Random load fluctuations and collapse probability of a power system operating near codimension 1 saddle-node bifurcation

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***Abstract*—For a power system operating in the vicinity of the power transfer limit of its transmission system, effect of stochastic fluctuations of power loads can become critical as a sufficiently strong such fluctuation may activate voltage instability and lead to a large scale collapse of the system. Considering the effect of these stochastic fluctuations near a codimension 1 saddle-node bifurcation, we explicitly calculate the autocorrelation function of the state vector and show how its behavior explains the phenomenon of critical slowing-down often observed for power systems on the threshold of blackout. We also estimate the collapse probability/mean clearing time for the power system and construct a new indicator function signaling the proximity to a large scale collapse. The new indicator function is easy to estimate in real time using data from PMU and SCADA information about power load fluctuations on the nodes of the grid. We discuss control strategies leading to the minimization of the collapse probability.**

***Index Terms*—Blackout prevention, emergency control, phasor measurements, power system stability, voltage stability, wide-area measurements and control.**

I. Introduction

It is well known that small stochastic fluctuations of power load, although usually negligible in the vicinity of a stable operating point, may potentially lead to a large scale cascading failure if the power system operates close to a saddle-node bifurcation point [1], [2]. Early detection and mitigation of such failures is a problem of utmost importance in contemporary world of constantly increasing power demand, where power grids often operate in a precritical regime.

Statistical properties of aggregated power load, their influence on static and dynamic characteristics of power systems remained the subject of extensive studies for the last three decades, see [1] for the review. A very considerable attention has been given to stability analysis of power networks based on Lyapunov theory of dynamical systems, as it was recognized early that the proximity of an operating point to saddle-node and/or Hopf bifurcations of power systems implies proximity to the threshold of instability of the base state [3]. In particular, saddle-node bifurcations were associated to the phenomenon of voltage collapse and loss of synchronism [4]. Correspondingly, a multitude of stability criteria based on the estimation of global Lyapunov functions of power systems have been introduced (see for example classical papers [5], [6], [7]), many corresponding indicator functions being already used in industry for preventive control and dynamic security assessment of power grids [1]. Despite its popularity, energy function analysis of power systems is not entirely free of drawbacks: (a) typically, a knowledge of the values of system variables on all nodes of the grid is needed to estimate the global Lyapunov function in a given operating regime, and (b) even if such tremendous amount of data is available in real time, the estimation process has a high computational cost, which becomes especially critical in the proximity to a large scale collapse.

Assuming in the present work that the operating point of the power system under consideration is close to a saddle-node bifurcation, we extend the standard approach to Lyapunov stability analysis of power systems, taking into account that for a typical power grid without any specific structural symmetries the center manifold is one-dimensional, i.e., saddle-node bifurcation has codimension 1. This observation allows us to explicitly calculate the autocorrelation function of system variables in the operating regime near the bifurcation point and find an approximate expression for the mean clearing time/probability of a large scale failure of the power system. Using the latter, we construct a new indicator function of proximity to the voltage collapse/loss of synchronism, which is significantly easier to estimate in real time than the global Lyapunov function.

This manuscript is organized as follows. In Sec. II we review the structure-preserving model used in the present paper to describe dynamic behavior of system variables in the vicinity of the saddle-node bifurcation point. The autocorrelation function of system variables is explicitly calculated in the Sec. III, while a new estimate for the mean clearing time is given in the Sec. IV. In the Sec. V we perform the validation of our model by checking the derived formulae against numerical simulations of the IEEE 39 bus (New England) power system described in [8]. Finally, the Sec. VI is devoted to discussion of control strategies for minimizing the collapse probability and conclusions.

II. The model

To describe dynamics of system variables in the vicinity of a saddle node bifurcation point, we use the system of coupled swing equations on (P, V) nodes of the grid

$$\frac{H_i}{\pi f_0}\frac{d^2\theta_i}{dt^2}+\alpha_i\frac{d\theta_i}{dt}=\sum_{j\sim i}\mathcal{Y}_{ij}V_iV_j\sin(\theta_i-\theta_j-\gamma_{ij})+P_{m,i} \quad (1)$$

