Convex Restriction of Power Flow Feasibility Set

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***Abstract*—Convex restriction of the power flow feasible set identifies the convex subset of power injections where the solution of power flow is guaranteed to exist and satisfy the operational constraints. In contrast to convex relaxation, convex restriction provides a sufficient condition and is particularly useful for problems involving uncertainty in power generation and demand. In this paper, we present a general framework of constructing convex restriction of an algebraic set defined by equality and inequality constraints, and apply the framework to power flow feasibility problem. The procedure results in a explicitly defined second order cone that provides a nearly tight approximation of the actual feasibility set for some of the IEEE test cases. In comparison to other approaches to the same problem, our framework is not relying on any simplifying assumptions about the nonlinearity and provide an analytical algebraic condition.**

***Index Terms*—Convex restriction, power flow feasibility, emergency control, security assessment**

I. Introduction

Power flow equation is one of the fundamental condition required in the steady-state analysis of the power grid. Power injections to the grid continuously vary with generation dispatch and demand, and the system operator has to assign the steady-state set point appropriately to ensure stability of the grid. Existence of steady-state equilibrium that respects the operational constraints such as voltage magnitude limits and reactive power limits is a necessary condition to ensure secure operation of the grid in a designed way [1], [2]. However, deriving a tractable sufficient condition for feasibility of power flow equation has remained as a challenge.

Power flow equation is a nonlinear equality constraint, which makes the feasible set generally non-convex and NP-hard [3], even for a radial network [4]–[6]. Convex relaxation of power flow equation has been studied extensively for solving Optimal Power Flow (OPF) [5]–[8]. Convex relaxation provides a necessary condition to satisfy the power flow equation and forms a set that contains the power flow feasible set. While the relaxation has the ability to check whether a solution is globally optimal in OPF problems, it does not guarantee the feasibility of the solution. There is a need for understand and characterize feasibility constraint and enforce them into OPF, especially with growing uncertain power injection from renewables [9], [10].

The search for tractable sufficient conditions for power flow solvability and feasibility started in [11] to find security region where the system is safe to operate. In recent years, number of progress has been made in this area, but its performance and generalization have been still lacking. Most of the work rely on certain modeling assumption such as radial structure [12]–[16], lossless network [17], [18], and decoupled power flow model [19]. Recent effort has been made to find a sufficient condition in a general network, but they still suffer from scalability and conservativeness. [20]–[22] In [23], the inner approximation with Brouwers fixed point theorem showed promising results in a general power network. One of the limitations of this approach was that it required solving non-convex optimization problem to find a locally maximal set that satisfies the self-mapping condition. In this paper, this limitation is alleviated by describing the set in a lifted space and give a closed form expression. Moreover, our approach in the lifted space is the union of all the self-mapping set that satisfies the Brouwer's fied point theorem. This allows drastic improvements against the conservativeness of the condition.

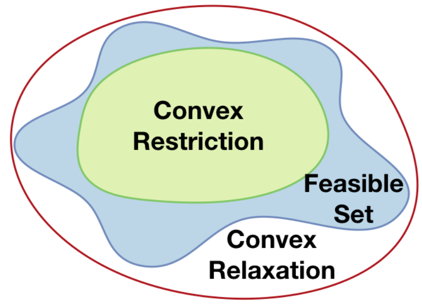

Fig. 1. Convex restriction and convex relaxation of feasible set over the controllable variable space.

The term *convex restriction* will be used to describe a convex set that is a sufficient condition for power flow feasibility. This paper proposes construction of convex restriction in a lifted space where the restriction become possible with envelopes over the nonlinearity. Convex restriction has features that are complementary to convex relaxation. Later, we will show that a concave envelope, which is also complementary to convex envelope, is needed for convex restriction. This envelope is counter-intuitive from the perspective of convex relaxation, but we will see that it enforces convexity to restriction. We frame our derivation of convex restriction for general equality and inequality constraints with variables divided into controllable and uncontrollable variables. Active and reactive power injections are considered as the controllable variables, and voltage magnitude and phase angles are considered uncontrollable variables. Convex restriction is derived for the controllable variables so that a convex optimization can be carried out while guaranteeing the feasibility of the solution. Our convex restriction is constructed around a base point, and we show that it is non-empty given a feasible base point. The current or planned operating point can be naturally used as the base point, and convex restriction can rigorously determine the robustness and margin to infeasibility from the base point.

