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Synchronization Stability of Lossy and Uncertain Power Grids

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***Abstract*— Direct energy methods have been extensively developed for the transient stability analysis and contingency screening of power grids. However, there is no analytical energy functions proposed for power grids with losses, which are normal in practice. This paper applies the recently introduced Lyapunov Functions Family approach to the certification of synchronization stability for lossy power grids. This technique does not rely on the global decreasing of the Lyapunov function as in the direct energy methods, and thus is possible to deal with the lossy power grids. We show that this approach is also applicable to uncertain power grids where the stable equilibrium is unknown due to possible uncertainties in system parameters. We formulate this new control problem and introduce techniques to certify the robust stability of a given initial state with respect to a set of equilibria.**

I. Introduction

A large number of agents in natural world can reach a common group objective through simply local interactions. Examples include flocking of birds, schooling of fish, and herding of animals. Such striking collective behaviors have blown a great research interest in many disciplines such as biology [1], social sciences [2], physics [3], computer science [4], and engineering [5].

This paper analyzes a collective property of power grids where a large number of generators reach a common angular velocity through their local interactions. This problem is known as frequency stability or synchronization stability. Formally, the multimachine power grids are characterized by the weighted graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ with nodes $\mathcal{V}=\{1,...,n\}$ representing generators, edges $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$, and positive weight matrix $\mathcal{A}$ including $a_{kj}=a_{jk}>0$ which denotes the strength of interaction between generators in each undirected edge $\{k,j\}\in\mathcal{E}$. The post-fault dynamics of each generator is characterized by the rotor angle δ_k and its angular velocity $\dot{\delta}_k$, and described by the so-called lossy swing model:

$$m_k\ddot{\delta}_k+d_k\dot{\delta}_k=P_k-\sum_{\{k,j\}\in\mathcal{E}}a_{kj}\sin(\delta_k-\delta_j+\alpha_{kj}) \quad (1)$$

The synchronization stability problem in this network of generators formally concerns the convergence of generators' angular velocities $\dot{\delta}_k$ to a synchronous velocity, while the generators' rotor angles δ_k converge to a stable equilibrium $\{\delta_1^*,...,\delta_n^*\}$ representing the desired operating condition. In the dynamics (1), the synchrony is enforced by the diffusive couplings $\sin(\delta_k-\delta_j+\alpha_{kj})$ between each generator with its neighbors, yet it is weakened by the heterogeneous toques P_k that drive the generators away from the synchronous velocity. Also, the nonlinear sinusoid couplings result in a system with multiple equilibria. As such, the synchrony can only obtained locally, instead of globally as in systems with linear couplings [6]–[9]. These rich dynamic properties make the synchronization problem of power grids challenging.

For multimachine power grids without losses, i.e. $\alpha_{kj}=0$ for all pair $\{k,j\}\in\mathcal{E}$, direct energy methods have been investigated to certify the synchronization stability of the system [10]–[14]. Exploiting the antisymmetric property of the couplings, i.e., $a_{kj}\sin(\delta_k-\delta_j)=-a_{jk}\sin(\delta_j-\delta_k)$, the energy function is proven to be always decreasing in the whole state space. As such, the system is guaranteed to converge to the stable equilibrium point from any initial state lying in the energy function level sets that do not contain any other equilibrium point. Much effort is then spent in determining the stability region as the largest energy level set by specifying the closest unstable equilibrium point (UEP) [15]–[17].

In the presence of losses there is, however, no analytical energy function proposed to guarantee the synchronization stability of the systems. The asymmetric property of the couplings, i.e., $a_{kj}\sin(\delta_k-\delta_j+\alpha_{kj})\neq\pm a_{jk}\sin(\delta_j-\delta_k+\alpha_{jk})$, causes the natural energy function to not decrease, and thus, the energy methods inapplicable [18], [14] (Chapter VI).

In this work we extend the recently introduced Lyapunov Functions Family method [19] to certifying the synchronization stability of lossy power grids. The principle of this method is to provide stability certificates by constructing a family of Lyapunov functions, which are generalizations of the classical energy function, and then find the best suited function in the family for each initial state. Since the nonlinear couplings among generators can be bounded by linear functions in a region around the equilibrium point, we can apply the well-known Popov stability methods to construct Lyapunov functions for the system [20], [21]. This nonlinearity separation method can be traced back to the pioneering work of Lur'e and Postnikov in 1944 [22]. Though the constructed Lyapunov functions are decreasing only in a finite neighborhood of the equilibrium point, instead of decreasing in the whole state space as the energy function, they can generally certify stability for a broader set of initial states compared to the energy function in the closest-UEP method [19]. Also, the LFF stability certificate is constructed via optimization-based techniques, rather than by identifying the UEPs as in the energy method which is known as an NP hard problem. In addition, the large family of possible Lyapunov functions allows efficient adaptation

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of the Lyapunov function to a given set of initial conditions. This adaptation can be seen as a counterpart of the problem of searching for the suitable controlling UEP in the energy method [23].

