

# Voltage Multi-stability in Distribution Grids with Power Flow Reversal

Hung D. Nguyen, *Student Member, IEEE*, and Konstantin Turitsyn, *Member, IEEE*

**Abstract**—In future distribution grids with high penetration of distributed generators more stable solutions may appear because of active or reactive power reversal. The systems can operate at very different states, and additional control measures may be required to ensure that it remains at the appropriate point. Under typical scenarios such as subjected to disturbances or load shedding, the system may be trapped at another stable equilibrium without violating any voltage constraints. In other words, the stable equilibrium is viable. This paper aims to analyze how the distribution system converge to the second stable and viable equilibrium. The dynamic simulations are carried out using Wolfram SystemModeler. The results are then analyzed and explained from various viewpoints giving better understanding of the system dynamics. In order to prevent undesired situations, new policies such as regulating reactive power compensations have been proposed. Moreover, a generic dynamic load modeling is developed to capture the essential dynamic characteristics of most of popular loads in power systems.

**Index Terms**—Voltage stability, multistability, power reversal, dynamic load modeling, distributed generation

## I. INTRODUCTION

Renewable energies may help reducing the fossil fuels consumption, but it also introduces various technical challenges to power systems. In particular the power flow reversal produced by distributed generation may cause the distribution systems to operate in very different conditions. There is a need to revisit voltage stability of the systems under such unconventional conditions.

The strong nonlinearities present in the power system determine the existence, multiplicity, and stability of the viable operating points [1], [2]. The nonlinear control loops inside individual system components are responsible for the voltage collapse [3]–[5] and loss of synchrony phenomena [6] that have caused some of the most severe blackouts in the recent history. Generally, the power flow equations that are commonly used for the description of steady states of the power system [7] may have multiple solutions [8], but in typical operating conditions, there always exists a high voltage solution that is considered a normal operating point [9].

The power flow equations solutions manifold has been studied rather extensively in the context of transmission grids. The existence of multiple solutions has been probably first identified in [10]. Series of works in the 80s have explored the relation between the solution manifold and voltage stability of the system [11], [12]. Efficient continuation-type algorithms

for finding the solutions have been proposed in [8]. The most advanced approaches towards the transient stability problem known under the name of “direct methods” are based on the analysis of saddle-node type of solutions of power flow equations [13]. However, most of the works focusing on multiplicity of solutions and their properties have targeted the transmission grids. There was little effort in understanding the solution manifold of distribution grids because vast majority of these networks operate in the conditions when only two solutions coexists for a given set of parameters. The structure of the manifold is very well captured by the textbook two-bus system used to describe the well-known nose-curve.

The structure of the solution manifold in distribution grids in reversed power flow regime is however poorly understood, although there are reasons to believe that it will be very different from the classical nose-curve type manifold. Although the direction of the power flow does not affect the qualitative properties of the solutions in linear (DC power flow) approximation, it becomes important when the nonlinearity is strong. The symmetry between the normal and reversed power flow solutions is broken because the losses that are the major cause of nonlinearity in the power flow equations are always positive. In traditional distribution grids the consumption of power and the losses have the same sign, while in the situation with reversed flows the processes of power injection and thermal losses are competing with each other. This competition may manifest itself in the appearance of new solutions of power flow equations that do not exist in the non-reversed power flow regime. From power engineering perspective this phenomena can be understood with the following argument. In the presence of power flow reversal the voltage is supported to be high enough for low voltage solutions to appear. This phenomena was observed by one of the authors in a recent work [14] but has not been explored in greater details since then.

Even for the traditional nose-curve scenario, the second low voltage solution may be stable under some conditions. This has been recognized for a long time [15]–[17]. Moreover, Venkatasubramanian in [17] noted that the situations in which the systems gets trapped at the second stable equilibrium have been observed. However, the relevance of the low voltage stable equilibrium did not draw much attention and/or has not been studied extensively because this stable equilibrium is neither viable nor convincingly verified numerically due to modelling difficulties. The main problem in the assessment of the stability is the highly complex nature of the load dynamics. The dynamic behavior of the loads are a result nonlinear interactions of millions of heterogeneous components that are

Hung D. Nguyen and Konstantin Turitsyn are with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA, 02139 USA e-mail: hunghtd@mit.edu and turitsyn@mit.edu. (see <http://www.mit.edu/~turitsyn/>).

poorly understood and not fully known to the operator of any given grid. At the same time the dynamic behavior has a direct effect on the stability properties that cannot be directly assessed via static power flow analysis [15]. In this work we attempt to address this problem by introducing a new form of the load models that is consistent with existing models in normal conditions but does not suffer from the convergence problems in abnormal situations.