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and power flow equations on (P,Q) nodes

$$P_{0,i}+\alpha_{p,i}\dot{\theta}_i+\beta_{p,i}V_i+T_{p,i}\dot{V}_i=\sum_{j\sim i}\mathcal{Y}_{ij}V_iV_j\sin(\theta_i-\theta_j-\gamma_{ij}), \tag{2}$$

$$Q_{0,i}+\alpha_{q,i}\dot{\theta}_i+\beta_{q,i}V_i+T_{q,i}\dot{V}_i=\sum_{j\sim i}\mathcal{Y}_{ij}V_iV_j\cos(\theta_i-\theta_j-\gamma_{ij}), \tag{3}$$

where as usual θ_i is a voltage phase on a bus i, H_i is an inertia constant for a generator on the node i, α_i denote frequency controls on the (P,V) nodes with generators, V_i is a voltage magnitude on a bus i (for the (P,V) nodes, $V_i=E_i$). Real and reactive power loads on (P,Q) nodes are generally (weakly changing) functions of frequency $\dot{\theta}_i$, voltage V_i and voltage change rate $\dot{V}_i$. If only dynamics of system variables at the vicinity of a stable operating point is to be considered, it is sufficient to keep first leading orders in the Taylor expansions of real and reactive power loads in powers of their arguments. This corresponds to the structure preserving model of [9] extended to the case of load dependence on the voltage change rate $\dot{V}_i$.

The loads have a fixed power factor k_i. They fluctuate with time, and load fluctuations on different nodes of the grid are statistically independent. We assume that their correlation properties are Gaussian:

$$\langle\delta P_i(t)\delta P_j(t')\rangle=B_i(t-t')\delta_{ij}. \tag{4}$$

Since the load power factor is fixed, fluctuations of reactive loads are related to (4) as

$$\langle\delta Q_i(t)\delta Q_j(t')\rangle=\frac{k_i^2-1}{k_i^2}B_i(t-t')\delta_{ij},$$

so that only fluctuations of δP_i are needed to be considered.

Note that (4) describes the autocorrelation function of *a stationary process*. The property of stationarity (or translational invariance in time) $t\to t+\delta t$, $t'\to t'+\delta t$ is only approximate, as behavior of loads and their random fluctuations features natural cycles (for example, day/night). However, we are only interested to study a short time scale (tens of seconds and minutes) behavior of the state vector, when (4) is perfectly applicable.

After solving the power flow equations, finding the base state and linearizing equations (1), (2), (3) about the stable operating point, one finds the matrix stochastic differential equation (SDE)

$$\mathcal{M}\ddot{x}+\mathcal{D}\dot{x}+\mathcal{K}x=\mathcal{A}x=\delta P. \tag{5}$$

Here x is the system state vector including voltage phases and magnitudes, the matrix $\mathcal{M}$ describes inertial properties of generators connected to the grid, the matrix $\mathcal{D}$ — primary frequency controls on (P,V) nodes as well as frequency and $\dot{V}$ dependence of power loads on (P,Q) nodes, and finally $\mathcal{K}$ is the power flow Jacobian encoding all static properties of the power system. The SDE (5) will be the main subject of our study.

III. Calculation of autocorrelation function

Generally, the autocorrelation function of the system vector x can be found as

$$\langle x(t)x^T(t')\rangle=\mathrm{Re}\int\frac{d\omega}{2\pi}e^{-j\omega(t-t')}\langle x(\omega)x^\dagger(\omega)\rangle, \tag{6}$$

where

$$\langle x(\omega)x^T(-\omega)\rangle=\mathcal{A}^{-1}(\omega)\mathcal{B}(\omega)(\mathcal{A}^\dagger(\omega))^{-1}$$

and the Fourier-transformed system matrix is

$$\mathcal{A}(\omega)=-\mathcal{M}\omega^2+j\mathcal{D}\omega+\mathcal{K}.$$

The correlation properties of fluctuating loads are given by the matrix

$$\mathcal{B}(\omega)=\langle\delta P(\omega)\delta P^\dagger(\omega)\rangle=B(\omega)\mathbf{1}.$$

The value of the integral (6) is determined by the singularities of the integrand in the complex ω plane, which in particular include zeros of $\det\mathcal{A}(\omega)$ and $\det\mathcal{A}(-\omega)$, as well as the singularities of $\mathcal{B}(\omega)$. In the regime of interest, dynamics of system variables is largely dictated by very slow modes (see below). Power loads fluctuate rapidly compared to the time scales of this slow dynamics, their correlation in time described by (4) becomes negligible, and one can simply write $\mathcal{B}(\omega)=\mathcal{B}\cdot\mathbf{1}$, where $B_{ij}=\int_0^{+\infty}dtB_{ij}(t)$.

