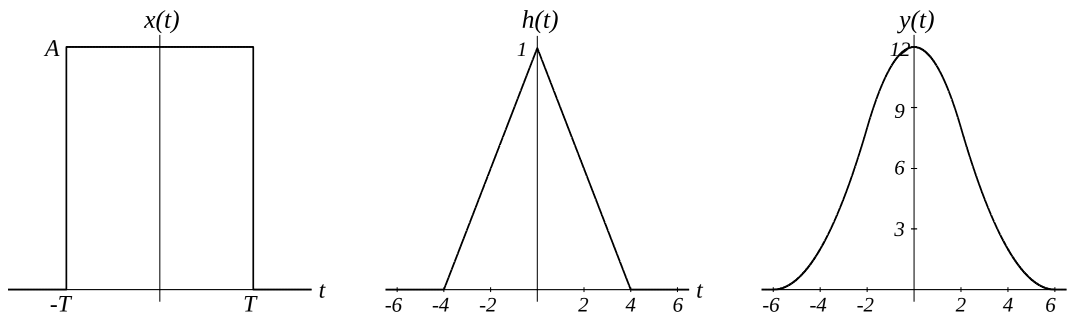
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

Final Exam Review Problems

Problem #1 *The purpose of this problem is to test your understanding of continuous-time convolution.*

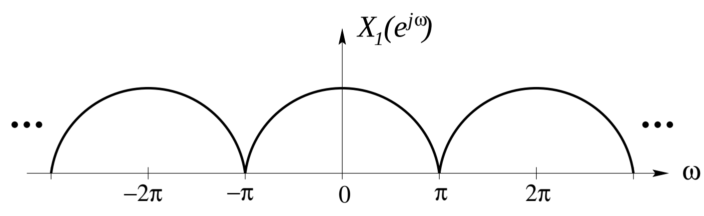
A CT LTI system has input $x(t)$, impulse response $h(t)$, and output $y(t)$ as shown below. Note that the scales for $x(t)$ are not necessarily the same as for $h(t)$ and $y(t)$.

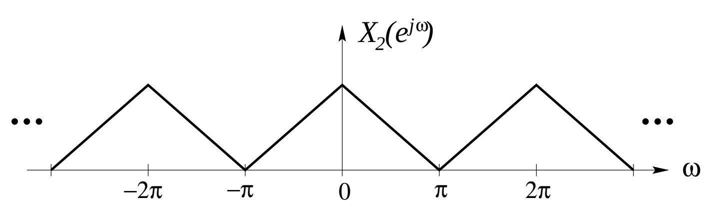


Determine the values of the parameters of $x(t)$: A and T.

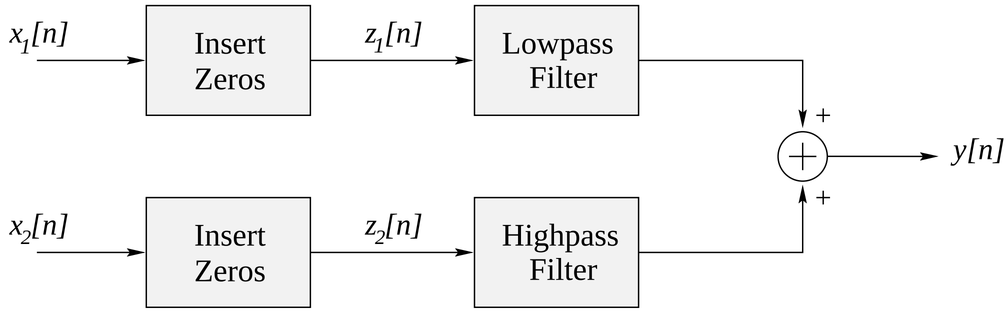
Problem #2 *The purpose of this problem is to test your understanding of discrete-time sampling.*

Suppose that we have two discrete-time signals, $x_1[n]$ and $x_2[n]$, that we wish to transmit simultaneously using frequency-division multiplexing. The problem is that each of the signals fills the entire frequency band. In particular, suppose that $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, the DTFTs of $x_1[n]$ and $x_2[n]$, are as shown below.