The paper is organized as follows. In part II introduce the Gröbner basis technique for characterization of the global solution manifold. This technique does not suffer from the convergence problems and is a convenient way of the manifold analysis at least for small enough systems. We use this technique to demonstrate the appearance of new load flow solutions under power flow reversal. In part III we introduce and explain the extension of the traditional load models applicable in the abnormal operating conditions. The load model is used to show the linear stability of some of the new solutions. In section IV we perform dynamic simulations on a three-bus model that illustrate the effect of multi-stability and possibility of system entrapment in undesirable low voltage states. The simulations illustrate both the stability of some regions of low voltage part of the nose curve as well as the effect of multi-stability at high loads.

## II. CHARACTERIZE POWER FLOW SOLUTION MANIFOLD

### A. Power flow solution branches

Traditional approaches for analysis of power flow solutions based on iterative methods like Newton-Raphson, continuation power flow, and their variations [7] are not suitable for identification of all branches of the solution manifold. By construction these methods find only one solution for given set of parameters. There is no systematic way of adjusting the initial conditions that would guarantee that all solutions branches are found. Several techniques have been introduced in the literature for identification of all power flow solutions, the optimal multiplier based method [18], and more recently a holomorphic embedding method [19]. An alternative approach proposed in this manuscript is based on the Gröbner basis technique applicable to any systems of polynomial equations. The introduction to the Gröbner basis approach can be found in [20], [21], while here we introduce the well-known Buchberger's algorithm for solving the set of polynomial equations. To our knowledge this techniques has been applied to power flow equation only in [22]–[24] but has not received wide adoption in the community.

In the following, we use the rectangular form of power flow equations. For the simplicity of explanations we restrict the discussion to radial configurations of  $n$  buses as shown in Figure 1. In this grid, the bus 1 is slack bus with voltage  $V_1 = 1\angle 0$ .

At bus  $i$ ,  $2 \leq i \leq n$ , let i)  $P_i$  and  $Q_i$  be active and reactive power consumed or generated at bus  $i$  (so  $P_i > 0$  corresponds to generation); ii)  $V_i = V_{iRe} + jV_{iIm}$  be the rectangular form

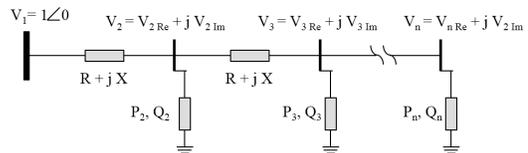


Fig. 1: A radial network

of bus voltage. The power flow equations can be expressed as follows [24]:

$$\begin{aligned} P_i &= \sum_{k=1}^n G_{ik}(V_{iRe}V_{kRe} + V_{iIm}V_{kIm}) \\ &+ \sum_{k=1}^n B_{ik}(V_{kRe}V_{iIm} - V_{iRe}V_{kIm}); \\ Q_i &= \sum_{k=1}^n G_{ik}(V_{kRe}V_{iIm} - V_{iRe}V_{kIm}) \\ &- \sum_{k=1}^n B_{ik}(V_{iRe}V_{kRe} + V_{iIm}V_{kIm}) \end{aligned} \quad (1)$$

where  $Y_{ik} = G_{ik} + jB_{ik}$  is an entry of the bus admittance matrix,  $Y$ .

For well-defined system of polynomial equations the Buchberger's algorithm transforms the system of equations in triangular form where the first equation is univariate polynomial, that can be solved using a wide range of single-variable polynomial solution techniques. Second equation expresses the second variable as a polynomial of the first one, the third equation provides a similar expression for the third variable and so on.

The Buchberger's algorithm is implemented in Wolfram Mathematica [20], [24] and other computer algebra software systems. We use the Mathematica software to transform the power flow equations into the following form:

$$\begin{aligned} \sum_{k=0}^N a_{1k}x_1^k &= 0 \\ \sum_{k=0}^N a_{2k}x_1^k + b_2x_2 &= 0 \\ &\dots \\ \sum_{k=0}^N a_{jk}x_{j-1}^k + b_jx_j &= 0 \\ &\dots \end{aligned} \quad (2)$$

where  $x_{2j} = V_{(n-j)Re}$  and  $x_{2j+1} = V_{(n-j)Im}$  and  $N = 2^{n-1}$ . The coefficients  $a_{jk}$  depend in general on the system parameters, specifically the active and reactive power consumptions, so the first equation establishes the algebraic relation between the voltage magnitude and power consumption levels.

