A Stability-constrained Optimization Framework for Lur'e Systems and Application in Power Grids

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***Abstract*—This paper addresses an issue of obtaining an optimal equilibrium for a Lur'e-type system which can guarantee the stability under some given disturbance. For this purpose, a new optimization framework is proposed. In this framework, the dynamic constraints, i.e. the differential algebraic equations (DAEs), are properly converted into a set of algebraic constraints. The existing methods formulate the stability by discretizing the DAEs, which is computationally intractable for large-scale Lur'e systems like power grids. Dissimilarly, the proposed approach generates the algebraic constraints of stability through Lyapunov theories rather than the discretization of the DAEs. In a transient stability-constrained optimal power flow (TSCOPF) problem of power systems, researchers look for a steady-state operating point with the minimum generation costs that can maintain the system stability under some given transient disturbances. The proposed approach is applied to develop a scalable TSCOPF framework for power systems. The TSCOPF model is tested on the IEEE 118-Bus power system.**

***Index Terms*—Lur'e system, Power system, Quadratic Lyapunov function, Stability-constrained optimization, TSCOPF.**

I. Introduction

STABILITY and optimality are two important aspects of an engineering system that pursued by system operators. On one hand, a basic framework for optimization problems consists of an algebraic objective function which represents the indices to be optimized and a set of algebraic constraints whose solutions (or equilibria) form the feasible set (or the set of condidate equilibria) [1]. Optimization technologies are attractive since the solutions of an optimization problem imply the "best performance" with respect to some given indices [2]. On the other hand, the system dynamics is generally formulated by a set of differential algebraic equations (DAEs). Stability, in brief, describes the system's ability of converging to a stable equilibrium after the disturbance disappears [3]. The connection between stability and optimality is that the equilibrium chosen by the optimization algorithm provides an initial point for the dynamic system trajectories. It is a consensus that, for a given fault, the stability of the system relies heavily on the initial point. An intuitive interpretation is that, with an equilibrium located close to the stability margin, the system is very easy to loss stability. This connection raises an important issue: how to obtain an equilibrium with some indices optimized which can also maintain the system's stability under some given disturbances?

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To obtain such a stability-guaranteed optimal solution, one needs to incorporate the stability criteria as constraints into the optimization framework. Unfortunately, the distinct gap between two mathematical processes prevents the direct implementation of stability constraints in the optimization framework. Consequently, most of the former researches investigated these two problems separately. A few of researchers have attempted to incorporate the DAE constraints into the steady-state optimization framework [4] and [5]. The general idea of these existing methods is discretizing one DAE into a set of algebraic equations with respect to small time steps, which is considered to be computationally intractable for large-scale dynamic systems.

To mitigate this problem, for Lur'e-type dynamic systems, this paper presented a novel framework of stability-constrained optimization which is independent from the basic principle of the existing methods. The proposed approach casts the system's dynamics into algebraic constraints in the steady-state domain based on a recent development of quadratic Lyapunov functions for Lur'e systems using the sector nonlinearity approach and the semidefinite programming (SDP) technique [6]. To better illustrate the proposed idea, we also introduce an application case in power systems, the transient stability-constrained optimal power flows (TSCOPF) [5] and [7].

Transient stability is the ability of the power system to maintain synchronism when subjected to a severe transient disturbance [8]. In transient stability assessment (TSA) of power systems, fault type, location, and clearing-time are pre-determined factors. Hence, what determines the system stability is the initial state. TSCOPF is an attractive technology to obtain an optimal power flow (OPF) [9] solution that can guarantee stability when the system suffers a transient disturbance. Aggressive introduction of renewable generation increases the overall stress of the power system [10], so the stability constraints will likely become the main barrier for transition to clean energy sources. Despite many decades of research, stability assessment is still the most computationally intensive task in power grid operation process. It is even more computationally intractable when stability constraints are considered in optimization problems.

In the past decade, the TSCOPF issues have attracted continual research efforts. In general, the existing TSCOPF methods are based on either DAE-discretization [5], [7], [11], and [12] or iterative algorithms where an independent TSA is required for each iteration [13] and [14]. However, these approaches are still not practical in real-time application due to the heavy computational burden they incur. We found that the developed framework is applicable for constructing an effective TSCOPF model if the dynamic model of power grids can be reformulated into the Lur'e form. Fortunately, our former work shows that it is possible to reformulate the dynamic power system model, which is described by the classical generator model with transfer conductance considered [15] and network preserved [16], into a classical Lur'e form.

The rest of the paper is organized as follow: In Section II, the proposed stability-constrained optimization framework is presented. Section III discusses the application of the proposed approach in TSCOPF of power grids. The novel TSCOPF framework is tested on the IEEE 118-bus system in Section IV. The novelty and limitation of the proposed approach as well as the future research are discussed in detail in the last section.

II. The Stability-constrained Optimization

A. Original Problem Formulations

For a steady-state nonlinear system (1b), we are usually interested in operating it at a point that some given index (1a) is optimized. For instance, power engineers pursue the solutions with the minimum generation costs in the OPF problems. The general formulation for such an optimization problem can be given as (1). Note that additional constraints may be needed due to actual requirements.

$$\min_{x} f(x) \tag{1a}$$
$$\text{s.t. } g(x) = 0 \tag{1b}$$
$$\underline{x} \leq x \leq \overline{x}, \tag{1c}$$

where x denotes the steady-state variable.

If the dynamic behaviors of the nonlinear system (1b) can be captured by the following Lur'e-type autonomous system with a nonlinear term of vanishing disturbance (2), we will develop a novel optimization framework for (1) to guarantee that the obtained solutions are locally asymptotically stable.

$$\dot{y} = Ay + B\phi(Cy) + \begin{cases} 0 & (t = t_0^-, t_c^+ \to \infty) \\ u(y) & (t = t_0^+ \to t_c^-) \end{cases}, \tag{2}$$

where y is a function of time t representing the dynamic-state variable with x as the origin, such that $y(t_0) = 0$, $g(y + x) = Ay - B\phi(Cy)$, and $\phi(0) = 0$; $u(y) = v(y + x)$ and $v(\cdot)$ is a derivable funciton; the nonlinearity $\phi : \mathcal{R} \to \mathcal{R}$ can be bounded by some local sectors. The nonlinear function ϕ is said to be in sector $[\gamma, \beta]$ if, for all $q \in \mathcal{R}$, $p = \phi(q)$ lies between γ and β as shown in Fig. 1.

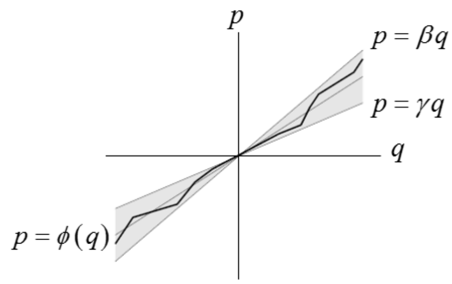


Fig. 1. The concept of sector bounds.