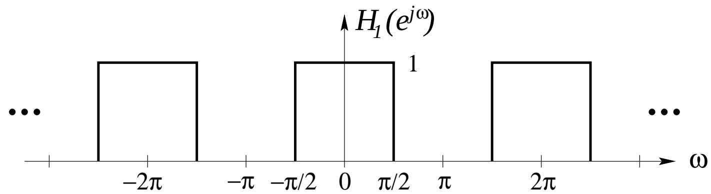


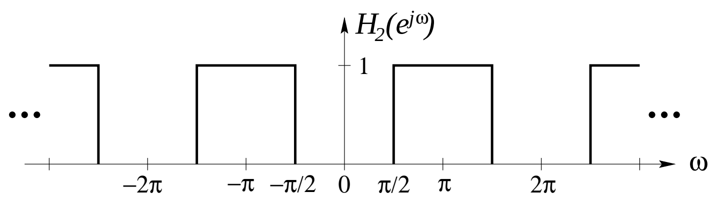


To perform the frequency-division multiplexing, a system with the following structure is proposed,



where the lowpass and highpass filters, $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ respectively, are as shown below.





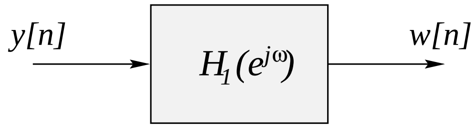
The signals $z_1[n]$ and $z_2[n]$ are obtained by inserting zeros between successive values of $x_1[n]$ and $x_2[n]$, respectively. This can be mathematically expressed as follows,

$$z_1[n] = \begin{cases} x_1\left[\frac{n}{2}\right] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$z_2[n] = \begin{cases} x_2\left[\frac{n}{2}\right] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

(a) Sketch the DTFTs of $z_1[n]$, $z_2[n]$, and $y[n]$.

(b) Suppose that $y[n]$ is passed through another lowpass filter whose frequency response is $H_1(e^{j\omega})$ given above. This is illustrated in the figure below. It is claimed that $x_1[n]$ can be recovered from the filter output $w[n]$. Show that the claim is valid, and describe how $x_1[n]$ can be recovered.



Problem #3 *The purpose of this problem is to test your understanding of the Laplace transform.*
An LTI system with system function $H(s)$ has input $x(t)$ and output $y(t)$. It is known that:

- When $x(t) = e^{-t}u(t)$, then

  $$y(t) \quad = \quad K\left(e^{-3t}u(t) + e^{t}u(-t)\right),$$

  where K is a constant that you will need to determine to solve the problem.

- When $x(t) = 1$ for all t, then $y(t) = \frac{8}{3}$ for all t.

Find $H(s)$ including its region of convergence (ROC).

Problem #4 *The purpose of this problem is to test your understanding of z-transforms.*
Determine the DT signal $x[n]$ given that the z-transform is

$$X(z) \quad = \quad \frac{1 + 3z^{-1}}{1 + 3z^{-1} + 2z^{-2}},$$

for $1 < |z| < 2$.

Problem #5 *The purpose of this problem is to test your understanding of discrete-time system functions.*
The system function of a DT LTI system is

$$H(z) \quad = \quad \frac{z^4}{z^4 - a^4},$$

where a is real and positive. It is known that

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

and that the unit-sample response of the system $h[n] = 0$ for all $n < N$ for some value of N.

(a) Sketch the pole/zero diagram of $H(z)$.

(b) Determine the region of convergence of $H(z)$ consistent with the information given.

(c) Determine the range of a that is consistent with the information given.

Problem #6 *The purpose of this problem is to test your understanding of pole-zero diagrams and Bode plots for continuous-time systems.*

For each pole-zero plot shown in Figure 1 below, find the frequency response, among those given in Figure 2, that could result from the pole-zero plot. The correct answers are among those given, and the same answer does not apply to more than one pole-zero plot.