More interestingly, in this paper we show that the LFF approach is applicable to uncertain power grids, in which the equilibrium is unknown due to the uncertainty in mechanical torques. Such uncertainty makes classical analysis and control approaches inapplicable, since these methods implicitly assume that the equilibrium is perfectly known. We explicitly formulate this new control problem of robust stability of systems with unknown equilibrium and present optimization-based techniques to construct the robust stability certificate of a given initial state with respect to a family of equilibria.

Among other works on lossy power systems, we note a recent study [24] that proposes to utilize network decomposition for transient stability analysis of lossy power grids based on Sum of Square programming. Also addressing the related problems in our work is the recent study on the stabilization of lossy power systems, in which excitation controllers are designed such that the closed system including the lossy power system and the control system is stable [25]. Our work is different from this work on that we certify the stability of lossy power systems without reliance on the controllers.

II. Lossy Multimachine Power Systems

In normal conditions, power grids operate at a stable equilibrium point. Under some fault or contingency scenarios, the system moves away from the pre-fault equilibrium point to some post-fault conditions. After the fault is cleared, the system experiences the transient dynamics. This work focuses on the transient post-fault dynamics of the power grids, and aims to develop computationally tractable certificates of transient stability of the system, i.e. guaranteeing that the system will converge to the post-fault equilibrium. In order to address this question we use a traditional swing equation dynamic model of a power system, where the loads are represented by the static impedances and the n generators have perfect voltage control and are characterized each by the rotor angle δ_k and its angular velocity $\dot{\delta}_k$. The dynamics of generators are described by a set of the so-called swing equations:

$$m_k\ddot{\delta}_k + d_k\dot{\delta}_k + P_{ek} - P_{mk} = 0, k = 1,..,n, \qquad (2)$$

where, m_k is the dimensionless moment of inertia of the generator, d_k is the term representing primary frequency controller action on the governor. P_{mk} is the effective dimensionless mechanical torque acting on the rotor and P_{ek} is the effective dimensionless electrical power output of the k^{th} generator. In the power grids with losses, the electrical power output is given by

$$P_{ek} = V_k^2 G_k + \sum_{j\in\mathcal{N}_k} V_k V_j Y_{kj}\sin(\delta_k - \delta_j + \alpha_{kj}). \qquad (3)$$

Here, $Y_{kj} = \sqrt{G_{kj}^2 + B_{kj}^2}$, where G_{kj} and B_{kj} are the (normalized) conductance and susceptance of the generator obtained by Kron-reduction with the loads removed from consideration. $\alpha_{kj} = \arctan(G_{kj}/B_{kj}) = \alpha_{jk}$ represents the lines with losses. Normally, $|\alpha_{kj}|$ is small but not negligible. The value V_k represents the voltage magnitude at the terminal of the k^{th} generator which is assumed to be constant. $\mathcal{N}_k$ is the set of neighboring generators of the k^{th} generator.

Substituting (3) into (2), we obtain the lossy model of the multimachine power systems in the form (1):

$$m_k\ddot{\delta}_k + d_k\dot{\delta}_k = P_k - \sum_{j\in\mathcal{N}_k} a_{kj}\sin(\delta_k - \delta_j + \alpha_{kj}) \qquad (4)$$

where $P_k = P_{mk} - V_k^2 G_k$ and $a_{kj} = V_k V_j Y_{kj}$. The desired operating point of this is unambiguously characterized by the angle differences $\delta_{kj}^* = \delta_k^* - \delta_j^*$ that solve the following system of power-flow like equations:

$$\sum_{j\in\mathcal{N}_k} V_k V_j Y_{kj}\sin(\delta_{kj}^* + \alpha_{kj}) = P_k \qquad (5)$$

Then, the set of swing equations (4) is equivalent with

$$m_k\ddot{\delta}_k + d_k\dot{\delta}_k = -\sum_{j\in\mathcal{N}_k} a_{kj}\big(\sin(\delta_{kj} + \alpha_{kj}) - \sin(\delta_{kj}^* + \alpha_{kj})\big) \qquad (6)$$

Formally, we consider the following problem.

> **Synchronization stability:** *Estimate the region of attraction of the stable equilibrium point* $\delta^* = [\delta_1^*, ..., \delta_n^*, 0, ..., 0]^T$, *i.e. the set of initial conditions* $\{\delta_k(0), \dot{\delta}_k(0)\}_{k=1}^n$ *starting from which the system* (6) *converges to the equilibrium* δ^*.