B. Quadratic Lyapunov Function for the Nominal System

The first step of developing the novel stability-constrained optimization framework is to construct a quadratic Lyapunov function for the nominal system of (2). For this purpose, we extend the classical circle criterion [3] and the related approaches presented in [6] to the case where the sector $[\gamma, \beta]$ is valid for bounding the nonlinearity ϕ in the the polytope $\mathcal{P} = \{y | \underline{z} \leq Cy \leq \overline{z}\}$.

Lemma 1. *The Lyapunov function $W(y) = y^T P y$ is decreasing along the trajectory of the nominal system of (2) whenever $y(t)$ is in the set $\mathcal{P}$, if:*
a) the nonlinearity ϕ satisfies the sector condition $(\phi(Cy) - \gamma Cy)^T(\phi(Cy) - \beta Cy) \leq 0$ for all $y \in \mathcal{P}$;
b) the positive definite matrix P satisfies the following linear matrix inequality (LMI)

$$\begin{bmatrix} A^T P + PA - C^T \gamma \beta C & PB + \frac{1}{2}(\gamma + \beta)C^T \\ B^T P + \frac{1}{2}(\gamma + \beta)C & -I \end{bmatrix} \preceq 0. \tag{3}$$

Proof: See Appendix A. □

Note that, to obtain a uniform Lyapunov function for the whole feasible set rather than a single equilibrium, the chosen sector $[\gamma, \beta]$ should be valid for all $y \in \mathcal{P}$. If $\phi(q) = [\phi_1(q_1), \dots, \phi_k(q_k)]^T$ represents multiple nonlinearities, one can customize the sector for each nonlinearity such that $\beta = \text{diag}(\beta_1, \dots, \beta_k)$ and $\gamma = \text{diag}(\gamma_1, \dots, \gamma_k)$.

If the obtained local quadratic Lyapunov function satisfies the following condition

$$\mathcal{P} \subseteq \mathcal{R} = \{y | \dot{W}(y) \leq 0\},$$

it is reasonable to search for the the minimum boundary energy W^{min} along the boundaries of polytope $\mathcal{P}$. We introduce a highly scalable technique of calculating W^{min}, which can be incorporated into the optmization framework, in this subsection. The original searching procedure of W^{min} can be described as

$$\begin{aligned} W^{min} &= \min_i W_i^{min} \\ W_i^{min} &= \min_y y^T P y \\ \text{s.t.} \quad & C_i^T y = z_i \end{aligned} \tag{4}$$

where C_i^T is the i-th row of matrix C, and $z_i = \overline{z}_i$ or $\underline{z}_i$. According to the first-order optimality conditions [17], the optimization problem (4) has a trivial solution $y^* = z_i P^{-1} C_i / (C_i^T P^{-1} C_i)$. Consequently, W^{min} can be obtained through the following simpler way

$$W^{min} = \min_i \{ \frac{\min\{\overline{z}_i^2, \underline{z}_i^2\}}{C_i^T P^{-1} C_i} \}.$$

Generally, in the y-coordinates, $\overline{z}_i = \Delta l$ and $\underline{z}_i = -\Delta l$, where Δl is positive. $\overline{z}_i = C_i^T x + \Delta l$ and $\underline{z}_i = C_i^T x - \Delta l$ in the x-coordinates. Hence, we can search for W^{min} in the x-coordinates though

$$W^{min} = \min_i \{ \frac{\min\{(C_i^T x + \Delta l)^2, (C_i^T x - \Delta l)^2\}}{C_i^T P^{-1} C_i} \}. \tag{5}$$

C. A Scalable Stability-constrained Optimization Framework

At $t = t_0$, system (2) suffers a disturbance and its trajectory $y(t)$ starts deviating from the origin (i.e. the equilibrium point). Let $y(t_c)$ denote the state when the disturbance is cleared. After the disturbance is cleared, the condition for that the system trajectory $y(t)$ can converge back to the origin is explored in this subsection. Based on the minimum Lyapunov value W^{min} obtained in the previous step, we introduce the stability certificate of the nominal system in (2) by applying LaSalle's *invariant principle*.

Lemma 2. *For any fault-cleared state $y(t_c)$ within the set Ω which is defined by*

$$\Omega = \{y(t_c) \in \mathcal{P} \mid W(y(t_c)) < W^{\min}\}, \tag{6}$$

the system trajectory of the nominal system (2) starting from $y(t_c)$ stays in the set Ω for all $t \geq 0$ and eventually will converge to the origin.

Proof: See Appendix B. □

If the fault clearing time t_c is sufficiently short, we can calculate the fault-cleared state using the following Taylor's series

$$\begin{aligned} y(t_c) &= y(t_0) + \frac{\dot{y}(t_0)}{1!}t_c + \frac{\ddot{y}(t_0)}{2!}t_c^2 + \cdots \\ &= 0 + \frac{v(x)}{1!}t_c + \frac{\dot{g}(x) + \dot{v}(x)}{2!}t_c^2 + \cdots, \end{aligned} \tag{7}$$

where $Ay(t_0) - B\phi(Cy(t_0)) = g(x) = 0$ and $u(y(t_0)) = v(x)$ as defined in Subsection II-A.

Constraints (5)-(7) together represent the dynamic stability certificate in the x-domain, i.e. the steady-state domain. By adding constraints (5)-(7) to problem (1), we have a stability-constrained optimization model for the nonlinear system (1b), which is denoted as (**P1**). Due to constraint (5), model (**P1**) is a bilevel optimization problem which is highly intractable. To overcome this issue, we proposed the following optimization model which is denoted as (**P2**)

$$\min_{x, W^{min}} F(x, W^{min}) = f(x) - \epsilon W^{min} \tag{8}$$

$$\text{s.t.} \quad (1b), (1c), (6), (7), \text{ and}$$

$$\begin{cases} W^{min} \leq \frac{(C_i^T x - \Delta l)^2}{C_i^T P^{-1} C_i} \\ W^{min} \leq \frac{(C_i^T x + \Delta l)^2}{C_i^T P^{-1} C_i} \end{cases}. \tag{9}$$

Theorem 1. *Optimal solution of (**P2**) is also optimal to optimization problem (**P1**).*

Proof: See Appendix C. □

The theorem 1 implies that one can obtain an exact locally optimal solution of problem (**P1**) by solving (**P2**) which is a single-level optimization problem and much easier to solve than (**P1**). Note that both (**P1**) and (**P2**) are nonconvex. Therefore, only locally optimal solutions can be guaranteed.

D. A Less-Conservative Convex Relaxation

It is a consensus that the Lyapunov-based methods for stability assessment are more or less conservative. Inequalities (9) are concave which are a special type of non-convex constraints. To reduce the computational burden introduced by (9) and the conservativeness of the proposed Lyapunov-based method, we relax the concave set (9) into its convex hull.