As described above, these equations can be solved one by one to get the voltage components on all the buses. There are many techniques to find all solutions of univariate polynomial equations that allow us to identify all the branches of the power flow equations. The first equation of (2) is first solved for all possible solutions of  $x_1 = V_{nIm}$ . Substituting each of the solutions  $V_{nIm}$  into the second equation of (2) allows to solve it for  $V_{nRe}$ . The iteration procedure continues until

$V_{2Re}$  is found. Finally, the voltage magnitude of any bus can be determined as:

$$|V_i| = \sqrt{V_{iRe}^2 + V_{iIm}^2} \quad (3)$$

### III. DYNAMIC LOAD MODEL

The stability of the different solution branches depends on the dynamical behavior of loads on individual buses. Load modeling is a mature field that has been developed for several decades [25]–[28]. Traditional models of load dynamics are based on combination of differential and algebraic equations for the load state: The algebraic equations in those systems describe the so-called constraint manifold and represent the equilibrium manifold of fast degrees of freedom that relax to equilibrium on time scales below the model resolution. Typically the fast degrees of freedom describe the instantaneous response of the network to the changing loading conditions. In most of the reported models describing the slow dynamics of the system on the tens of seconds-minutes timescales, the instantaneous response of the network is modelled via nonlinear load-flow equations similar to (1). The state of the loads in those models is described by the values of active and reactive power consumption that change according to some dynamic law. Although these models are actively used in the community and have been validated in wide range of normal operating conditions, they are not always applicable to abnormal situations when the solution of algebraic power flow equations does not exist. Similarly, this model can be ill defined in a situation when power flow equations have multiple solutions. These problems are purely mathematical and arise when the implicit modeling assumptions about the dynamics of fast and slow degrees of freedom no longer hold. In order to overcome these problems we introduce an alternative representation of load models that is equivalent to the standard models in normal operating conditions but does not suffer from convergence problems in abnormal situations.

Within our modeling framework the state of each individual load  $k$  is described by its instantaneous conductance  $g_k$  and susceptance  $b_k$  values. The instantaneous values of active and reactive power consumption levels are equal to  $p_k = g_k|V_k|^2$  and  $q_k = b_k|V_k|^2$ . The steady state (static) consumption levels of the loads are given by the functions  $P_k^s(|V_k|, t)$  and  $Q_k^s(|V_k|, t)$  that may represent arbitrary dependence on bus voltage. The explicit dependence of time here represents the change in active/reactive power demand of the load. Also, for renewable generators it may represent the changing environmental conditions, like wind or solar irradiance. We assume that the dynamics of instantaneous conductance and susceptance levels  $g_k(t), b_k(t)$  is described by first order differential equations that depend only on local values of voltage magnitude  $|V_k|$  and :

$$\dot{g}_k = f_k^g(g_k, |V_k|), \quad (4)$$

$$\dot{b}_k = f_k^b(b_k, |V_k|). \quad (5)$$

The steady-state value of active and reactive power imply that the zeros of the functions  $f_k^g, f_k^b$  have to satisfy the following

conditions for any value of the bus voltage  $|V|$ :

$$f_k^g(g_k = \frac{P_k^s(|V|, t)}{|V|^2}, |V|) = 0, \quad (6)$$

$$f_k^b(b_k = \frac{Q_k^s(|V|, t)}{|V|^2}, |V|) = 0. \quad (7)$$

Moreover, we assume that the equilibria  $p_k = P_k^s, q_k = Q_k^s$  are stable for fixed values of bus voltage  $|V_k|$ , which implies that  $\partial f_k^g/\partial g_k, \partial f_k^b/\partial b_k < 0$  for  $p_k = P_k^s, q_k = Q_k^s$ . This implies, that each individual load is stable when connected to a fixed voltage slack bus. Note, that this conditions do not imply the overall stability of the operating point.