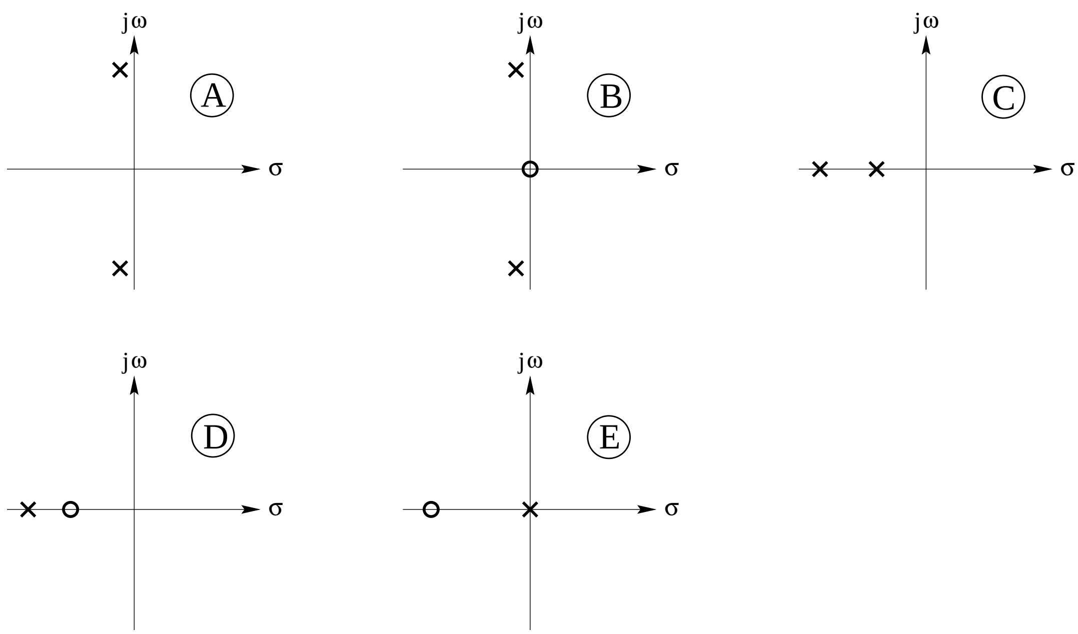


Figure 1: Pole-zero diagrams for $H(s)$.