Theorem 2. *Set Ψ is the convex hull of set ψ, where*

$$\psi = \left\{ (x, W^{min}) \;\middle|\; \begin{array}{l} W^{min} \leq \frac{((C_i^T x - \Delta l)^2}{C_i^T P^{-1} C_i} \\ W^{min} \leq \frac{(C_i^T x + \Delta l)^2}{C_i^T P^{-1} C_i} \\ \underline{C_i^T x} \leq C_i^T x \leq \overline{C_i^T x} \end{array} \right\}$$

$$\Psi = \left\{ (x, W^{min}) \;\middle|\; \begin{array}{l} W^{min} \leq \frac{(\overline{C_i^T x} - 2\Delta l) C_i^T x + \Delta l^2}{C_i^T P^{-1} C_i} \\ W^{min} \leq \frac{(\underline{C_i^T x} + 2\Delta l) C_i^T x + \Delta l^2}{C_i^T P^{-1} C_i} \\ \underline{C_i^T x} \leq C_i^T x \leq \overline{C_i^T x} \end{array} \right\},$$

where W^{min} is nonnegative and $-2\Delta l \leq \underline{C_i^T x} \leq 0 \leq \overline{C_i^T x} \leq 2\Delta l$.

Proof: See Appendix D. □

Remark 1. A pictorial interpretation of the proposition is given in Fig. 2 from which we can observe that set Ψ is the convex hull [18] of ψ. Compared with the concave constraints in ψ, the linear constraints in Ψ are easier to compute. Searching for the optimal solution over set Ψ will result in a slightly bigger W_{min}, which will reduced the conservativeness of the Lyapunov method.

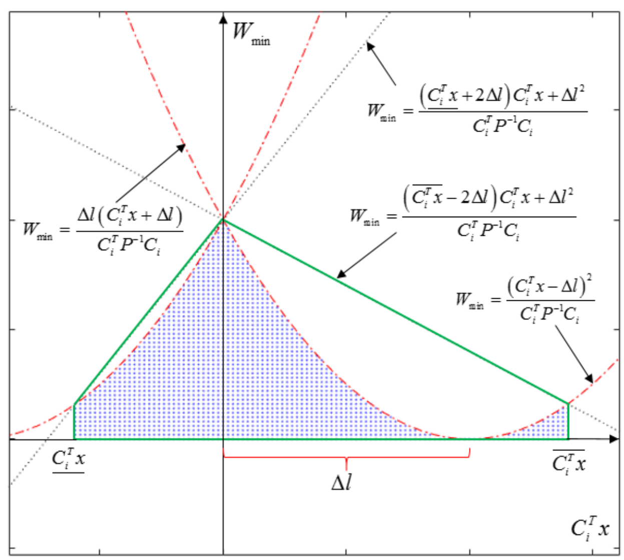


Fig. 2. The convex hull relaxation of concave constraints (9). The shaded area denotes set ψ while the area with green boundaries is set Ψ.

As a result, the proposed stability-constrained optimization frame for nonlinear system (1b) is given by

$$\begin{aligned} &\min \ (8) \\ &\text{s.t.} \ (1b), (6), (7), \text{ and } \Psi. \end{aligned} \tag{10}$$

Remark 2. The constraint (6) is convex while Ψ is a convex polytope. If there exist convex relaxations for the original system, i.e. equation (1b), $v(x)$, $\dot{v}(x)$, and $f(x)$, it is straight forward to develop a convex stability-constrained optimization model for system (1b).

III. Application in Power Grids: TSCOPF

In this section, we apply the framework (10) to power systems to obtain a novel optimization model for the transient stability-constrained optimal power flow. It has been verified in [15] and [16] that the procedure of obtaining quadratic Lyapunov functions introduced in Subsection II-B is valid for the power grid when its angle dynamics is described by the classical generator model with the network preserved. Based on this dynamic power system model, this paper further considers the transfer conductances of transmission lines. Note that, being different from the previous works, the quadratic Lyapunov function of the dynamic power system constructed in this section is valid for the whole feasible set rather than just a given post-fault equilibrium.

A. Steady-state Model of Power Grids

Let $\mathcal{N}_G$, $\mathcal{N}_L$, $\mathcal{N}$, and $\mathcal{B}$ denote the sets of generator buses, load buses, all buses, and all branches of the network respectively. V and θ are the vectors of bus voltage magnitudes and phase angles respectively in the steady-state domain. Let $x = [V^T \ \theta^T]^T$ and Y denote the vector of steady-state variables and a set of alterable parameters, the steady-state power network model, i.e. power flow model, is given by

$$p^G - p^L - g^p(x,Y) = 0 \tag{11a}$$
$$q^G - q^L - g^q(x,Y) = 0 \tag{11b}$$

where

$$\begin{aligned} g_i^p(V,\theta,Y) &= V_i \sum_j V_j (G_{ij}\cos(\theta_i-\theta_j) + B_{ij}\sin(\theta_i-\theta_j)) \\ &= V_i \sum_{j\in\mathcal{N}_i} V_j |Y_{ij}| \sin(\theta_{ij}+\alpha_{ij}) \\ g_i^q(V,\theta,Y) &= V_i \sum_j V_j (G_{ij}\sin(\theta_i-\theta_j) - B_{ij}\cos(\theta_i-\theta_j)). \end{aligned}$$

G_{ij} and B_{ij} are the conductance and susceptance of line ij respectively, and $Y_{ij} = G_{ij} + jB_{ij}$, $\alpha_{ij} = \arctan(G_{ij}/B_{ij})$, $i \in \mathcal{N}$. Note that the rotor velocity ω of a generator is assumed to be constant in steady domain with a uniform value of nearly 1 per unit. Consequently, it is not considered as a variable in the steady-state power flow model.

B. Lur'e-form Dynamic Model with Multiple Nonlinearities

In this subsection, we introduce the dynamic model of power systems and its Lur'e-form reformulation with multiple nonlinearities. In the dynamic domain, let δ and ω denote the vectors of bus phase angles and rotor velocities respectively. They are functions of time and $\delta_i(t_0)$=θ_i, $\omega_i(t_0) \approx 1$ ($i \in \mathcal{N}$). The dynamic model of angle stability is given as

$$\dot{\delta}_i = \omega_i - \omega_i(t_0),\ i \in \mathcal{N}_G \tag{12a}$$
$$d_i\dot{\delta}_i = -p_i^L - g_i^p(V,\delta,Y),\ i \in \mathcal{N}_L \tag{12b}$$
$$m_i\dot{\omega}_i = p_i^G - g_i^p(V,\delta,Y) - d_i(\omega_i - \omega_i(t_0)),\ i \in \mathcal{N}_G \tag{12c}$$

where p_i^G and p_i^L represent generator output and load at bus i respectively; m_i denotes the generator moment of inertia and d_i represents the damping coefficient of generator or load.