To address this problem we use a sequence of techniques originating from nonlinear control theory that are most naturally applied in the state space representation of the system. Hence, we view the multimachine power systems (6) as a system with the state space vector $x = [x_1^T, x_2^T]^T$, in which x_1 is the vector of angle deviations from equilibrium, $x_1 = [\delta_1 - \delta_1^*, ..., \delta_n - \delta_n^*]^T$, and x_2 is the vector of angular velocities, $x_2 = [\dot{\delta}_1, ..., \dot{\delta}_n]^T$. Let $M_{n\times n} = \mathrm{diag}(m_1, ..., m_n), D_{n\times n} = \mathrm{diag}(d_1, ..., d_n)$. We define the block diagonal matrix Z of size $n \times 2|\mathcal{E}|$ as $Z = \mathrm{diag}(Z_1, ..., Z_n)$, where $Z_k = [(a_{kj})_{j\in\mathcal{N}_k}]$. Let the matrix E be the $2|\mathcal{E}| \times n$ matrix such that $E[\delta_1, ..., \delta_n]^T = [(\delta_1 - \delta_j)_{j\in\mathcal{N}_1}, ..., (\delta_n - \delta_j)_{j\in\mathcal{N}_n}]^T$. Define the vector of nonlinearity F of size $2|\mathcal{E}|$ as $F(Ex_1) = [(\sin(\delta_{1j} + \alpha_{1j}) - \sin(\delta_{1j}^* + \alpha_{1j}))_{j\in\mathcal{N}_1}, ..., (\sin(\delta_{nj} + \alpha_{nj}) - \sin(\delta_{nj}^* + \alpha_{nj}))_{j\in\mathcal{N}_n}]$.

With these notations, the set of equations (6) can be rewritten in a compact form as follows:

$$\dot{x} = Ax - BF(Cx), \qquad (7)$$

where

$$A = \begin{bmatrix} O_{n\times n} & I_{n\times n} \\ O_{n\times n} & -M^{-1}D \end{bmatrix},$$
$$B = [O_{n\times 2|\mathcal{E}|} \;\; M^{-1}Z]^T, C = [E \;\; O_{2|\mathcal{E}|\times n}].$$

Here, O represents the zero matrix and $I_{n\times n}$ the identity matrix of size $n \times n$. The key advantage of this state

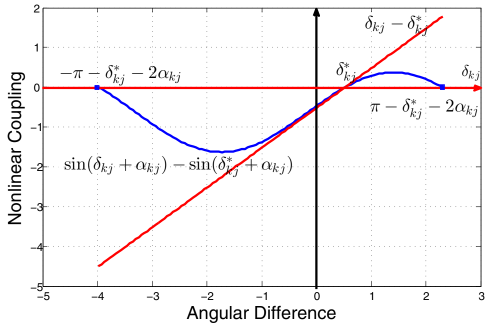

Fig. 1. Bounding of nonlinear sinusoidal interaction by two linear functions as described in (8)

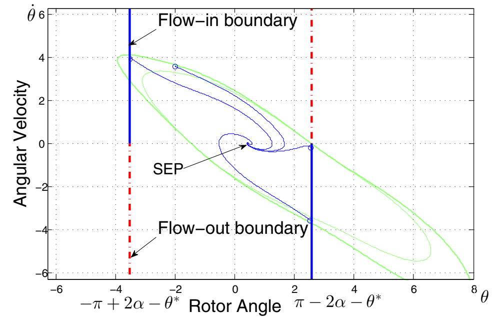

Fig. 2. Estimations of stability region of lossy power system by the LFF method. Stability region estimates are intersection of the Lyapunov function sublevel sets (green solid lines) and the flow-in boundary of the polytope $\mathcal{P}$ defined by inequalities $-\pi + 2\alpha - \delta^* \leq \delta \leq \pi - 2\alpha - \delta^*$.

space representation of the system is the clear separation of nonlinear terms that are represented as a "diagonal" vector function composed of simple univariate functions applied to individual vector components. This simplified representation of nonlinear interactions allows us to naturally bound the nonlinearity of the system by linear functions, as will be shown in the next section.

III. Lyapunov Functions Family Approach

This paper proposes a family of Lyapunov functions to certify the synchronization stability for the system (7). The construction of this Lyapunov functions family is based on the linear bounds of the nonlinear couplings which are clearly separated in the state space representation (7). From Fig. 1, we observe that

$$\begin{aligned} 0 &\leq (\delta_{kj} - \delta_{kj}^*)(\sin(\delta_{kj} + \alpha_{kj}) - \sin(\delta_{kj}^* + \alpha_{kj})) \\ &\leq (\delta_{kj} - \delta_{kj}^*)^2, \end{aligned} \tag{8}$$

for any $-\pi - 2\alpha_{kj} \leq \delta_{kj} + \delta_{kj}^* \leq \pi - 2\alpha_{kj}$. Hence, for any $|\delta_{kj} + \delta_{kj}^*| \leq \pi - 2\alpha_{kj}$, we have the nonlinear bounds (8) for both nonlinear couplings corresponding to δ_{kj} and δ_{jk}.

