterior of a circle outside the outermost pole, or (b) $H(z)$ is proper (degree of numerator does not exceed degree of denominator.)
2. A LTI system is stable iff the ROC of $H(z)$ includes the unit circle $|z| = 1$.
3. A causal LTI system is stable if all poles lie inside the unit circle.

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Characterization of LTI systems using z-transform

The z-transform is often in the form

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \end{aligned}$$

This comes from the LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

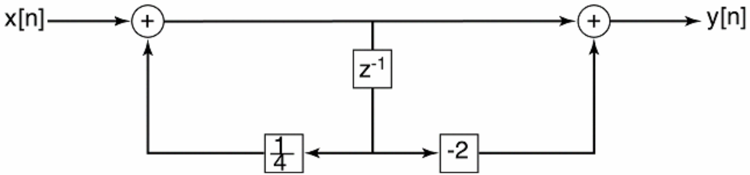
Characterization of LTI systems using z-transform

Find the z-transform $H(z)$ of the an LTI system for which $x[n]$ and $y[n]$ satisfy the LCCDE

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

Block diagram representation

Find the z-transform $H(z)$ of the an LTI system for which $x[n]$ is the input and $y[n]$ is the output.



Find the unit sample response $h[n]$.