The transient disturbances that a power system may encounter include faults on transmission facilities, loss of generation, and loss of large loads [8]. Generally, the disturbance will be cleared after a short period. Hence, each type of disturbance corresponds to the variation of some parameters with respect to time, of which the details are given by

$$(G_{ij}, B_{ij}) = \begin{cases} (G'_{ij}, B'_{ij}) & t = t_0^- \\ (G''_{ij}, B''_{ij}) & t = t_0^+ - t_c^- \\ (G'''_{ij}, B'''_{ij}) & t = t_c^+ - \infty \end{cases} \tag{13a}$$

$$p_i^G = \begin{cases} p_i^G & (t = t_0^-, t_c^+ \to \infty) \\ 0 & (t = t_0^+ \to t_c^-) \end{cases} \tag{13b}$$

$$p_i^L = \begin{cases} p_i^L & (t = t_0^-, t_c^+ \to \infty) \\ 0 & (t = t_0^+ \to t_c^-) \end{cases}, \tag{13c}$$

where $t_c^- = \infty$ if the fault is permanent.

In dynamic system (12), the classical model is adopted for generators and the network is preserved with transfer conductance considered. A important assumption of the classical generator model is that the voltage V keeps constant during transient. Hence, we only need to consider δ and ω as state variables in the dynamic domain. To mitigate the inaccuracy of the classical generator model, we only require the voltage magnitude to remain constant during the fault-on transient and obtain the post-fault equilibrium by solving the power flow equations (11). Let the vectors $[V^T \ \theta^T]$ and $[V^{*T} \ \theta^{*T}]$ denote the solution of steady-state system (11) with the pre-fault and post-fault parameters respectively. They are considered as the pre-fault and post-fault equilibrium points respectively for the dynamic system (12). If the typology of the power system can be restored after the fault is cleared, we have $[V^{*T} \ \theta^{*T}]$ = $[V^T \ \theta^T]$. Considering the post-fault system as the nominal system, we re-define the dynamic state variables as the deviations of rotor angles and velocities from the post-fault equilibrium, i.e. $y = [\delta_1 - \theta_1^*, \ldots, \delta_{|\mathcal{G}|} - \theta^*_{|\mathcal{N}_G|}, \delta_{|\mathcal{N}_G|+1} - \theta^*_{|\mathcal{N}_G|+1}, \ldots, \delta_{|\mathcal{N}|} - \theta^*_{|\mathcal{N}|}, \omega_1 - 1, \ldots, \omega_{|\mathcal{N}_G|} - 1]^T$. Let E be the incidence matrix of the directed graph $\mathcal{G}(\mathcal{N},\mathcal{B})$, so that $E[\delta_1, \ldots, \delta_{|\mathcal{N}|}]^T = [(\delta_i - \delta_j)_{\{i,j\}\in\mathcal{B}}]^T$. Let the matrix C be $E[I_{|\mathcal{N}|\times|\mathcal{N}|} \ O_{|\mathcal{N}|\times|\mathcal{N}_G|}]$. Then

$$Cy = E[\delta_1 - \theta_1^*, \ldots, \delta_{|\mathcal{N}|} - \theta^*_{|\mathcal{N}|}]^T = [(\delta_{ij} - \theta^*_{ij})_{\{i,j\}\in\mathcal{B}}]^T.$$

Based on the dynamic variable vector y defined above, we can rewrite the dynamic model (12) in the form of (2) and first have

$$A = \begin{bmatrix} O_{|\mathcal{N}_G|\times|\mathcal{N}|} & I_{|\mathcal{N}_G|\times|\mathcal{N}_G|} \\ O_{|\mathcal{N}|-|\mathcal{N}_G|\times|\mathcal{N}|} & O_{|\mathcal{N}|-|\mathcal{N}_G|\times|\mathcal{N}_G|} \\ O_{|\mathcal{N}_G|\times|\mathcal{N}|} & -M_1^{-1}D_1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} O_{|\mathcal{N}_G|\times|\mathcal{B}|}; & S_1M^{-1}E^T; & S_2M^{-1}E^T \end{bmatrix},$$

where $S_1 = [O_{|\mathcal{N}|-|\mathcal{N}_G|\times|\mathcal{N}_G|} \quad I_{|\mathcal{N}|-|\mathcal{N}_G|\times|\mathcal{N}|-|\mathcal{N}_G|}]$, and $S_2 = [I_{|\mathcal{N}_G|\times|\mathcal{N}_G|} \ O_{|\mathcal{N}_G|\times|\mathcal{N}|-|\mathcal{N}_G|}]$; $M_1 = \mathrm{diag}(m_1, \ldots, m_{|\mathcal{N}_G|})$, $D_1 = \mathrm{diag}(d_1, \ldots, d_{|\mathcal{N}_G|})$ and $M = \mathrm{diag}(d_{|\mathcal{N}_G|+1}, \ldots, d_{|\mathcal{N}|}, m_1, \ldots, m_{|\mathcal{N}_G|})$ represent the matrices of moment of inertia, frequency controller action on governor, and frequency coefficient of load respectively.

Then, consider the vector of nonlinear interactions ϕ in the simple trigonometric form

$$\phi_k(C_k y) = V_i^* V_j^* |Y_{ij}'''|(\sin(\delta_{ij} + \alpha_{ij}) - \sin(\theta_{ij}^* + \alpha_{ij})),$$

where C_k is the kth row of C and the terms $(V_i^* V_j^* |Y_{ij}'''|)$ can be regarded as parameters for any given post-fault equilibrium during fault-on transient. In next subsection, we will show that, by defining the nonlinear interactions ϕ in this way, one can customize sector bounds for each nonlinearity and obtain valid quadratic Lyapunov functions for the whole feasible region rather than a single equilibrium point.

Finally, based on the time-variant parameters given in (13), the perturbation term can be expressed as

$$\begin{cases} V_i \sum_j V_j(\Delta B_{ij}'\sin\theta_{ij} + \Delta G_{ij}'\cos\theta_{ij}) & t = t_0^- \\ V_i \sum_j V_j(\Delta B_{ij}''\sin\theta_{ij} + \Delta G_{ij}''\cos\theta_{ij}) & t = t_0^+ - t_c^- \\ 0 & t = t_c^+ - \infty \end{cases}$$

$$\begin{cases} 0 & t = t_0^-, t_c^+ \to \infty \\ -p_i^G & t = t_0^+ \to t_c^- \end{cases}$$

$$\begin{cases} 0 & t = t_0^-, t_c^+ \to \infty \\ p_i^L & t = t_0^+ \to t_c^- \end{cases},$$

where $\Delta G_{ij}' = G_{ij}' - G_{ij}'''$, $\Delta B_{ij}' = B_{ij}' - B_{ij}'''$, $\Delta G_{ij}'' = G_{ij}'' - G_{ij}'''$, and $\Delta B_{ij}'' = B_{ij}'' - B_{ij}'''$. Note that, if the system typology can be restored after the fault is cleared, we have $(G_{ij}''', B_{ij}''') = (G_{ij}', B_{ij}')$.

C. Lyapunov Functions and Fault-on Trajectories

In traditional transient-stability analysis of power systems, the Lyapunov function is constructed based on a given post-fault equilibrium point [15] and [16]. However, the Lyapunov function obtained in this way is not valid for the TSCOPF framework since the post-fault equilibrium is a solution of TSCOPF which is unknown before solving the TSCOPF. This subsection aims at constructing an effective common Lyapunov function for the whole feasible region in the steady-state domain by carefully designing the sector bounds.