One simple example of the load model that satisfies all the conditions above and used throughout this paper is presented below:

$$\tau_{gk} \dot{g}_k = -(g_k|V_k|^2 - P_k^s), \quad (8)$$

$$\tau_{bk} \dot{b}_k = -(b_k|V_k|^2 - Q_k^s). \quad (9)$$

where  $\tau_{gk}$  and  $\tau_{bk}$  are time constants correspond to the load at bus  $k$ .

One can easily check that this model satisfies the equilibrium and stability conditions described above. The main advantage of the proposed model is the simple form of the network equations that have unique solution for any state of the system. This form of network equations is based on the implicit assumption that the instantaneous levels of the load admittances change slower in comparison to the electromagnetic transients in the network. In this case one can model the electromagnetic dynamics of the networks with a simple equilibrium Kirchhoff laws. This assumption naturally holds for most of the known loads and generators, except, probably fast power electronic devices like inverters. However, these devices can be also incorporated in the model by introducing dynamical equations with small time constants.

Note, that our model is just another form of the traditional dynamic load models introduced originally by [26], [29]:

$$\dot{P}_d + f(P_d, V) = g(P_d, V) \dot{V} \quad (10)$$

Here  $P_d$  is the instantaneous power, that is denoted by  $p_k = g_k|V_k|^2$  in our notations and  $V$  is the bus voltage magnitude, referred to as  $|V_k|$  in equations (4,12). The more specific form of these equations, known as exponential recovery model was introduced by Karlsson and Hill in [26], [29]:

$$T_p \dot{P}_d + P_d = P_s(V) + k_p(V) \dot{V} \quad (11)$$

We can recover the model (10) from equation (8) by taking the derivative of  $g_k|V_k|^2$ . This results in the following expression:

$$\dot{p}_k + \frac{p_k - P_k^s(|V_k|)}{\tau_{gk}} |V_k|^2 = 2 \frac{p_k}{|V_k|} \frac{d}{dt} |V_k| \quad (12)$$

### IV. DYNAMIC SIMULATIONS OF A THREE THREE-BUS NETWORK

In this and the second sections, we perform dynamic simulations of two simple networks to show that the two stable equilibria of load dynamic equations may coexist at the same

time, and that the distribution system may become entrapped at the lower voltage equilibrium.

First, we consider a three-bus network as shown in Figure 2 with bus 1 being the slack bus and buses 2 and 3 representing the dynamic loads with distributed generation exporting reactive power. This system could represent the future distribution grids with the inverters of PVs panels participating in voltage regulation (see [30] for further discussions of this proposal). Alternatively, it could represent highly capacitive grid, for example involving long underground cables.

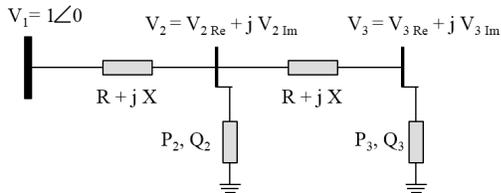


Fig. 2: A three-bus network

The dynamic equations for the loads have the following form:

$$\tau_1 \dot{g} = -(p - P^s) \quad (13)$$

$$\tau_2 \dot{b} = -(q - Q^s) \quad (14)$$

We assume that the steady-state active and reactive power levels don't depend on the bus voltage, so that  $P^s$  and  $Q^s$  depend only on time. In other words, the loads can be classified as constant power loads, that attempt to achieve the given power demand levels  $P^s(t), Q^s(t)$ . This assumption is clearly a simplification of the real loads. However, it is a common assumption in most of the classical voltage stability studies, and also may be a good approximation of power systems with aggressive VAR compensation or power grids interconnected through fast voltage source converters (VSC).

In our simulations we use  $\tau_1 = \tau_2 = 0.07 s$  for the load at bus 2 and  $\tau_1 = \tau_2 = 0.03 s$  for the load at bus 3. The actual values are not very important, we chose them for convenience of presentation, but physically the fast dynamic loads could correspond to power electronics regulating the voltage levels on consumer side. Both of the power lines between buses 1 – 2 and 2 – 3 have the same values of resistance and inductance:  $r = 0.1464 p.u.$  and  $x = 0.5160 p.u.$  The load power consumption levels are defined as:  $P_i = P_{G_i} + P_{L_i}$  and  $Q_i = Q_{G_i} + Q_{L_i}$ ; where  $i$  is the load number,  $i = 2, 3$ .  $P_{G_i}$  and  $Q_{G_i}$  are the active and reactive powers produced from distributed generators at bus  $i$ , whereas,  $P_{L_i}$  and  $Q_{L_i}$  are the active and reactive powers consumed at bus  $i$ . The power levels have negative values if the bus is generating power, whereas, positive values indicate that the bus is consuming power.