Rest of the paper is organized as follows. In Section II, feasibility problem is formulated for power flow equation. Convex restriction for a general constraint is discussed in

Section III. The proposed method is applied to the power flow equation with visualization IV. Section V concludes with discussions.

II. Convex Restriction of Feasibility Set: Formulation and Preliminaries

A. Power Flow Equation and Operational Constraints

Consider a power network as a directed graph $N(\mathcal{V}^+, \mathcal{E})$ where each node $\mathcal{V}^+ = \{0\} \cup \mathcal{V}$ represents bus with $\mathcal{V} = \{1, ..., |\mathcal{V}|\}$, and $\{0\}$ being the slack bus. The slack bus has fixed voltage magnitude and phase angle. $\mathcal{E} \subseteq \mathcal{V}^+ \times \mathcal{V}^+$ represents a transmission lines. The set of PV buses are denoted by $\mathcal{G} \subseteq \mathcal{V}$, and the set of PQ buses are denoted $\mathcal{L} \subseteq \mathcal{V}$, and $\mathcal{V} = \mathcal{G} \cup \mathcal{L}$. Consider AC power flow equation with operational constraints:

$$\begin{aligned} p_i &= \sum_{k=1}^{n} v_i v_k (G_{ik} \cos\theta_{ik} + B_{ik} \sin\theta_{ik}), \quad i \in \mathcal{V}, \\ q_i &= \sum_{k=1}^{n} v_i v_k (G_{ik} \sin\theta_{ik} - B_{ik} \cos\theta_{ik}), \quad i \in \mathcal{V}, \end{aligned} \tag{1}$$

$$p_i^{min} \le p_i \le p_i^{max}, \qquad i \in \mathcal{G} \tag{2a}$$
$$q_i^{min} \le q_i \le q_i^{max}, \qquad i \in \mathcal{G} \tag{2b}$$
$$v_i^{min} \le v_i \le v_i^{max}, \qquad i \in \mathcal{L} \tag{2c}$$
$$\vartheta_e^{min} \le \theta_{ik} \le \vartheta_e^{max}, \qquad e = (i,k) \in \mathcal{E}. \tag{2d}$$

A line connecting bus k and l are indexed by either e or (i, k). The phase difference is denoted by $\theta_{ik} = \theta_i - \theta_k$. The phase difference bound may be introduced as a stability constraint, but it could be also removed by setting $\vartheta_e^{min} = -\pi$ and $\vartheta_e^{max} = \pi$. The variables involved in the power flow equations are voltage magnitude, phase angle, active and reactive power at every bus. In most analysis, the interest of the system operator is making decision over the controllable variables, which will be denoted by u. Controllable variables can include active power generation and voltage magnitude setting at generator buses, or active and reactive power at load buses controlled by load-shedding actions. The uncontrollable variables x include voltage phase angle or voltage magnitude at PQ buses, which are the internal state of the system. The system operators need to decide the set point on the controllable variable subject to the power flow feasibility set in equation (1) and (2) based on their objective. Our objective is to find the maximal convex restriction around some base point, which is a convex set that is contained in the feasibility set. A general framework for constructing convex restriction will be presented, and then an example in power flow feasibility will be shown.

B. General Formulation

Consider the following general nonlinear equality and inequality constraint with control variable $u \in \mathbb{R}^m$ and uncontrollable variable $x \in \mathbb{R}^n$,

$$F_i(x, u) = 0, \quad i = 1, ..., n \tag{3a}$$
$$h_i(x, u) \le 0, \quad i = 1, ..., r. \tag{3b}$$

where $F_i(x, u) : (\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ and $h_i(x, u) : (\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ are smooth functions.

In this paper, the feasibility of active power injection at PV buses and active and reactive power at PQ buses is considered. The controllable space considered is $u = \begin{bmatrix} p_{\mathcal{V}}^T & q_{\mathcal{L}}^T \end{bmatrix}^T$ with internal state $x = \begin{bmatrix} v_{\mathcal{V}}^T & \theta_{\mathcal{V}}^T & q_{\mathcal{G}}^T \end{bmatrix}^T$.