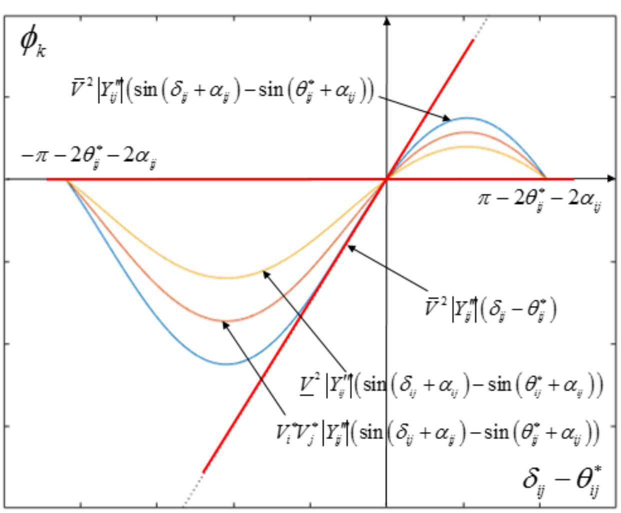


Fig. 3. The designed sector bounds for the nonlinearities in system (12).

We can observe from Fig. 3 that, within the polytope $\mathcal{P} = \{\delta_{ij} | -\pi - \theta_{ij}^* - 2\alpha_{ij} \leq \delta_{ij} \leq \pi - \theta_{ij}^* - 2\alpha_{ij}, (i,j) \in \mathcal{B}\}$, the nonlinear terms $\phi_k(C_k y)$ ($k \in \mathcal{B}$ is the serial number corresponding to branch (i,j) in the branch set $\mathcal{B}$) are bounded by the sectors $[0, \beta_k]$ regardless of the post-fault equilibrium if $\beta_k = \bar{V}^2 |Y_{ij}'''|$. It has been pointed out in [16] that, for the structure-preserving model, both γ_k and β_k ($\forall k$) should be nonzero since the matrix A is not strictly stable for this case. As a result, we use a small positive value ξ for γ instead of 0. By recalling the LMI (3) with multiple nonlinearities, we have the following LMI for the dynamic model (12) of power systems

$$\begin{bmatrix} A^T P + PA - C^T \gamma \beta C & PB + \frac{1}{2}(\gamma + \beta) C^T \\ B^T P + \frac{1}{2}(\gamma + \beta) C & -I_{|\mathcal{B}| \times |\mathcal{B}|} \end{bmatrix} \preceq 0, \tag{14}$$

where P is a $(|\mathcal{N}| + |\mathcal{N}_G|) \times (|\mathcal{N}| + |\mathcal{N}_G|)$ positive definite matrix, matrices A, B, and C are defined in Subsection II-B $\beta = \text{diag}(\beta_1, \ldots, \beta_k)$, and $\gamma = \text{diag}(\xi, \ldots, \xi)$. By solving the LMI (14), one can obtain a common quadratic Lyapunov function of dynamic system (12) which is valid for all potential post-fault equilibia within the polytope $\mathcal{P}$.

Substitute the parameters of system (12) into expression (7) and neglect the terms whose orders are higher than 3, we have the approximate fault-on trajectories of power grids:

$$\delta_i(t_c) = \theta_i - \frac{d_i t_c^2}{2! M_i} K_i, i \in \mathcal{N}_G \tag{15a}$$

$$\delta_i(t_c) = \theta_i - \frac{2! + t_c^2}{2! d_i} K_i, i \in \mathcal{N}_L \tag{15b}$$

$$\omega_i(t_c) = 1 + (\frac{d_i t_c^2}{2! M_i^2} - \frac{t_c}{M_i}) K_i, i \in \mathcal{N}_G, \tag{15c}$$

where $K_i = V_i \sum_j V_j(\Delta B_{ij}\sin\theta_{ij} + \Delta G_{ij}\cos\theta_{ij})$, $\Delta G_{ij} = G_{ij}'' - G_{ij}'$, and $\Delta B_{ij} = B_{ij}'' - B_{ij}'$. For the sake of simplicity, the 3rd-order term in (15b) is also omitted. Note that, with high-speed relays, the fault can generally be cleared within 6 cycles (i.e. 0.1 s). Therefore, a 3rd-order Taylor's series is considered sufficiently accurate for approximating the fault-on trajectories of power systems. Based on the approximation of fault-on trajectories (15) and the dynamic state variable vector y defined in the third paragraph of Subsection III.B, we can cast the fault-clearing state (point) $y(t_c)$ in the dynamic domain into a function of the pre- and post-fault equilibira x, x^* in the steady domain

$$y(t_c) = h(x, x^*). \tag{16}$$

D. A Novel TSCOPF Framework for Power Systems

Based on the definitions and preparations in Subsections III.A-C, we develop a novel TSCOPF framework for power grids by applying the stability-constraint optimization framework (10) proposed in Section II. The developed TSCOPF model can be expressed as

$$\begin{aligned}
\min_{p^G, W_{min}} \quad & f = \sum_i (a_{i1}(p_i^G)^2 + a_{i2}p_i^G) - \epsilon W^{min} && \text{(17a)}\\
\text{s.t.} \quad & \text{(6), (16), and} && \\
& p^G - p^L - g^p(x, Y') = 0 && \text{(17b)}\\
& q^G - q^L - g^q(x, Y') = 0 && \text{(17c)}\\
& S(x, Y') \le \bar{S} && \text{(17d)}\\
& \underline{V}, \underline{E}[\theta] \le V, E[\theta] \le \bar{V}, \bar{E}[\theta] && \text{(17e)}\\
& p^G - p^L - g^p(x^*, Y''') = 0 && \text{(17f)}\\
& q^G - q^L - g^q(x^*, Y''') = 0 && \text{(17g)}\\
& S(x^*, Y''') \le \bar{S} && \text{(17h)}\\
& \underline{V}, \underline{E}[\theta] \le V^*, E[\theta^*] \le \bar{V}, \bar{E}[\theta] && \text{(17i)}\\
& \underline{p}^G, \underline{q}^G \le p^G, q^G \le \bar{p}^G, \bar{q}^G && \text{(17j)}\\
& \begin{cases} W^{min} \le \frac{\pi(\pi + 2\theta_{ij}^* + 2\alpha_{ij})}{C_i^T P^{-1} C_i} \\ W^{min} \le \frac{\pi(\pi - 2\theta_{ij}^* - 2\alpha_{ij})}{C_i^T P^{-1} C_i} \end{cases}, && \text{(17k)}
\end{aligned}$$

where $S_{ij}(x, Y) = Y_{ij}^2 V_i^2 V_j^2$ $(i, j \in \mathcal{N}, ij \in \mathcal{E})$ representing the apparent power in transmission line ij. The summation term in (17a) is the generation costs which are to be minimized in an OPF problem. Constraints (17b)-(17e) and (17f)-(17i) denote the power flows of pre- and post-fault systems respectively. Constraints (17k) is the specialization of its general form Ψ for power systems. One of them is redundant if the system restores after the fault is cleared. The algebraic constraints (6), (16), and (17k) on the steady-state variable x together depict the stability region of a given transient disturbance.