#### A. The transient from the high voltage equilibrium to the low voltage one

We consider the following scenario that leads to the entrapment of the system in low voltage equilibrium.

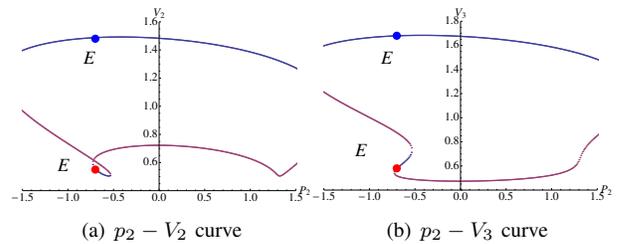


Fig. 3: Voltage multi-stability in a three-bus network

The bus 3 has base demand level  $P_3^s = -0.75 p.u.$  and  $Q_3^s = -0.45 p.u.$  which corresponds to a capacitive load producing active power. The base active and reactive power demands of load 2 are also given as  $P_2^s = -0.7 p.u.$ ,  $Q_2^s = -0.9 p.u.$  While keeping  $Q_2^s, P_3^s, Q_3^s$  fixed to equal to the base level and changing the active demand level at bus 2,  $P_2^s$ , we can plot different  $PV$  curves as in Figure 3 with two stable equilibria. In these plots, the blue dot segments represent stable equilibria and the red dot segments - the unstable ones, all observed for different values of  $P_2^s$ . The large blue dot represents the high voltage stable equilibrium,  $E_H$ , and the large red one marks the low voltage one,  $E_L$ . The following scenario initiates the transition of the system from the high voltage equilibrium to the low voltage one.

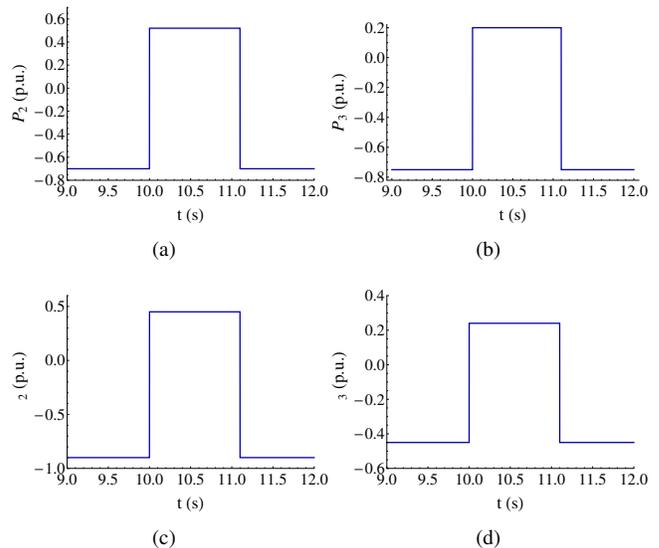


Fig. 4: The power demands at bus 2 and bus 3 during the first disturbance

Assume that initially the system is operating at the high voltage equilibrium,  $E_H$ . In the dynamic simulation, the preferred operating condition can be reached by choosing appropriate initial conditions in the neighborhood of the steady state. A suggested initial condition is  $g_2 = -0.324 p.u.$ ,  $b_2 = -0.416 p.u.$ ,  $g_3 = -0.269 p.u.$ ,  $b_3 = -0.161 p.u.$  The transient of the system is shown in Figure 5 following the blue arrows.

After the system reaches the high voltage equilibrium, a large disturbance occurs, i.e. distributed generation partly is

lost. For example it could represent the cloud covering the PV panels with a shadow. Therefore, the loads change their modes from generating to consuming both active and reactive powers as in Figure 4. As a result, the system starts to diverge from the high voltage stable equilibrium,  $E_H$ , and approach the low voltage one,  $E_L$ . This progress is recorded in Figure 5 following the red arrows. The transient dynamics of the system dies out around  $t \approx 20$  s.