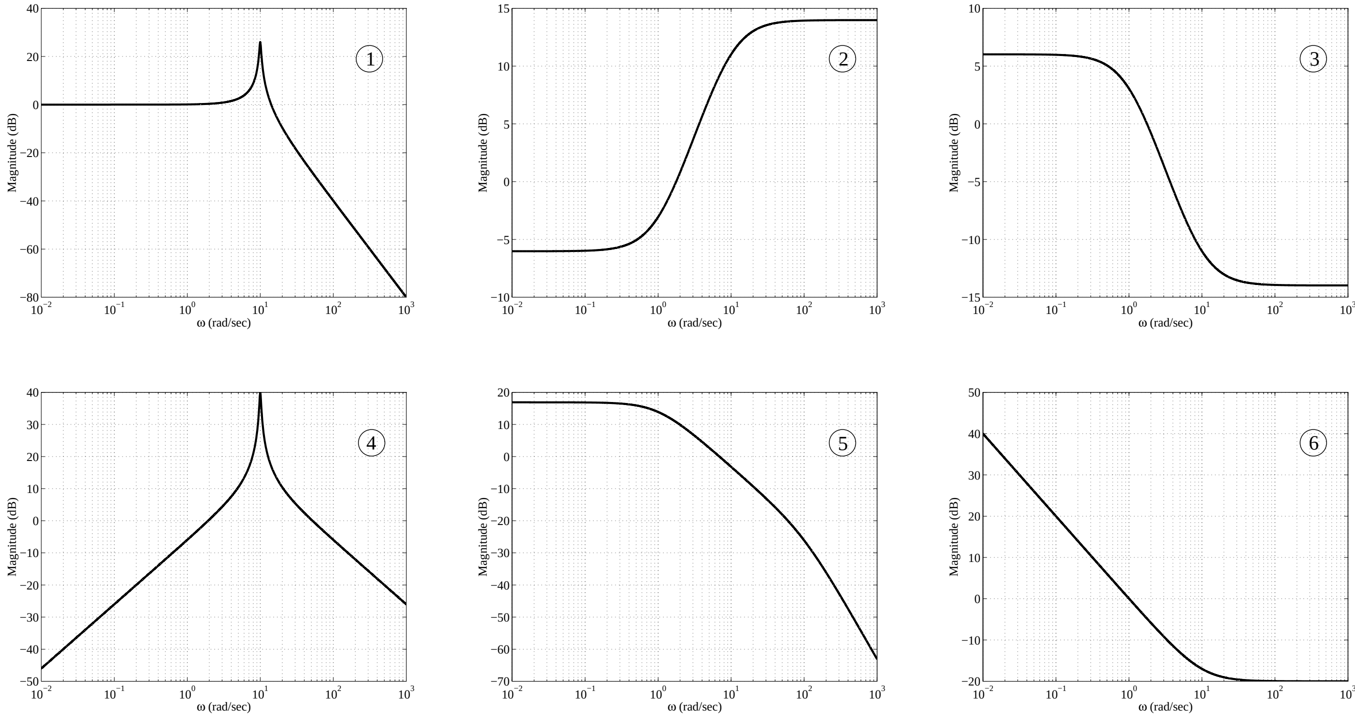


Figure 2: Bode plot magnitudes (*i.e.* $20\log_{10}|H(j\omega)|$ versus ω on a log scale).

Problem #7 *The purpose of this problem is to test your ability to sketch Bode plots of a CT LTI system.*

Consider a stable CT LTI system whose frequency response, $H(j\omega)$ is given as below:

$$H(j\omega) = \frac{j\omega - 10}{(j\omega + 100)((j\omega)^2 + 0.2(j\omega) + 1)}.$$

Sketch the Bode plot of the system.

Problem #8 *The purpose of this problem is to test your ability to determine the response of an LTI system to an input.*

(a) The input of a CT LTI system is $x(t) = e^{-t}$ for all t, and the impulse response of the system is $h(t) = \delta(t) - 2e^{-2t}u(t)$. Determine the output $y(t)$.

(b) The input of a CT LTI system is

$$x(t) = \frac{\sin(4\pi t)}{4\pi t},$$

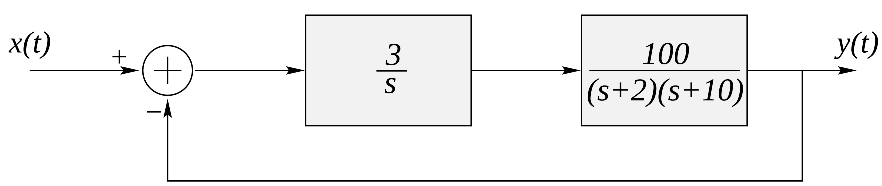
and the impulse response is

$$h(t) = \left(\frac{\sin(2\pi(t-2))}{2\pi(t-2)}\right)^2.$$

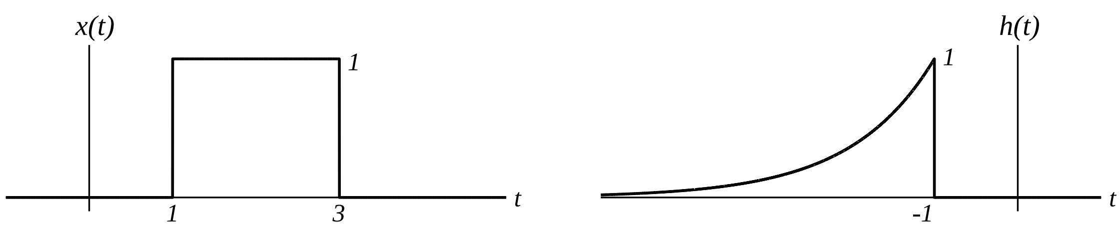
Determine the output $y(t)$.