Remark 3. All variables in the TSCOPF model (17) are in the steady-state domain. Consequently, it is reasonable to consider the region specified by constraints(6), (16), and (17k) as the projection of the transient stability region with respect to a given fault onto the steady-state domain.

Remark 4. There exist some convex relaxations for the power flows (17b)-(17c) and (17f)-(17h), for instance the quadratic convex relaxation presented in [19]. Since the K_i in (15) has the same form as the power flow equations, it is easy to extend the convex relaxations of power flows to constraint (16). We can obtain a convex TSCOPF model by replacing the nonconvex constraints (16), (17b)-(17c) and (17f)-(17h) with their convex relaxations.

IV. Case Study

A. Introduction to the Test System

We test the proposed TSCOPF framework (17) on one of the most commonly used test transmission grids, the IEEE 118-bus system, which consists of 19 generators, 35 synchronous condensers, 177 lines, 9 transformers, and 91 loads [20] as shown in Fig. 4. The dynamic data, i.e. generator moment of inertia m_i $(i \in \mathcal{N}_G)$ and the damping coefficient d_i $(i \in \mathcal{N}_G \cup \mathcal{N}_L)$, comes from [21]. The system is stable under a wide range of transient faults with the original load profile. To create two heavy-loaded test cases, we scale up the demand at each load bus by factors of 1.6 and 1.9 respectively.

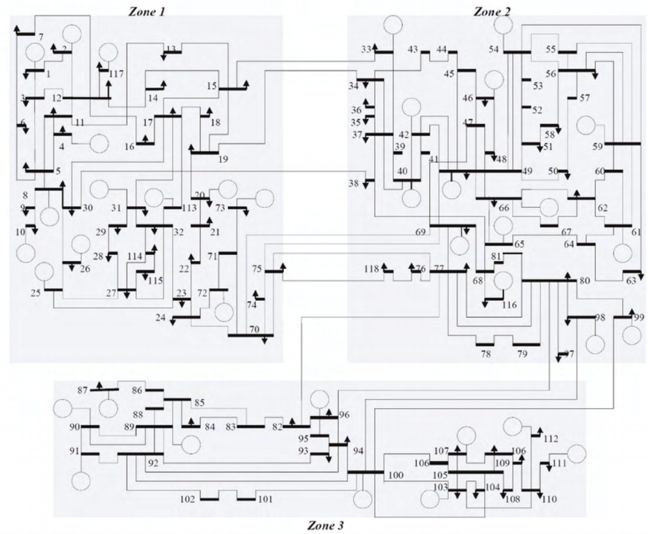


Fig. 4. IEEE 118-bus test system.

B. Case Design and Results

The results of two scenarios are compared to evaluate the effectiveness of the proposed TSCOPF framework. In the first scenario, we solve the problem (18) to obtain a solution of the original OPF while problem (17) is considered in the second scenario. Note that constraints (16) and (18b)-(18e) are related to a specific fault. In this case study, we consider a symmetric bus-to-ground fault at Node 8 since a symmetric fault is the most severe type of fault involving the largest current.

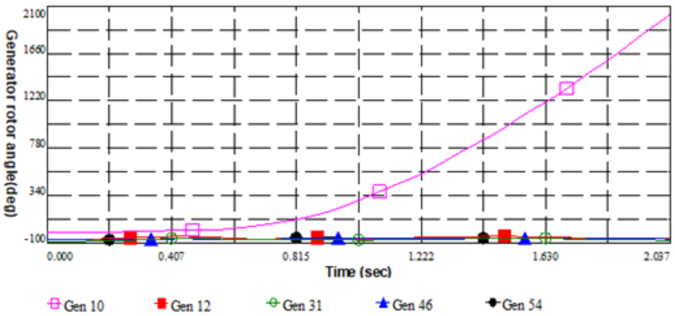
$$\begin{aligned}
&\min_{p^G} \ f = \sum_i (a_{i1}(p_i^G)^2 + a_{i2}p_i^G) \\
&\text{s.t.} \ (18b) - (18e), \ \text{and} \, (18j)
\end{aligned} \quad (18)$$

The accuracy of the fault-on trajectory approximation (15) relies on the fault duration. To assess the sensitivity of the proposed TSCOPF model to the fault clearing time, we consider fault clearing times of 0.167s (i.e. 10 cycles) and 0.100s (i.e. 6 cycles) for the 1.6 times and 1.9 times load cases respectively. The optimal solutions, including the required CPU times, of the original OPF and the TSCOPF are tabulated in Table I. Problems (18) and (17) are solved by the nonlinear solver IPOPT (version 3.12.4) [22] through the optimization package JuMP in Julia (version 0.5.2) [23]. Before that, we obtain a uniform quadratic Lyapunov function for (17) using a MATLAB toolbox YALMIP [24] by calling the SDP solver MOSEK (version 7.1.0.34) [25]. A MAC computer with a 64-bit Intel i7 dual core CPU at 2.40 GHz and 8 GB of RAM was used to solve the optimization cases.

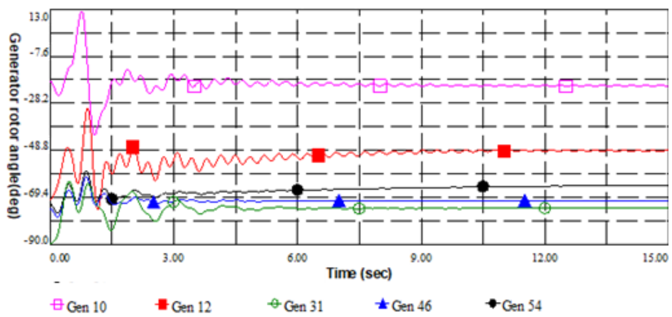
TABLE I
Optimal Solutions

Load factor	Scenario	Objective value (k Dollars)	CPU time (secs)
1.6	Original OPF	2341.0998	8.986
	TSCOPF	2350.5974	15.815
1.9	Original OPF	2895.7271	9.571
	TSCOPF	2909.8168	18.298

With the initial solutions (i.e. the optimal solutions) obtained in the previous step, transient stability analysis is conducted using DSA Toolbox [26] in a Windows computer with a 64-bit Intel i7-7700 4 cores CPU at 3.60 GHz and 16 GB of RAM. The system responses of the two scenarios are plotted in Fig. 5 and Fig. 6 respectively.