Later, at  $t_d = 20$  s the second disturbance in  $P_2^s$  occurs that changes the demand  $P_2^s$  to some lower value  $P_2^s = -0.56$  p.u. for 2 s. As shown in Figure 6, the system first moves away from the low voltage equilibrium following the blue arrows then returns back to the same equilibrium following the blue dashed arrows. Our numerical experiments with several disturbances with different amplitudes and durations indicate that the low voltage equilibrium is nonlinearly stable.

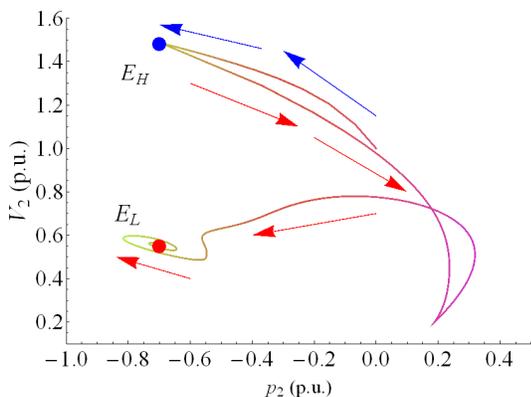


Fig. 5: The PV curve at bus 2,  $t < 20$  s

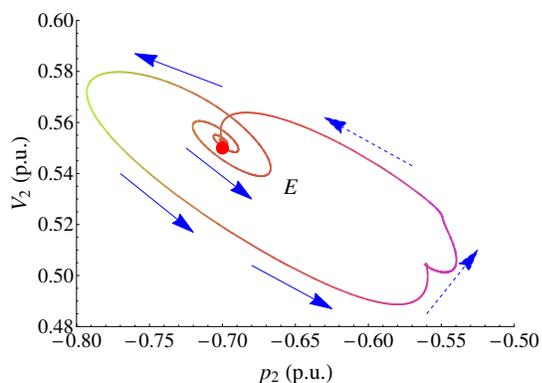


Fig. 6: The PV curve at bus 2,  $20$  s  $\leq t \leq 30$  s

Figure 7 shows how the voltage at bus 2 and bus 3 illustrates changes between two voltage levels.

### B. Recovering to the high voltage stable equilibrium from the low voltage one

In this section, we introduce another scenario which shows how the system can be forced back to the desirable working conditions at the high voltage equilibrium,  $E_H$  after being entrapped at the low voltage stable equilibrium,  $E_L$ .

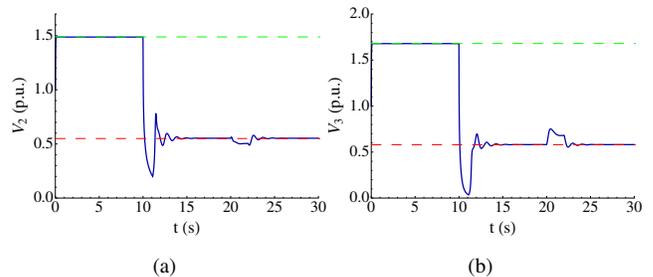


Fig. 7: Voltages at bus 2 and bus 3 for the small second disturbance,  $t \leq 30$  s

From section IV-A, the system is entrapped at the low voltage equilibrium at  $t \approx 20$  s. After that, the second large disturbance happens cause active power demand level at bus 2,  $P_2^s$ , to increase for 2 s. The similar disturbance can happen when PVs generation reduces. Consequently, the system returns to the high voltage equilibrium with the trajectory shown in Figure 8. The instantaneous power consumptions and voltages at bus 2 and bus 3 are shown in Figure 9 and Figure 10.

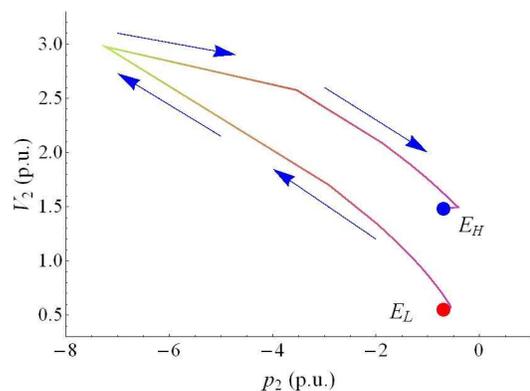


Fig. 8: The PV curve at bus 2,  $20$  s  $\leq t \leq 30$  s

## V. CONCLUSION

In this work we have shown that distribution grids with active or reactive power flow reversal can have multiple stable solutions of the power flow problems. We proposed a novel technique for characterization of all the solution branches based on the Gröbner basis approach. The stability of the low voltage solution branches has been demonstrated with the dynamic simulations based on the new representation of the standard load models that does not suffer from the divergence problems. More analysis is required to establish new regulations that prevent the entrapment in future power grids with high penetration of DG and power electronics.