(c) The input to a DT LTI system is $x[n] = 2^n u[-n]$, and the unit sample response is $h[n] = 0.5^n u[n]$. Determine the output $y[n]$.

(d) The input to the feedback system shown below, is a step, *i.e.* $x(t) = u(t)$. Determine the steady-state value of the output, *i.e.* $\lim_{t\to\infty} y(t)$.



(e) The input $x(t)$ and the unit impulse response $h(t) = e^{t+1}u(-t-1)$ of a CT LTI system are shown below. Determine the output $y(t)$ for $t = 1$.



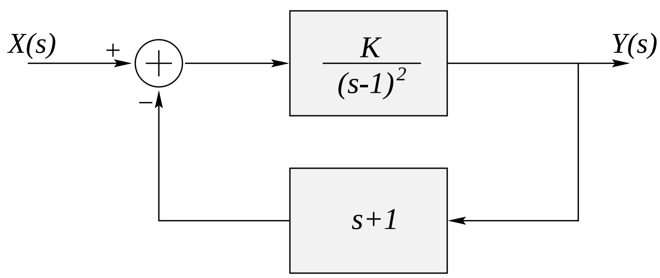
Problem #9 *The purpose of this problem is to test your understanding of discrete-time systems.* A causal DT LTI system has input sequence $x[n] = (1/2)^n u[n]$, output sequence $y[n]$, and system function $H(z)$, where

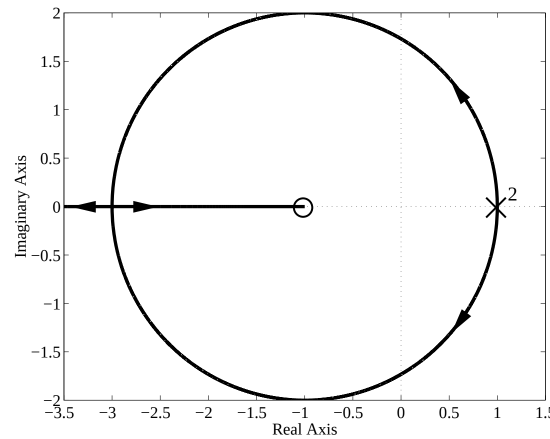
$$H(z) = \frac{1 - z^{-1}}{1 - 2z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3}}.$$

Determine the value of $y[1]$. [HINT: the solution to this problem using partial fraction expansion is more time-consuming than using other methods.]

Problem #10 *The purpose of this problem is to test your understanding of continuous-time feedback.*

A causal CT LTI feedback system is shown below along with a root-locus diagram as K varies from 0 to $+\infty$. The arrowheads on the locus indicate the direction of increasing values of K. The "2" near the pole indicates that this is a second-order pole.





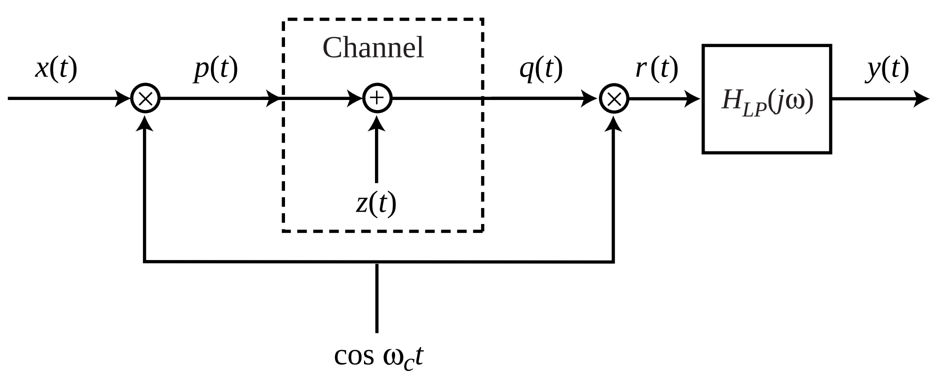
(a) Find the closed-loop system function $H(s) = Y(s)/X(s)$.