Naturally, the singularity closest to the real ω axis determines behavior of the autocorrelation function (6) at late times. To identify it, we note that at a saddle-node bifurcation point several eigenvalues of the power flow Jacobian $\mathcal{K}$ vanish. There exists only one such vanishing eigenvalue $\epsilon\to 0$ if the center manifold of the power system is one-dimensional [12]. Performing the eigenvalue decomposition of the inverse power flow Jacobian

$$\mathcal{K}^{-1}=\sum_i b_i\Lambda_i^{-1}a_i^T,$$

where Λ_i are eigenvalues of $\mathcal{K}$ and a_i and b_i are corresponding left and right eigenvectors, we see that if the operating point of the power system is close to the saddle-node bifurcation point, $\mathcal{K}^{-1}$ is approximately given by

$$\mathcal{K}^{-1}=\frac{1}{\epsilon}ba^T+\dots, \tag{7}$$

where ϵ is the eigenvalue of $\mathcal{K}$ vanishing at the bifurcation point. Dots denote the contribution of all the other eigenvalues of $\mathcal{K}$, which is at most of the order $\mathcal{O}(1)$ in powers of ϵ (although for large grids it can be numerically large). We are particularly interested in the situation when the 2-norm $||\frac{1}{\epsilon}ba-\mathcal{K}^{-1}||_2\ll||\mathcal{K}^{-1}||_2$ and the asymptotic representation (7) holds well.

In the case under consideration, the leading singularity of the integrand in (6) coincides with a zero of $\det\mathcal{A}(\omega)$ (or $\det\mathcal{A}(-\omega)$ depending on the sign of the time difference $t-t'$). Such singularity is a simple pole by assumption that the center manifold of the power system is one-dimensional. Solving the equation $\det\mathcal{A}(\omega)=\det(-\mathcal{M}\omega^2+i\mathcal{D}\omega+\mathcal{K})=0$ by means of perturbation theory in ϵ, one finds that the mode determining behavior of the autocorrelation function at late times is given by

$$\omega_0 \approx -\frac{j}{\mathrm{Tr}(\mathcal{D}\mathcal{K}^{-1})} = -\frac{j\epsilon}{a^T\mathcal{D}b} + \mathcal{O}(\epsilon^2). \tag{8}$$

Estimating the integral (6) in the vicinity of this leading singularity, one finally finds (to the leading order in ϵ)

$$\langle x(0)x^T(t)\rangle = \frac{b(a^T\mathcal{B}a)b^T}{2\epsilon a^T\mathcal{D}b}\exp\left(-\frac{\epsilon t}{a^T\mathcal{D}b}\right). \tag{9}$$

Note that in this operating regime near the codimension 1 saddle-node bifurcation point *dynamics* of system variables $x(t)$ is directly related to *static* characteristics of the power system such as the lowest eigenvalue ϵ of the power flow Jacobian.

As follows from (9), the right eigenvector b of the power flow Jacobian $\mathcal{K}$ determines the preferred direction in the phase space of a power system near a codimension 1 saddle-node bifurcation [12], [11]. In turn, the left eigenvector a identifies the nodes, where load fluctuations are important for the stochastic dynamics of the state vector.

As the operating point of the power system approaches the power transfer limit and the lowest eigenvalue ϵ of the power flow Jacobian gets smaller, the rate of decay of the autocorrelation function (9) decreases quickly. This observation is known as the phenomenon of critical slowing-down observed during large scale failures in power grids [13], [14].

Finally, we would also like to emphasize that fluctuations of the leading mode, the component of the system vector x along b direction, become strongly amplified as ϵ decreases, since the amplitude of the leading mode near the bifurcation is proportional to $\epsilon^{-1/2}$. One of the main goals of system control in this regime is to decrease the amplitude of the leading mode by adjusting the values of the matrix element $a^T\mathcal{D}b$ (and, if possible, $a^T\mathcal{B}a$). This can be done using primary frequency control on (P,V) nodes as well as utilizing resources of load following on (P,Q) nodes as we discuss below.