In the future works we plan to extend the Gröbner basis approach to large systems with the help of reduction and approximation techniques. The extension of this approach could be useful for development of nonlinear dynamic equivalents of large-scale distribution grids and the effects of distributed controls on the overall system stability. Incorporation of dynamic

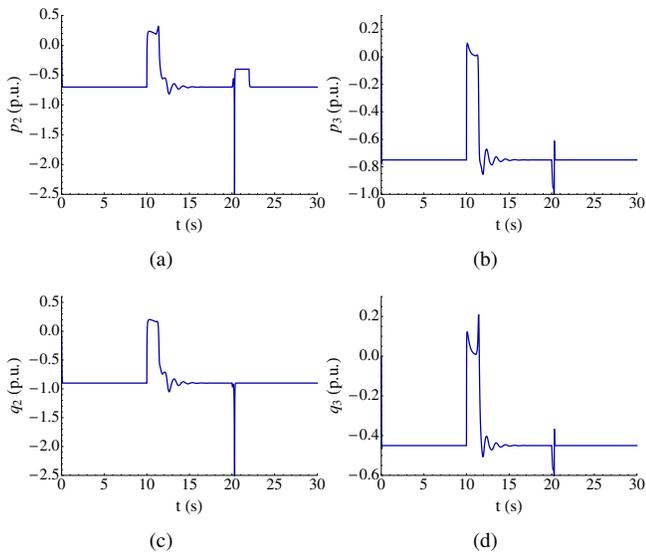


Fig. 9: The power consumptions at bus 2 and bus 3,  $t \leq 30$  s

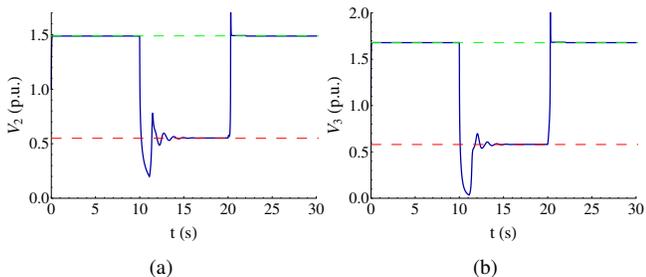


Fig. 10: Voltage at bus 2 and bus 3 for the large second disturbance,  $t \leq 30$  s

stability analysis in the planning stage decision making process may require new stability criteria that guarantee the stability in the presence of load uncertainty. The sufficient criteria can be developed with the help of the more general D-stability framework and will be reported in the future works.

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## REFERENCES

- [1] I. Hiskens, "Analysis tools for power systems-containing with nonlinearities," *Proceedings of the IEEE*, vol. 83, no. 11, pp. 1573–1587, Nov 1995.
- [2] I. Hiskens and D. Hill, "Energy functions, transient stability and voltage behaviour in power systems with nonlinear loads," *Power Systems, IEEE Transactions on*, vol. 4, no. 4, pp. 1525–1533, Nov 1989.
- [3] I. Dobson, H. Glavitsch, C.-C. Liu, Y. Tamura, and K. Vu, "Voltage collapse in power systems," *Circuits and Devices Magazine, IEEE*, vol. 8, no. 3, pp. 40–45, 1992.
- [4] I. Dobson and L. Lu, "Voltage collapse precipitated by the immediate change in stability when generator reactive power limits are encountered," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 39, no. 9, pp. 762–766, 1992.
- [5] T. Van Cutsem and C. Vournas, *Voltage stability of electric power systems*. Springer, 1998, vol. 441.