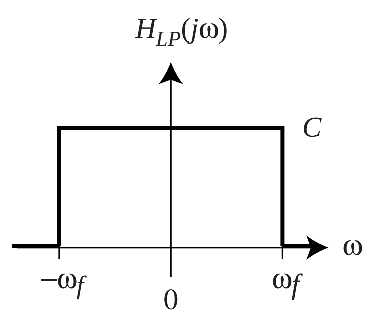
(b) Find the range of values of $K > 0$ for which the system is BIBO stable.

(c) For some value of K, the impulse response of the system has the form $h(t) = Ate^{-\alpha t}u(t)$. Find the corresponding values of K, α, and A.

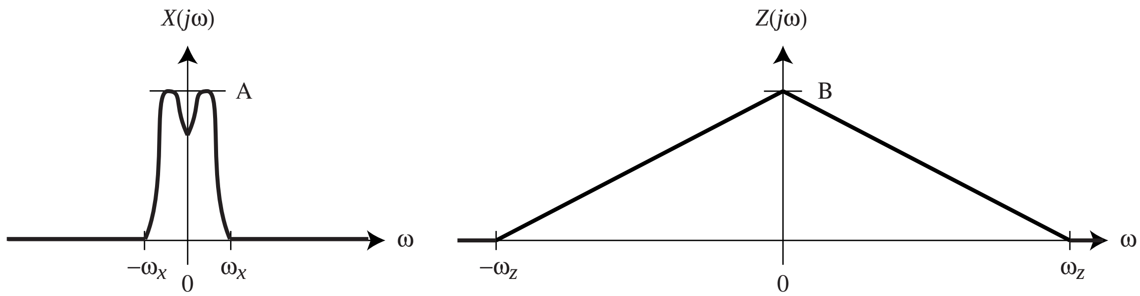
Problem #11 *The purpose of this problem is to test your understanding of continuous-time communication system.*

The following transmission system is intended to allow a signal $x(t)$ to be transmitted through a "channel" that also carries other signals represented by $z(t)$.





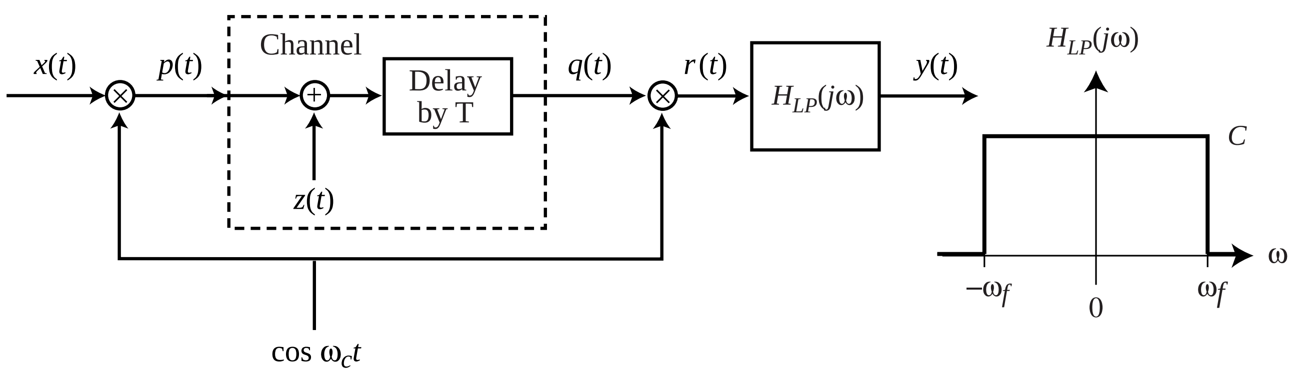
Both $x(t)$ and $z(t)$ are bandlimited, and their Fourier transforms $X(j\omega)$ and $Z(j\omega)$ are real, as sketched below. Notice that the bandwidth ω_z of $Z(j\omega)$ is much greater than the bandwidth ω_x of $X(j\omega)$.



(a) We wish to determine parameters for the transmission system so that the output $y(t)$ is equal to the input $x(t)$. For this part of the problem, determine the range of values of ω_c for which $y(t)$ can be made equal to $x(t)$. Explain.