IV. Mean clearing time and probability of voltage collapse

It is also possible to explicitly calculate the mean clearing time for an operating point close[1] to the bifurcation point. Namely, considering a scalar reduced system variable $z(t)$ defined according to

$$z(t) = b^T x(t), \tag{10}$$

one finds the following equation of motion for it:

$$a^T\mathcal{M}b\cdot\ddot{z} + a^T\mathcal{D}b\cdot\dot{z} + \epsilon z + a^T\Gamma bb\cdot z^2 + \ldots = a^T\delta P, \tag{11}$$

where the matrix $\Gamma_{ijk} = \frac{\partial\mathcal{K}_{ij}}{\partial\theta_k}$ is the first derivative of the power flow Jacobian w.r.t. the system variables and $\ldots$ denote higher derivatives of $\mathcal{K}$. The equation (11) is a scalar SDE describing the process of activation of an over-damped unharmonic oscillator. Therefore, one can simply apply the classical result by Kramers [18] to calculate the probability of collapse/the mean clearing time:

$$t_{\mathrm{mct}} \approx \frac{2\pi a^T\mathcal{D}b}{\epsilon}\exp\left(\frac{\epsilon^3 a^T\mathcal{D}b}{3(a^T\Gamma bb)^2(a^T\mathcal{B}a)}\right) \tag{12}$$

Estimates similar to this one were previously found using Lyuapunov stability analysis of power systems (see for example [6], [7]). Yet, the expression (12) is different from these well-known results in one important respect: the mean clearing time (12) depends only on a small number of parameters of the power system, unlike the global Lyuapunov function which is a functional of dynamical characteristics of the power system. This is so because the total number of degrees of freedom of a power system gets enormously reduced in the vicinity of bifurcation, and all the information about the power system becomes aggregated into a small number of quantities. These quantities (such as matrix elements $a^T\mathcal{D}b$ and $a^T\mathcal{B}a$ or the lowest eigenvalue ϵ of the power flow Jacobian) entering (12) can be straightforwardly estimated from the static state analysis using the measurements of the correlation function $\langle x(0)x^T(t)\rangle$ of the state vector (containing the information about the vector b, the matrix element $a^T\mathcal{D}b$ and ϵ) and local SCADA data of power consumption on individual nodes of the grid (which encode information about the matrix $\mathcal{B}$) and therefore do not require *any a priori knowledge* of the power system structure. Such measurements will make it easier to estimate the mean clearing time for the power system in real time using the expression (12).

V. Numerical simulation of IEEE 39 power system

We shall now perform the verification of the model described in the Section II comparing the expression (9) for the correlation function with results of numerical simulations of the IEEE 39 (New England) power system. The strategy is as follows. First, the saddle-node bifurcation point is localized using continuation power flow procedure [15] implemented in PSAT Toolbox for Matlab [16], then it is identified more precisely using MATPOWER 4.1 library for Matlab [17]. An operating point close to the bifurcation is chosen, static power flow equations — solved using MATPOWER, the power flow Jacobian — calculated, and finally stochastic differential equations of the Section II are solved using the implicit Euler algorithm. The number N_r of realizations of the random load fluctuations is 1000 (we consider only a stationary regime where averaging the system variables $x(t)$ over realizations of the statistical ensemble (4) of random loads is expected to be equivalent to averaging of $x(t)$ over time).

The load parameter $\lambda = 2.12$ is chosen (if $\lambda = 1$ corresponds to the operating point described in [8]). This corresponds to a 0.9% displacement in terms of the generated real power as compared to the critical regime.[2] As usual, the bus 31 is a slack bus. The state vector includes voltage phases on the nodes 1-30, 32-39 and voltage magnitudes — on the nodes 1-29. The same $\mathcal{M}$ matrix chosen as presented in [8], with the base frequency $f_0 = 60$ Hz. The $\mathcal{D}$ matrix is

[1] But not too close as will become clear from the subsequent discussion.

[2] The total generated real power at $\lambda_{\mathrm{crit}} \approx 2.13569843$ is approximately 13.67 GW.

diagonal with components $d = 1$ p.u. corresponding to voltage phases on (P, Q) buses, $d = 0.1$ p.u. corresponding to voltage magnitudes on (P, Q) buses and $d = 10$ p.u. — to phases on (P, V) buses. The loads are allowed to fluctuate only on the buses 3, 10 and 21, with the same characteristic amplitudes of fluctuations $\sqrt{B} = 0.1$ p.u. For our choice of λ the smallest eigenvalue of the power flow Jacobian $\mathcal{K}$ is $\epsilon \approx 0.1811$ p.u, with the ratio of 2-norms $||\mathcal{K}^{-1} - \epsilon^{-1} ba||_2 / ||\mathcal{K}^{-1}||_2 \approx 0.0717$. Therefore, the expansion in powers of ϵ is feasible, albeit ϵ is not too small by itself (only smallness of dimensionless quantities dictates where the perturbation theory is applicable).