- [6] J. Machowski, J. Bialek, and J. Bumby, *Power system dynamics: stability and control*. John Wiley & Sons, 2011.
- [7] P. Kundur, *Power System Stability and Control*, New York, 1994.
- [8] W. Ma and J. Thorp, "An efficient algorithm to locate all the load flow solutions," *Power Systems, IEEE Transactions on*, vol. 8, no. 3, pp. 1077–1083, Aug 1993.
- [9] H.-D. Chiang and M. Baran, "On the existence and uniqueness of load flow solution for radial distribution power networks," *Circuits and Systems, IEEE Transactions on*, vol. 37, no. 3, pp. 410–416, 1990.
- [10] A. Klos, S. Mikotajczyk, and A. Kaminaki, "Non-uniqueness and stability of load flows," in *Proc. PSCC VI, Darmstadt, 1978*, pp. 704–710.
- [11] Y. Huang, S. Iwamoto, "Detection of possible voltage collapse buses in stressed power systems based on information from multiple solutions in load flow calculations," *Electrical Engineering in Japan*, vol. 119, no. 1.
- [12] A. Yokoyama and Y. Sekine, "Multisolutions for load flow problem of power system and their physical stability test," *Electrical Engineering in Japan*, vol. 100, no. 5.
- [13] H.-D. Chiang, *Direct methods for stability analysis of electric power systems: theoretical foundation, BCU methodologies, and applications*. John Wiley & Sons, 2011.
- [14] D. Wang, K. Turitsyn, and M. Chertkov, "Distflow ode: Modeling, analyzing and controlling long distribution feeder," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*. IEEE, 2012, pp. 5613–5618.
- [15] T. Overbye, "Effects of load modelling on analysis of power system voltage stability," *International Journal of Electrical Power & Energy Systems*, vol. 16, no. 5, pp. 329–338, 1994.
- [16] S. Corsi and G. Taranto, "Voltage instability - the different shapes of the 'nose'," in *Bulk Power System Dynamics and Control - VII. Revitalizing Operational Reliability, 2007 iREP Symposium*, Aug 2007, pp. 1–16.
- [17] V. Venkatasubramanian, H. Schattler, and J. Zaborszky, "Voltage dynamics: study of a generator with voltage control, transmission, and matched mw load," *Automatic Control, IEEE Transactions on*, vol. 37, no. 11, pp. 1717–1733, Nov 1992.
- [18] S. Iwamoto and Y. Tamura, "A load flow calculation method for ill-conditioned power systems," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-100, no. 4, pp. 1736–1743, 1981.
- [19] A. Trias, "The holomorphic embedding load flow method," in *Power and Energy Society General Meeting, 2012 IEEE*, 2012, pp. 1–8.
- [20] D. A. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 3/e.
- [21] B. Buchberger, "Gröbner bases: A short introduction for systems theorists," in *Computer Aided Systems Theory, EUROCAST 2001*. Springer, 2001, pp. 1–19.
- [22] J. Castro and A. Montes, "Solving the load flow problem using gröbner basis."
- [23] J. Ning, W. Gao, G. Radman, and J. Liu, "The application of the groebner basis technique in power flow study," in *North American Power Symposium (NAPS), 2009*, 2009, pp. 1–7.
- [24] R. G. Kavasseri and P. Nag, "An algebraic geometric approach to analyze static voltage collapse in a simple power system model," in *Fifteenth National Power Systems Conference*, 2008, pp. 482–487.
- [25] C. Concordia and S. Ihara, "Load representation in power system stability studies," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-101, no. 4, pp. 969–977, April 1982.
- [26] D. Karlsson and D. Hill, "Modelling and identification of nonlinear dynamic loads in power systems," *Power Systems, IEEE Transactions on*, vol. 9, no. 1, pp. 157–166, Feb 1994.
- [27] F. J. Meyer and K. Lee, "Improved dynamic load model for power system stability studies," *Power Engineering Review, IEEE*, vol. PER-2, no. 9, pp. 49–50, Sept 1982.
- [28] W. Xu and Y. Mansour, "Voltage stability analysis using generic dynamic load models," *Power Systems, IEEE Transactions on*, vol. 9, no. 1, pp. 479–493, Feb 1994.
- [29] D. J. Hill, M. Pal, X. Wilsun, Y. Mansour, C. Nwankpa, L. Xu, and R. Fischl, "Nonlinear dynamic load models with recovery for voltage stability studies. discussion. authors' response," *IEEE Transactions on Power Systems*, vol. 8, no. 1, pp. 166–176, 1993.
- [30] K. Turitsyn, P. Sulc, S. Backhaus, and M. Chertkov, "Options for control of reactive power by distributed photovoltaic generators," *Proceedings of the IEEE*, vol. 99, no. 6, pp. 1063–1073, 2011.