Exploiting this nonlinearity bounding, we propose to use the convex cone of Lyapunov functions defined by the following system of Linear Matrix Inequalities for positive, diagonal matrices K, H of size $2|\mathcal{E}| \times 2|\mathcal{E}|$ and symmetric, positive matrix Q of size $2n \times 2n$:

$$\begin{bmatrix} A^T Q + QA & R \\ R^T & -2H \end{bmatrix} \leq 0, \tag{9}$$

where $R = QB - C^T H - (KCA)^T$. For every pair Q, K satisfying (9) the corresponding Lyapunov function is:

$$\begin{aligned} V(x) = \frac{1}{2} x^T Q x &- \sum K_{\{k,j\}} \cos(\delta_{kj} + \alpha_{kj}) \\ &- \sum K_{\{k,j\}} \delta_{kj} \sin(\delta_{kj}^* + \alpha_{kj}). \end{aligned} \tag{10}$$

Here, the summation goes over all the pair $\{k, j\} \in \mathcal{E}$, with differentiating between $\{k, j\}$ and $\{j, k\}$. Note, the Lyapunov functions defined by (10) have the same structure as the energy function, and the energy function is a member of this Lyapunov functions family. However, in this paper to establish the synchronization stability certificate for the system (6) we only exploit the property that these Lyapunov functions are decreasing in a neighborhood of the equilibrium point, instead of decreasing in the whole state space as in the energy method. This makes our method different from the energy method.

By similar proof as in [19], we have the following theorem stating the decay of Lyapunov function in the polytope $\mathcal{P}$ defined by the inequalities $|\delta_{kj} + \delta_{kj}^*| \leq \pi - 2\alpha_{kj}$.

Theorem 3.1: In the polytope $\mathcal{P}$, the Lyapunov function defined by (10) *is decaying along the trajectory of* (7)*, i.e., $V(x(t))$ is decaying whenever $x(t)$ evolves inside $\mathcal{P}$.*

A. Constructions of Invariant Sets

In this section, we propose two techniques to construct the invariant sets of the system (7) inside $\mathcal{P}$. The first approach to construct an invariant set of the system (7) in the polytope $\mathcal{P}$ is based on the minimization of the Lyapunov function $V(x)$. We divide the boundary $\partial\mathcal{P}_{kj}$ corresponding with the equality $|\delta_{kj} + \delta_{kj}^*| = \pi - 2\alpha_{kj}$ into two segments $\partial\mathcal{P}_{kj}^{in}$ and $\partial\mathcal{P}_{kj}^{out}$ where the system trajectory goes in and goes out $\mathcal{P}$. The flow-in boundary segment $\partial\mathcal{P}_{kj}^{in}$ is defined, as in Fig. 2, by $|\delta_{kj} + \delta_{kj}^*| = \pi - 2\alpha_{kj}$ and $\delta_{kj}\dot\delta_{kj} < 0$, while the flow-out boundary segment $\partial\mathcal{P}_{kj}^{out}$ is defined by $|\delta_{kj} + \delta_{kj}^*| = \pi - 2\alpha_{kj}$ and $\delta_{kj}\dot\delta_{kj} \geq 0$.

Now we define the minimization of the function $V(x)$ over the union $\partial\mathcal{P}^{out}$ of the flow-out boundary segments $\partial\mathcal{P}_{kj}^{out}$:

$$V_{\min} = \min_{x \in \partial\mathcal{P}^{out}} V(x), \tag{11}$$

The corresponding invariant set is defined as:

$$\mathcal{R} = \{x \in \mathcal{P} : V(x) < V_{\min}\}. \tag{12}$$

The decay property of Lyapunov function in the polytope $\mathcal{P}$ ensures that the system trajectory cannot meet the boundary segments $\{x : V(x) = V_{\min}\}$ and $\partial\mathcal{P}_{kj}^{out}$ of the set $\mathcal{R}$. By definition, once the system trajectory meets the boundary segment $\partial\mathcal{P}_{kj}^{in}$, it can only go in the polytope $\mathcal{P}$. Hence, the system (7) cannot escape $\mathcal{R}$. We have the following theorem for the convergence property of the system to the stable equilibrium point (similar proof as in [19]).

Theorem 3.2: From any initial state x_0 in the invariant set $\mathcal{R}$ defined by (12)*, the system trajectory will converge to the stable equilibrium point δ^*.*