The numerical results for the autocorrelation function $\langle z(t)z(t')\rangle$ of the reduced system variable $z(t) = b^T x(t)$ and the scalar autocorrelation function $\langle x^T(t)x(t')\rangle$ are presented on the Fig. 1. The full correlation function of the system vector $\langle x^T(t)x(t')\rangle$ clearly follows the one of reduced variable $\langle z(t)z(t')\rangle$ except the initial short interval of time, where contributions from sub-leading modes into the autocorrelation function remain large. This confirms our expectation that fluctuations of system variables orthogonal to the direction of the vector b become suppressed in the vicinity of the saddle-node bifurcation. Note that for any given finite number of realizations N_r the autocorrelation function $\langle x^T(t)x(t')\rangle$ fluctuates stronger than $\langle z(t)z(t')\rangle$ because it also accounts for fluctuations of the system vector orthogonal to the vector b.

For our choice of parameters the value of the amplitude of the correlation function $\langle x^T(t)x(t')\rangle$ is $a^T\mathcal{B}a/2\epsilon a^T\mathcal{D}b \approx 5.96 \cdot 10^{-4}$, while the best fit value recovered from 1 is $5.26 \cdot 10^{-4}$. The autocorrelation decay time $\tau = a^T\mathcal{D}b/\epsilon \approx 17.3$ sec also coincides well with the best fit value $\tau \approx 17.0$ sec and an integral estimate for the average correlation time $\int_0^{t_f} dt \langle x^T(t)x(t')\rangle / \langle x^T(t)x(t)\rangle \approx 18.1$ sec.

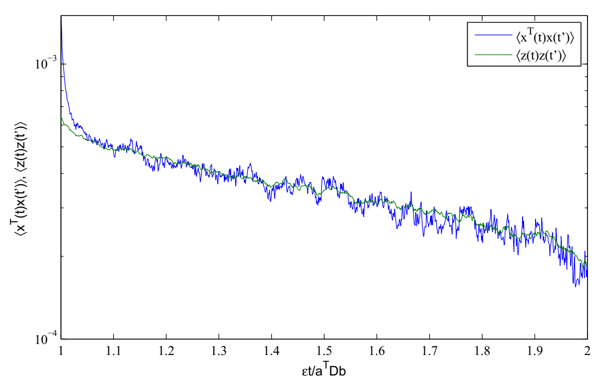

Fig. 1. Relation between the logarithms of the autocorrelation function $\langle z(t)z(t')\rangle$ of the reduced system variable $b^T x(t)$ (red) and the full autocorrelation function $\langle x^T(t)x(t')\rangle$ (blue). The initial moment of time t is chosen to be $t = a^T\mathcal{D}b/\epsilon$.

VI. Discussion and path forward

In the present paper we discuss a general power grid without any particular structural symmetries operating near the power transfer limit of its transmission system. A saddle-node bifurcation present in the phase space of such a system typically has codimension 1, which implies that dynamics of state variables near such bifurcation point is essentially determined by a single degree of freedom which we denoted as reduced system variable. We consider how exactly stochastic power load fluctuations influence this dynamics, explicitly calculating (a) the autocorrelation function of the state vector (9) and (b) the probability of a large scale failure of the power system in the vicinity of the saddle-node bifurcation point/the mean clearing time for the system (12). Although such an operating regime was extensively studied in literature for the last 30 years (see [1] and references therein), both results (a) and (b) are new to our knowledge. Also, using (a) we were able to quantify the phenomenon of critical slowing-down recently discovered in power grids operating on the threshold of a large scale failure [13], [14].

Using the expression for the mean clearing time (12), one can introduce a new indicator function (13) signaling proximity of a power system to a large scale failure. Indeed, as follows from the expression (12), collapse probability is exponentially small when the expression

$$I_c = \frac{\epsilon^3 a^T\mathcal{D}b}{(a^T\Gamma bb)^2 a^T\mathcal{B}a} \gg 1 \tag{13}$$

is large and vise versa. Even when the operating point of the power system is close to the power transfer limit (i.e., ϵ nearly vanishes), the mean clearing time could still be exponentially large if the matrix element $a^T\mathcal{D}b$ is large and/or the matrix element $a^T\mathcal{B}a$ — small.