(b) Given a value of ω_c in the range specified in (a), determine the range of values of ω_f and the value of C for which $y(t) = x(t)$. Your expressions may contain ω_c and/or parameters of the Fourier transforms $X(j\omega)$ and $Z(j\omega)$. Briefly explain your reasoning.

(c) Consider next what would happen if the "channel" also had appreciable delay, as in the following diagram.



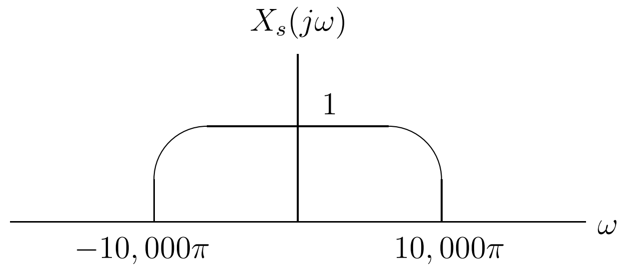
Find an expression for the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

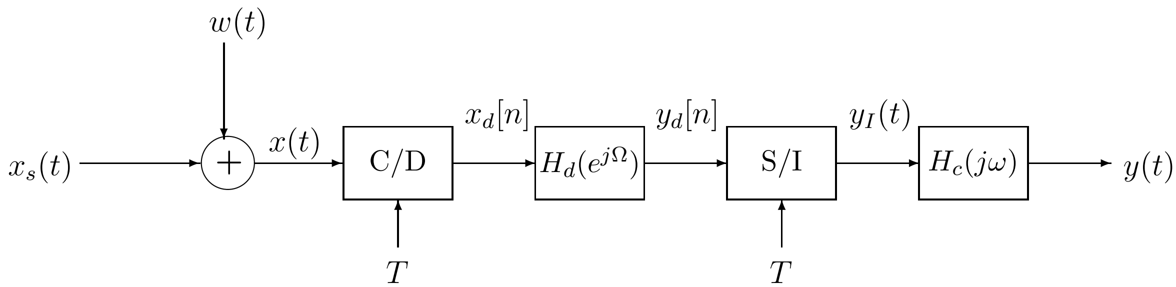
assuming that the parameters are chosen as in (a) and (b).

Problem #12 *The purpose of this problem is to test your understanding of DT processing of CT signals, Geometric evaluation for frequency response of DT LTI system.*

Consider a continuous-time signal $x_s(t)$ with Fourier Transform $X_s(j\omega)$ that is bandlimited to $10,000\pi$ radians/sec as shown below:



The signal $x_s(t)$ is contaminated by a noise signal $w(t) = \cos 15000\pi t$. We desire to remove the unwanted noise component using the system shown below:

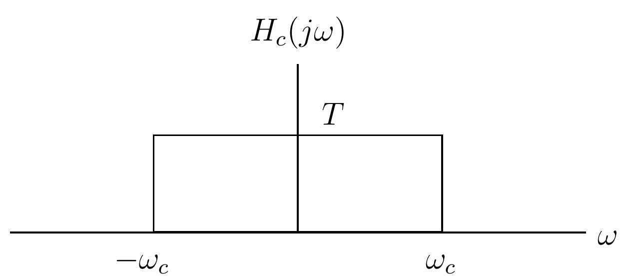


Please note the following:

- The C/D block is an ideal continuous-time to discrete-time converter with sampling period T (i.e., $x_d[n] = x(nT)$).
- $H_d(e^{j\Omega})$ is the frequency response of a discrete-time filter.
- The S/I block converts discrete-time samples into an impulse train where

$$y_I(t) = \sum_{k=-\infty}^{\infty} y_d[k]\delta(t - kT).$$

- $H_c(j\omega)$ is the frequency response of an ideal lowpass filter with cutoff frequency ω_c as depicted below:

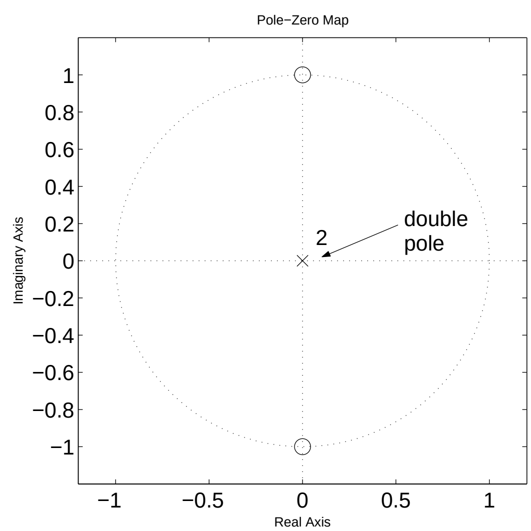


(a) For this part *only* assume that $H_d(e^{j\Omega}) = 1$ and that $\omega_c = 11,000\pi$. What is the maximum value $T_{\max}$ such that $y(t) = x_s(t)$ for all $T < T_{\max}$? Explain carefully.

For the remainder of the problem, assume that $T = 1/30,000$ and that $\omega_c = 30,000\pi$.

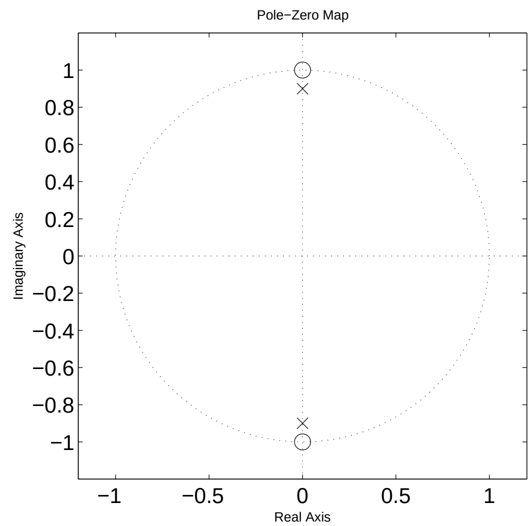
(b) The signal $x_d[n]$ is the result of performing an ideal C/D conversion on the noise-corrupted signal $x(t) = x_s(t) + w(t)$. Sketch $X_d(e^{j\Omega})$, the Fourier transform of $x_d[n]$, for $-2\pi \le \Omega \le 2\pi$. Clearly label your sketch.

We wish to design a simple filter $H_d(e^{j\Omega})$ such that the continuous-time output $y(t)$ has no harmonics at $\pm 15,000\pi$ radians/sec. One system which will achieve this goal is a stable, causal system $H_d(z)$ with pole-zero diagram given below. The dotted circle indicated on the diagram is the unit circle and the "2" indicates that there are two poles at the origin.



(c) Given that $H_d(e^{j0}) = 1$, find an expression for $|H_d(e^{j\Omega})|$ in terms of *real* functions (i.e., your final answer should not contain expressions involving $e^{j\alpha\Omega}$ where α is a real number). Sketch $|H_d(e^{j\Omega})|$ for $-2\pi \le \Omega \le 2\pi$.

In a 6.003 TA meeting, Taka claims that if we create a new system $G_d(z)$ in which we replace the two poles at the origin with poles at $z = \pm 0.9j$, then $|G_d(e^{j\Omega})|$ will be *flatter* than $|H_d(e^{j\Omega})|$ for $|\Omega| \le \pi/4$ (i.e., $|G_d(e^{j\Omega})| \approx$ constant for $|\Omega| \le \pi/4$). The pole-zero diagram for the $G_d(z)$ is shown below.



Sybor disagrees with Taka. He thinks that $|G_d(e^{j\Omega})|$ will be *less* flat than $|H_d(e^{j\Omega})|$.

(d) Who is correct? Argue your answer by making a rough sketch of $|G_d(e^{j\Omega})|$ for $-2\pi \leq \Omega \leq 2\pi$, given that $G_d(e^{j0}) = 1$. Provide a brief comparison (no more than 25 words) of $|H_d(e^{j\Omega})|$ and $|G_d(e^{j\Omega})|$