The second approach to certification of stability does not involve finding the value of $V_{\min}$ at all. We consider a scenario when the initial state x_0 is inside the polytope $\mathcal{P}$, but too far away from the equilibrium δ^* such that the approaches described above fail to find a Lyapunov function certifying $V(x_0) < V_{\min}$. In this case, it is still possible to certify that the trajectory $x(t)$ only evolves inside $\mathcal{P}$. Indeed, let $\lambda > 0$ be a small constant, for example $\lambda = 0.01$. Consider the polytope $\mathcal{Q} \subset \mathcal{P}$, which is defined by the inequalities $|\delta_{kj} + \delta_{kj}^*| \leq \pi - 2\alpha_{kj} - \lambda$. Let $\Phi_{kj}^{\pm}$ be the boundary of $\mathcal{Q}$ corresponding to the equality $\delta_{kj} + \delta_{kj}^* = \pm(\pi - 2\alpha_{kj} - \lambda)$. In order to enforce the system to evolve inside $\mathcal{Q}$, we consider the following optimizations:

$$\begin{aligned} c_{kj}^{\max} &= \max C_{\{k,j\}} Ax \\ \text{subject to: } & V(x) \leq a, x \in \Phi_{kj}^{+}, \\ d_{kj}^{\min} &= \min C_{\{k,j\}} Ax \\ \text{subject to: } & V(x) \leq a, x \in \Phi_{kj}^{-}, \end{aligned} \tag{13}$$

where $a > 0$ is a constant and $V(x)$ is a member of LFF. In Appendix VII-A, we prove the following theorem.

Theorem 3.3: Assume that $c_{kj}^{\max} < 0$ and $d_{kj}^{\min} > 0$ for all pairs $\{k, j\} \in \mathcal{E}$. Then, from any x_0 in the set

$$\mathcal{R}^* = \{x \in \mathcal{Q} : V(x) \leq a\}, \tag{14}$$

the system trajectory $x(t)$ will only evolve in $\mathcal{R}^*$ and converge to the equilibrium point δ^*.

We note that the conditions in Theorem 3.3 hold when a equals the minimum value of $V(x)$ taken over the boundary of the polytope $\mathcal{P}$. To enlarge the stability region $\mathcal{R}^*$ we may utilize some heuristic algorithms in which we increase the value of a from $V_{\min}$ with a small amount ϵ in each step until the conditions in Theorem 3.3 are not satisfied.

Remark 3.1: So far, we have presented two stability certificates to verify if the multimachine power system (2) converges from the initial state x_0 to the stable equilibrium point $[\theta_1^*, ..., \theta_n^*, 0, ..., 0]^T$. According to the first certificate given by Theorem 3.2, we need to check if the initial state x_0 is in the stability region $\mathcal{R}$, i.e., if $x_0 \in \mathcal{P}$ and $V(x_0) < V_{\min}$. By the second certificate given by Theorem 3.3, we need to check if $x_0 \in \mathcal{Q}$ and $c_{kj}^{\max} < 0$ and $d_{kj}^{\min} > 0$ for all pairs $\{k, j\} \in \mathcal{E}$, in which $c_{kj}^{\max}$ and $d_{kj}^{\min}$ are defined by (13) with a replaced by $V(x_0)$.

Remark 3.2: Solutions (Q, K) of the LMIs (9) provide us with a family of Lyapunov functions $V(x)$ and the corresponding estimations of stability region $\mathcal{R}_S(Q, K)$. The best estimation can be obtained as the union of $\mathcal{R}_S(Q, K)$ over all the solutions (Q, K) of the LMIs (9) and all the formulations described.

Remark 3.3: The inscription of the union of stability region $\mathcal{R}(Q, K)$ over all the solutions (Q, K) of the LMIs (9) is computationally difficult since there are usually infinite solutions of the LMIs (9). However, the large cone of possible Lyapunov functions allow us to find a Lyapunov function that is best suited for a given initial state $x_0 \in \mathcal{P}$ or family of initial states. We can apply the stability certificate in Theorem 3.2 and use the same algorithm in [19] for the adaptation of Lyapunov functions to a given initial state x_0. For the simple case of 2-bus system, we can see in Fig. 3 that this algorithm allows us to quickly find the suitable Lyapunov function in the family for each initial state.

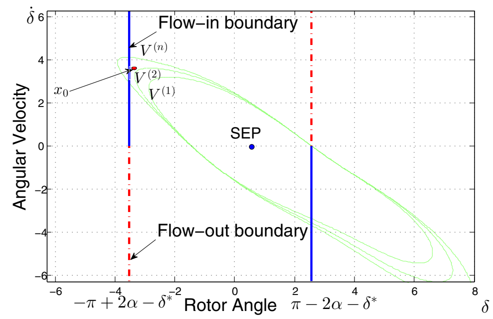

Fig. 3. Adaptation of the Lyapunov functions to the contingency scenario

IV. Robust Stability of Uncertain Power Grids

For practical applications it is desirable to construct Lyapunov functions that certify the stability of the system even if the vector of mechanical torques P_k is not known in advance. As such, the equilibrium calculated by (5) is also unknown. This makes the stability certification in the previous section difficult to verify since the Lyapunov function (10) is dependent on the equilibrium. In this section, we extend the stability certificates in Theorems 3.2 and 3.3 and present techniques to certify the synchronization stability for a set of unknown equilibria. The main motivation to consider this problem is that, in practice P_k is changing in time, and thus the robust stability certificate for a set of equilibria may be utilized to "off-line" certify the stability of the system without repeating the stability assessment in each time step.