As follows from (13), a strategy for a system operator to keep the power system close to the power transfer limit of its transmission system while minimizing the probability of a large scale collapse would be to maximally utilize resources of load following [21] (effectively reducing the matrix element $a^T\mathcal{B}a$) and/or adjust, if it is possible, parameters of the primary frequency control on (P, V) nodes (increasing the value of the matrix element $a^T\mathcal{D}b$). Note that $a^T\mathcal{D}b$ cannot be increased indefinitely: such increase for example would imply a growth of the correlation time $\tau = a^T\mathcal{D}b/\epsilon$ leading in turn to possible issues with interference between primary and secondary frequency controls.

There exist several reasons why the operating regime of a power system discussed in the paper seems rather attractive despite the fact that the operating point is located near the threshold of system instability. First of all, in such a regime, the power grid and its transmission system are utilized with the best effectiveness possible: the throughput of the transmission system is maximal, nearly all energy which the grid is capable to produce is consumed (assuming the minimal power losses in the transmission system). Therefore, such operating regime of the grid is optimal from the economical point of view given the system remains under control and the operating costs of control systems are not too high. Second, synchronism of the power system actually holds rather well in the regime under consideration. Indeed, the degree of synchronism is determined by the correlation function of the operating frequency $\langle \delta\omega_i(t)\delta\omega_j(t')\rangle$, which for the nodes i with large representation in the vector b is given to the leading order in

ϵ by

$$\langle \delta\omega_i(t)\delta\omega_j(t')\rangle \approx \frac{\epsilon b_i(a^T\mathcal{B}a)b_j}{(a^T\mathcal{D}b)^3}\exp\left(-\frac{\epsilon|t-t'|}{a^T\mathcal{D}b}\right). \quad (14)$$

Control countermeasures necessary to maximize the value of the indicator function (13) and minimize the collapse probability are the ones which also decrease the amplitude of the correlation function (14).

Finally, since in the regime under consideration dynamics of the power system is approximately described by a single degree of freedom z, the system state can be recovered from the autocorrelation function of the state vector (9). In turn, the latter can be straightforwardly determined using the real time data streams from PMU and SCADA. To extract such characteristics of the power system as the eingenvector b from the measured values of (9), one needs first to find the autocorrelation function of the fluctuating power loads (4). This can be done using SCADA data from multifunction/smart meters installed on separate nodes of the grid. SCADA data in particular contain information about instantaneous values of the loads $P_i(t)$. A rather low sampling rate of SCADA data can be allowed. Indeed, the sampling rate of SCADA data determines the high frequency cutoff for the function $\mathcal{B}(\omega)$. However, as follows from (6), fast fluctuations of δP_i provide subdominant contribution to the function $\mathcal{B}(\omega)$ and the integral (6) in the operating regime of interest.

As for the PMU implementation, if no apriori information about topology and parameters of the power system is given, full observability of the power system by PMU is required for the complete recovery of the leading system mode and parameters of the grid from the autocorrelation function (9). Roughly $N_{PMU} \approx N/4$ PMUs are required to achieve the full observability, where N is the total number of nodes of the grid [22], and there are always multiple options for their placement. For example, full observability of the IEEE 39 power system discussed above can be achieved by installing phasor measurement units on the nodes 3, 6, 10, 16, 20, 23, 29, 37 and 39. On the other hand, if topology of the power grid and its parameters are at least partially known, the nodes of the grid contributing the most to the eigenvectors a and b can be identified, and PMUs are only only needed to be installed to monitor the system state variables on the dominant nodes.

The same applies to the problem of system identification [23]: as we have discussed, measurements of the autocorrelation function $\langle x(0)x^T(t)\rangle$ using PMU data feeds directly provide information about the matrix elements $a^T\mathcal{D}b$ and $a^T\mathcal{B}a$, the lowest eigenvalue of the power flow Jacobian, corresponding left and right eigenvectors a and b. Therefore, such measurements can potentially allow for the approximate recovery of the system matrix in the vicinity of the bifurcation point or at least its part responsible for the dynamics of the leading mode of the state vector. We shall discuss this observation more extensively elsewhere.

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