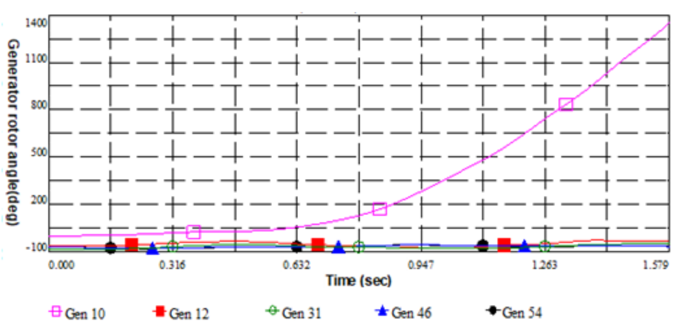


(a) Original OPF

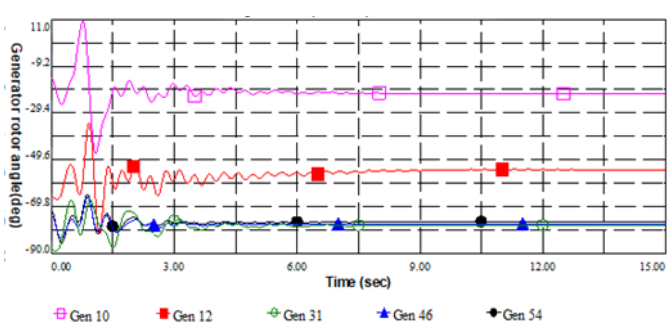


(b) TSCOPF

Fig. 5. System Responses of the case with load factor of 1.6.



(a) Original OPF



(b) TSCOPF

Fig. 6. System Responses of the case with load factor of 1.6.

C. *Analysis*

Based on the numerical results obtained in Subsection IV-B, we have the following observations:

(1) The system remains stable after the fault is cleared if it is operated at the equilibrium obtained by solving the proposed TSCOPF. The solution of the original OPF can not guarantee the stability even though the bus-to-ground fault at Bus 8 is cleared very fast.

(2) One can obtain a stability-guaranteed solution by solving the TSCOPF with the cost of a higher (however not significantly higher) computational time, since only a limited number of nonlinear and nonconvex constraints need to be added to the original OPF to construct the TSCOPF. Compared with the existing methods, this is one of the most significant advantages of the proposed TSCOPF framework. To be more precisely, the added constraints include a set of power flow equations (18f)-(18i) with respect to the post-fault equilibrium, the approximation of fault-on trajectories (16) which has a similar form as the power flows with a much smaller size, and some linear or convex constraints (18k) and (6).

(3) The feasible set of TSCOPF is a subset of that of the original OPF. Consequently, it is direct to know that the cost, namely the optimal objective value, of the TSCOPF is higher than that of the original OPF. As shown in Table I, the costs are increased by less than 5%, which can be claimed acceptable. Due to the convex relaxation (18k), the conservativeness of the proposed approach is effectively reduced.

V. Conclusions and Future Work

This paper proposes a stability-constrained optimization framework for a type of nonlinear systems whose dynamics can be described by a Lur'e system. Unlike the existing methods which are based on either DAE-discretization or iterative algorithms where an independent stability assessment is required for each iteration, the introduced framework is developed based on Lyapunov stability theories. One of the primary advantages of the developed approach is more computational tractable than the existing methods. To illustrate the application values of the proposed framework, it has been successfully applied in power grids to develop a novel TSCOPF model.

The numerical study on the IEEE-118 test system demonstrates that the prosed TSCOPF framework can effectively obtain a stability-guarantee optimal solution with acceptable computational burden. However, There are many directions can be pursued to push the introduced stability-constrained optimization framework to the online application level. First and foremost, methods of constructing uniform quadratic Lyapunov functions for the nominal system of (2) with less conservativeness need to be explore. Although the quadratic form Lyapunov fucntions are computationally effective, they may be conservative for stability assessment of many dynamic systems. It is necessary to customize the procedure introduced in Subsection II-B for specific dynamic systems to construct quadratic Lynapunov functions with less conservativeness.

The non-convex nature of many Lur'e type systems prevents the application of powerful convex optimization approaches.

It is valuable to explore effective convex relaxations to make the stability-constrained optimization framework more computationally tractable for the purpose of online application. From the perspective of power grids, the dynamic system with higher-order generator and load models is no longer of Lur'e type and have multi-variate nonlinear terms. It is also necessary to extend the construction from sector-bounded nonlinearities to more general norm-bounded nonlinearities [27].

Appendix A
Proof of Lemma 1

The derivative of W along the trajectories of the nominal system in (2) is given by

$$\begin{aligned}\dot{W}(y) &= \dot{y}^T P y + y^T P \dot{y} \\ &= y^T (A^T P + PA) y + y^T PB\phi + \phi^T B^T P y \\ &= \begin{bmatrix} y \\ \phi \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} y \\ \phi \end{bmatrix}.\end{aligned}$$

Denote the above block matrix as $BM1$. The sector bound condition (a) is equivalent to

$$\begin{bmatrix} y \\ \phi \end{bmatrix}^T \begin{bmatrix} \gamma\beta C^T C & -\frac{(\gamma+\beta)}{2} C^T \\ -\frac{(\gamma+\beta)}{2} C & I \end{bmatrix} \begin{bmatrix} y \\ \phi \end{bmatrix} \leq 0.$$

Denote the above block matrix as $BM2$. By observing the LMI in (3), we realize that $LMI(3) = BM1 - BM2 \preceq 0$, which means $BM1 \preceq BM2 \preceq 0$. As a result, $\dot{W}(y) \leq 0$.

Appendix B
Proof of Lemma 2

Suppose y^* is a point on one of the edges of the polytope $\mathcal{P}$. Due to the definition of W^{min}, we have $W(y^*) \geq W^{min}$. Hence, the system is not able to evolve from the fault clearing point $y(t_c)$ to y^* since $W(t_c) \leq W^{min} \leq W(y^*)$. The Lyapunov function value $W(y)$ can only decrease along the system trajectory since $\dot{W} \leq 0$ within polytope $\mathcal{P}$. as a result, the system trajectory will stay within the region Ω or even converge to the origin as t goes to infinity.

Appendix C
Proof of Theorem 1

Assume that $\tilde{s} = [\tilde{x}\ \tilde{W}_{min}]^T$ is an optimal solution of (**P2**):

i. $\tilde{s}$ is feasible to (**P1**).

First, we will show that any optimal solution of (**P2**) is optimal to problem (4). Suppose that $\tilde{W}_{min}$ does not make equal sign hold in any of (9), which means $\tilde{s}$ is not feasible to (4). It suffices to show there exists another feasible solution of (**P2**), $\hat{s} = [\tilde{x}\ \tilde{W}_{min} + \Delta W]^T$, where ΔW is an arbitrarily small positive value. We have

$$F(\hat{s}) - F(\tilde{s}) = -\epsilon \Delta W \leq 0,$$

which contradicts the optimality of $\tilde{s}$. In other words, for any solution $\tilde{s}$ in which no equal sign holds in any inequality of (9), one can always choose a ΔW to construct another feasible solution $\hat{s}$ of (**P2**) with a smaller/better objective value until equal sign holds in one of the equalities of (9). Such an solution is exactly the optimal solution of problem (4). Optimal solution $\tilde{s}$ of (**P2**) being optimal to (4) implies that it is feasible to (**P1**) since (**P1**) is a bilevel optimization problem with (4) as the lower-level problem.

ii. $\tilde{s}$ is also optimal to (**P1**).