We also note that in practice, when the system parameters change, then the bus angles at the stable operating point are usually still close to each other, i.e, the angular differences at the stable operating point are small regardless of the parameter changes. As such, we consider robust stability for the set of equilibria whose angular differences are small. Formally, we consider the following problem:

> **Robust stability:** *Certify the stability of the system* (4)*, in which the mechanical torques P_k are unknown such that the stable equilibrium point $\delta^* = [\delta_1^*, ..., \delta_n^*, 0, ..., 0]^T$ is in the polytope Θ defined by the inequalities $|\delta_{kj}^*| \leq \Delta_{kj}$, where $\Delta_{kj} > 0$ is a constant.*

First, we construct a polytope in which the Lyapunov function is decreasing for any equilibrium point in the set Θ. This polytope is actually the common set of the polytope $\mathcal{P}(\delta^*)$ corresponding to each equilibrium δ^*. Note, that the matrices A, B, C in (7) are independent on the mechanical torques P_k. Hence, the matrices Q, K obtained by solving (9) are also independent on P_k. The Lyapunov function (10) is now a function of two variables $\bar{x} = [\delta_1, ..., \delta_n, \dot{\delta}_1, ..., \dot{\delta}_n]$ and δ^*. By Theorem 3.1 we have $\dot{V}(\bar{x}, \delta^*) \leq 0$ for all $\bar{x}$ in the polytope $\mathcal{P}(\delta^*)$ defined by the inequalities $|\delta_{kj} + \delta_{kj}^*| \leq \pi - 2\alpha_{kj}$. Hence, $\dot{V}(\bar{x}, \delta^*) \leq 0$ for all δ^* in the set Θ and $\bar{x}$ in the polytope $\underline{\mathcal{P}}$ defined by the inequalities $|\delta_{kj}| \leq \pi - 2\alpha_{kj} - \Delta_{kj}$. We note that in practice α_{kj} is small and $|\delta^*_{kj}| < \pi/2$, i.e. $\Delta_{kj} < \pi/2, \forall\{k,j\}$. Hence, $\pi - 2\alpha_{kj} - \Delta_{kj}$ is around $\pi/2$, and thus the polytope $\underline{\mathcal{P}}$ cover most contingency scenarios in practice, where δ_{kj} is kept to be less than $\pi/2$ by actions of protective relays.

We now present the robust stability certificates based on the stability certificate given in Theorem 3.2. The proof of this lemma is provided in Appendix VII-B.

Lemma 4.1: Consider the system (4) *whose the stable equilibrium point δ^* is unknown, but is in the polytope Θ. Consider an initial state $\bar{x}_0$ in the polytope $\underline{\mathcal{P}}$. Suppose that there exist matrices Q, K satisfying* (9) *and*

$$\min_{\bar{x}\in\partial\underline{\mathcal{P}}} \Big[\frac{1}{2}(\bar{x}^T Q\bar{x} - \bar{x}_0^T Q\bar{x}_0) - \sum K_{\{k,j\}}(\cos(\delta_{kj}+\alpha_{kj}) - \cos(\delta_{kj0}+\alpha_{kj}))\Big] > \max_{\bar{x}\in\partial\underline{\mathcal{P}},\delta^*\in\Delta} \Big[\delta^{*T} Q(\bar{x}-\bar{x}_0) + \sum K_{\{k,j\}}(\delta_{kj}-\delta_{kj0})\sin(\delta^*_{kj}+\alpha_{kj})\Big]. \quad (15)$$

Then, the system will converge from the initial state $\bar{x}_0$ to the equilibrium point δ^* for any $\delta^* \in \Delta$.

Also, we have the robust stability certificate based on Theorem 3.3. Let $\lambda > 0$ be a small constant. Consider the polytope $\underline{\mathcal{Q}} \subset \underline{\mathcal{P}}$, which is defined by the inequalities $|\delta_{kj}| \leq \pi - 2\alpha_{kj} - \Delta_{kj} - \lambda$. Let $\Psi^{\pm}_{kj}$ be the boundary of the polytope $\underline{\mathcal{Q}}$ corresponding to the equality $\delta_{kj} = \pm(\pi - 2\alpha_{kj} - \Delta_{kj} - \lambda)$. For the initial state $\bar{x}_0$ in the polytope $\underline{\mathcal{Q}}$, we consider the following optimizations:

$$\begin{aligned} c^{\max}_{kj} &= \max C_{\{k,j\}} A\bar{x} \quad (16)\\ \text{subject to: } & V(\bar{x},\delta^*) \leq V(\bar{x}_0,\delta^*), \bar{x}\in\Psi^+_{kj}, \delta^*\in\Delta,\\ d^{\min}_{kj} &= \min C_{\{k,j\}} A\bar{x}\\ \text{subject to: } & V(\bar{x},\delta^*) \leq V(\bar{x}_0,\delta^*), \bar{x}\in\Psi^-_{kj}, \delta^*\in\Delta. \end{aligned}$$

Here, $V(x,\delta^*)$ is a member of LFF with the corresponding matrices Q, K obtained by solving the LMIs (9). We have the following theorem for the robust stability of the system (4), the proof of which is similar to Theorem 3.3 and omitted.