Suppose $\tilde{s}$ is not optimal to (**P1**), then there exist a feasible solution to (**P1**), $\hat{s} = [\hat{x}\ \hat{W}_{min}]^T$, that is in the vicinity of $\tilde{s}$ satisfying

$$f(\hat{x}) - f(\tilde{x}) \leq 0.$$

It is easy to verify that $\hat{s}$ is also feasible to (**P2**) and satisfies

$$F(\hat{s}) - F(\tilde{s}) = f(\hat{x}) - f(\tilde{x}) + \epsilon(\tilde{W}_{min} - \hat{W}_{min}) \leq 0. \quad (19)$$

Note that ϵ is an arbitrarily small value. Hence, it is reasonable to assume that the term $\epsilon(\tilde{W}_{min}$ - $\hat{W}_{min})$ is not comparable to $(f(\hat{x})$ - $f(\tilde{x}))$, which means $(F(\hat{x})$ - $F(\tilde{x}))$ has the same sign as $(f(\hat{x})$ - $f(\tilde{x}))$. Condition (19) contradicts the optimality of $\tilde{s}$ to (**P2**). Namely, $\tilde{s}$ is an local minimum of (**P1**) if it is an local minimum of (**P2**).

So far, the theorem has been proved by contradiction.

Appendix D
Proof of Theorem 2

For the sake of convenience, we replace the terms $C_i^T x$ and $C_i^T P^{-1} C_i$ with X and $1/\lambda$ respectively. The notation $CONV(A)$ means the convex hull of set A.

i. $CONV(\psi) \subseteq \Psi$.

For any $X^* \in [\underline{X}, 0]$, we have

$$\begin{aligned}\psi(X^*) &= \{W^{min} | 0 \leq W^{min} \leq \lambda(X^* + \Delta l)^2\} \\ \Psi(X^*) &= \{W^{min} | 0 \leq W^{min} \leq \lambda((\underline{X} + 2\Delta l)X^* + \Delta l^2)\}.\end{aligned}$$

It is direct to know that $\psi(X^*) \subseteq \Psi(X^*)$ since $(X^* + \Delta l)^2 \leq (\underline{X} + 2\Delta l)X^* + \Delta l^2)$. Similarly, for any $X^* \in [0, \overline{X}]$, we have the same conclusion, which implies $\psi \subseteq \Psi$ and that Ψ is a convex relaxation of ψ. Since convex hull is defined as the intersection of all convex relaxations of a non-convex set [18], we have $CONV(\psi) \subseteq \Psi$.

ii. $CONV(\psi) \supseteq \Psi$.

If a linear inequality is valid for a given set Ω_A, it will also be valid for any subset of Ω_A. Note that a linear inequality is valid for a set means the inequality is satisfied by all its feasible solutions [28]. On the other way round, according to the properties of supporting hyperplanes [29], Ω_B is said to be a subset of Ω_A if Ω_A is convex and any valid linear inequality of Ω_A is also valid for Ω_B [30]. Let $s = [X\ W^{min}]^T$ and suppose that $\alpha s \geq \beta$ is any valid linear cut for $CONV(\psi)$, this cut should be also valid for all the points in ψ. To prove that $CONV(\psi) \supseteq \Psi$ (i.e. Ψ is a subset of $CONV(\psi)$), we try to show that $\alpha s \geq \beta$ is valid for all the edges of Ψ.

The convex set Ψ has five edges of which the formulations are given as

$$\Psi_1 = \left\{ (X, W^{min}) \left| \begin{array}{l} W^{min} = \lambda((\overline{X} - 2\Delta l)X + \Delta l^2) \\ W^{min} \leq \lambda((\underline{X} + 2\Delta l)X + \Delta l^2) \\ \underline{X} \leq X \leq \overline{X} \end{array} \right. \right\},$$

$$\Psi_2 = \left\{ (X, W^{min}) \left| \begin{array}{l} W^{min} \leq \lambda((\overline{X} - 2\Delta l)X + \Delta l^2) \\ W^{min} = \lambda((\underline{X} + 2\Delta l)X + \Delta l^2) \\ \underline{X} \leq X \leq \overline{X} \end{array} \right. \right\},$$

$$\Psi_3 = \left\{ (X, W^{min}) \;\middle|\; \begin{array}{l} W^{min} \le \lambda((\overline{X} - 2\Delta l)X + \Delta l^2) \\ W^{min} \le \lambda((\underline{X} + 2\Delta l)X + \Delta l^2) \\ X = \underline{X} \end{array} \right\},$$

$$\Psi_4 = \left\{ (X, W^{min}) \;\middle|\; \begin{array}{l} W^{min} \le \lambda((\overline{X} - 2\Delta l)X + \Delta l^2) \\ W^{min} \le \lambda((\underline{X} + 2\Delta l)X + \Delta l^2) \\ X = \overline{X} \end{array} \right\},$$

$$\Psi_5 = \left\{ (X, W^{min}) \;\middle|\; \begin{array}{l} W^{min} = 0 \\ \underline{X} \le X \le \overline{X} \end{array} \right\}.$$

As an example, we show that the cut $\alpha s \ge \beta$ is valid for edge Ψ_1 in this paragraph. It is easy to verify that the two points $s_1 = (0, \lambda \Delta l^2)$ and $s_2 = (\overline{X}, \lambda(X^* - \Delta l)^2)$ are located in both ψ and Ψ_1. That means the cut $\alpha s \ge \beta$ is valid for these two points and we have $\alpha s_1 \ge \beta$ and $\alpha s_2 \ge \beta$. Let $s^* = (X^*, W^*_{min})$ denote any given point in set Ψ_1. For any given s^*, there exists a value c $(0 \le c \le 1)$ satisfying $s^* = cs_1 + (1-c)s_2)$. It suffices to verify this statement by substituting $s^* = ((1-c)\overline{X}, c\lambda\Delta l^2 + (1-c)\lambda(X^* - \Delta l)^2)$ into the first equation in Ψ_1. As a result, we have

$$\alpha s^* = c\alpha s_1 + (1-c)\alpha s_2 \ge c\beta + (1-c)\beta = \beta,$$

which means the linear cut is also valid for any given point in Ψ_1 and, consequently, valid for Ψ_1.

Using the same method, the readers are able to prove that the linear cut $\alpha s \ge \beta$ is valid for all the other four edges of Ψ and, consequently, valid for the whole convex set Ψ. Hence, $CONV(\psi) \supseteq \Psi$.

iii. $\Psi = CONV(\psi)$.

$CONV(\psi) \subseteq \Psi$ and $CONV(\psi) \supseteq \Psi$ together imply $\Psi = CONV(\psi)$.

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