Lemma 4.2: Consider the system (4) *whose the stable equilibrium point δ^* is unknown, but is in the polytope Θ. Consider an initial state $\bar{x}_0$ in the polytope $\underline{\mathcal{Q}}$. Assume that the optimum values defined in* (16) *satisfy that $c^{\max}_{kj} < 0$ and $d^{\min}_{kj} > 0$ for all pairs $\{k,j\} \in \mathcal{E}$. Then, the system trajectory $x(t)$ will only evolve in the polytope $\underline{\mathcal{Q}}$ and converge to the equilibrium point δ^* for any $\delta^* \in \Delta$.*

V. Simulation results

The effectiveness of the LFF approach can be most naturally illustrated on a classical 2-bus with easily visualizable state-space regions. This system is described by a single 2-nd order differential equation: $m\ddot{\delta} + d\dot{\delta} + a\sin(\delta+\alpha) - P = 0$. For this system, $\delta^* = \arcsin(P/a) - \alpha$ is the only stable equilibrium point (SEP). For numerical simulations, we choose $m = 1$ p.u., $d = 1$ p.u., $a = 0.8$ p.u., $P = 0.4$ p.u., $\alpha = 0.05$ and thus $\delta^* = \pi/6 - 0.05$. Figure 2 shows

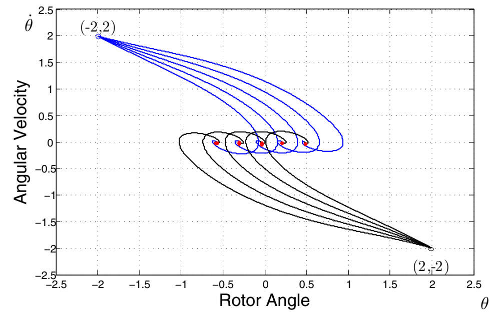

Fig. 4. Robust stability of the contingency scenario $\{\delta_0 = -2, \dot{\delta}_0 = 2\}$ when the SEP is unknown and in the set $-\pi/6 - 0.05 \leq \delta^* \leq \pi/6 - 0.05$.

the stability regions estimated by the LFF approach. It can be seen that the proposed method can certify stability of the lossy power system for a broad set of contingency scenarios. Figure 3 shows the adaptation of the Lyapunov function to the contingency scenario defined by the initial state x_0. It can be seen that the algorithm results in Lyapunov functions providing increasingly large stability regions until we obtain one stability region containing x_0.

Consider the case when the mechanical input P is unknown and in the set $-0.4 \leq P \leq 0.4$. The equilibrium δ^* then belongs to the set $-\pi/6 - \alpha \leq \delta^* \leq \pi/6 - \alpha$. By the robust stability certificate in Lemma 4.1, it can be checked that the contingency defined by the initial state $\{\delta_0 = -2, \dot{\delta}_0 = 2\}$ is stable with respect to any equilibrium point in the set $-\pi/6 - \alpha \leq \delta^* \leq \pi/6 - \alpha$. Figure 4 confirms this anticipation.

VI. Conclusions

This paper applied the LFF method to certify the synchronization stability of lossy power grids, which is impossible by the standard energy methods. The proposed method was based on constructing a family of generalized classical energy functions and adapting these functions to the initial states. Unlike energy function and its variations, these Lyapunov functions are only decreasing in a finite polytope around the stable equilibrium point, but can still certify a broad set of fault and contingency scenarios. We presented optimization-based techniques to explicitly construct the stability certificates. We also showed that the proposed method is well applicable for uncertain power grids with unknown equilibrium points. We solved this problem by providing robust stability certificates for a set of equilibrium points. Such certificates are however conservative, and improvement of the method should be made in the future.

VII. Appendix

A. Proof of Theorem 3.3

Consider an initial state x_0 in the set $\mathcal{R}^*$. We note that the set $\mathcal{R}^* = \mathcal{Q} \cap \{x : V(x) \leq a\}$. Hence, the boundary of the set $\mathcal{R}^*$ includes the segment $\{x \in \mathcal{R}^* : V(x) = a\}$ and the segments $\{x \in \mathcal{R}^* \cap \Phi^{\pm}\}$. Since $\dot{V}(x) \leq 0$ for all $x \in \mathcal{Q} \subset \mathcal{P}$, the system trajectory cannot escape the set $\mathcal{R}^*$ through the segment $\{x \in \mathcal{R}^* : V(x) = a\}$.

Consider the segments $\{x \in \mathcal{R}^* \cap \Phi^{\pm}\}$. On $\Phi^{\pm}$, we have $\delta_{kj} + \delta^*_{kj} = \pm(\pi - 2\alpha_{kj} - \lambda)$. Note that

$$\dot{\delta}_{kj} = C_{\{k,j\}}(Ax - BF(Cx)) = C_{\{k,j\}}Ax \qquad (17)$$

Since $c^{\max}_{kj} < 0, d^{\min}_{kj} > 0$ for all pairs $\{k, j\} \in \mathcal{E}$, we conclude that $\dot{\delta}_{kj} < 0$ for all x in the segment $\{x \in \mathcal{R}^* \cap \Phi^+\}$ and $\dot{\delta}_{kj} > 0$ for all x in the segment $\{x \in \mathcal{R}^* \cap \Phi^-\}$. Hence, the system trajectory $x(t)$ cannot escape the set $\mathcal{R}^*$ through the boundary $\{x \in \mathcal{R}^* \cap \Phi^{\pm}\}$. This means that once the system trajectory meets the boundary $\{x \in \mathcal{R}^* \cap \Phi^{\pm}\}$, it will go back $\mathcal{R}^*$. Therefore, $x(t)$ only evolves within $\mathcal{R}^*$.

From Theorem 3.1 and that $\mathcal{R}^* \subset \mathcal{Q} \subset \mathcal{P}$, we have $\dot{V}(x(t)) \leq 0$ for all t. By LaSalle's Invariance Principle, we conclude that $x(t)$ will converge to the set $\{x : \dot{V}(x) = 0\}$, which means that the system trajectory will converge to the stable equilibrium point δ^* or to some point x^* lying on the boundary of $\mathcal{P}$. But $x(t)$ only evolves in the polytope $\mathcal{Q}$, which is strictly inside the polytope $\mathcal{P}$. Therefore, the system will converge to the stable equilibrium point δ^*.

B. Proof of Lemma 4.1

Since $\dot{V}(\bar{x}, \delta^*) \leq 0$ for all $\bar{x} \in \underline{\mathcal{P}}$ and $\delta^* \in \Delta$, Theorem 3.2 ensures that the system will converge from the initial state $\bar{x}_0$ to the equilibrium point δ^* if $\min_{\bar{x} \in \partial \underline{\mathcal{P}}} V(\bar{x}, \delta^*) > V(\bar{x}_0, \delta^*)$ for all $\delta^* \in \Delta$. Note, the Lyapunov function (10) is expressed as a function of $\bar{x}$ and δ^*:

$$\begin{aligned}
&V(\bar{x}, \delta^*) = 0.5(\bar{x} - \delta^*)^T Q(\bar{x} - \delta^*) \\
&- \sum K_{\{k,j\}}(\cos(\delta_{kj} + \alpha_{kj}) + \delta_{kj} \sin(\delta^*_{kj} + \alpha_{kj})) \\
&= 0.5\bar{x}^T Q\bar{x} + 0.5\delta^{*T} Q\delta^* - \delta^{*T} Q\bar{x} \\
&- \sum K_{\{k,j\}}(\cos(\delta_{kj} + \alpha_{kj}) + \delta_{kj} \sin(\delta^*_{kj} + \alpha_{kj})) \qquad (18)
\end{aligned}$$

As such

$$\begin{aligned}
&\min_{\bar{x} \in \partial \underline{\mathcal{P}}} V(\bar{x}, \delta^*) - V(\bar{x}_0, \delta^*) = \min_{\bar{x} \in \partial \underline{\mathcal{P}}} \Big[0.5(\bar{x}^T Q\bar{x} - \bar{x}_0^T Q\bar{x}_0) \\
&- \sum K_{\{k,j\}}(\cos(\delta_{kj} + \alpha_{kj}) - \cos(\delta_{kj0} + \alpha_{kj})) - \\
&\big\{\delta^{*T} Q(\bar{x} - \bar{x}_0) + \sum K_{\{k,j\}}(\delta_{kj} - \delta_{kj0}) \sin(\delta^*_{kj} + \alpha_{kj})\big\}\Big] \\
&\geq \min_{\bar{x} \in \partial \underline{\mathcal{P}}} \Big[0.5(\bar{x}^T Q\bar{x} - \bar{x}_0^T Q\bar{x}_0) \\
&- \sum K_{\{k,j\}}(\cos(\delta_{kj} + \alpha_{kj}) - \cos(\delta_{kj0} + \alpha_{kj}))\Big] - \\
&\max_{\bar{x} \in \partial \underline{\mathcal{P}}} \Big[\delta^{*T} Q(\bar{x} - \bar{x}_0) + \sum K_{\{k,j\}}(\delta_{kj} - \delta_{kj0}) \sin(\delta^*_{kj} + \alpha_{kj})\Big]
\end{aligned}$$

Hence, if (15) holds, then we have $\min_{\bar{x} \in \partial \underline{\mathcal{P}}} V(\bar{x}, \delta^*) > V(\bar{x}_0, \delta^*)$ for all $\delta^* \in \Delta$, and the system is robustly stable.

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