The feasibility set will be denoted as $\mathcal{N} = \mathcal{F} \cap \mathcal{H}$, which is an intersection of the equality constrained set $\mathcal{F} = \{(x, u) \mid F_i(x, u) = 0, \ i = 1, ..., n\}$ and inequality constrained set $\mathcal{H} = \{(x, u) \mid h_i(x, u) \le 0, \ i = 1, ..., r\}$.The projection of set $\mathcal{N}$ to controllable space u is defined as $\mathcal{N}_u = \{u \mid \exists\, x \text{ such that } (x, u) \in \mathcal{N}\}$. Similarly, $\mathcal{H}_u$ and $\mathcal{F}_u$ is defined as projection of equality and inequality constrained set to the controllable variable space. The goal of this paper is to find the convex restriction of the projection of feasibility set to the controllable variable space, $\mathcal{N}_u$. If $u \in \mathcal{F}_u$, then u is solvable, and $\mathcal{F}_u$ is referred as *solvability set*. Similarly, if $u \in \mathcal{N}_u$, then u is feasible, and $\mathcal{N}_u$ referred as *feasibility set*.

C. Convex Restriction of Inequality Constraint

First, let us consider the convex restriction of inequality constraints. This case is much simpler than the convex restriction with nonlinear equality constraint. Suppose a vector of functions $\overline{h}(x, u)$ and $\underline{h}(x, u)$ establishing bounds on individual components as in

$$\underline{h}_k(x, u) \le h_k(x, u) \le \overline{h}_k(x, u). \tag{4}$$

$\underline{h}_k(x, u)$ and $\overline{h}_k(x, u)$ are referred as under-estimator and over-estimator of $h_k(x, u)$, respectively. Let us consider and compare convex restriction and convex relaxation in this inequality constraint, $\mathcal{H} = \{(x, u) \mid h_k(x, u) \le 0, \ k = 1, ..., r\}$.

Lemma 1. Consider under and over-estimators $\underline{h}_k(x, u)$ and $\overline{h}_k(x, u)$ that satisfy Equation (4). If both estimators are convex, then

$$\overline{\mathcal{H}} = \{(x, u) \mid \underline{h}_k(x, u) \le 0, \ k = 1, ..., r\} \tag{5}$$

is a convex relaxation, and

$$\widehat{\mathcal{H}} = \{(x, u) \mid \overline{h}_k(x, u) \le 0, \ k = 1, ..., r\} \tag{6}$$

is a convex restriction of $\mathcal{H}$.

Proof. $\underline{h}_k(x, u) \le 0$ is a necessary, and $\overline{h}_k(x, u) \le 0$ is a sufficient condition for $h_k(x, u) \le 0$, so $\widehat{\mathcal{H}} \subseteq \mathcal{H} \subseteq \overline{\mathcal{H}}$. Moreover, $\widehat{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are convex sets because the over-estimator and under-estimators are convex functions. □

Throughout the paper, convex restriction of $\mathcal{H}$ will be denoted by $\widehat{\mathcal{H}}$. In literatures in convex relaxation, a "concave" envelope refers to a concave under-estimator, and a "convex" envelope refers to a convex over-estimator [24]. In this paper, we redefine these terms so that an envelope is a *convex envelope* if the enclosed region is a convex set, and *concave envelope* if the complement of the enclosed region is a convex set. With this definition, convex over-estimator and concave under-estimator are *concave envelopes*, and concave over-estimator and convex under-estimator are *convex envelope* as illustrated in Figure 2. The intersection of *convex envelope* forms a convex region and is widely used for relaxation [8]. The *concave envelope* is found to be useful in enforcing convexity to the restriction of feasibility set.

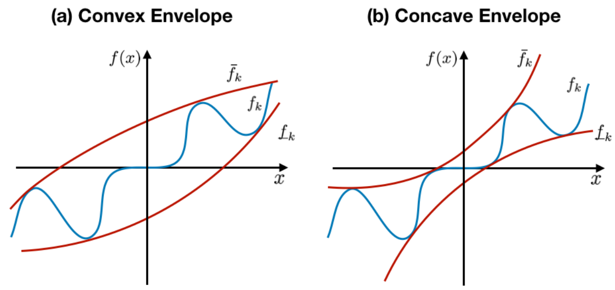

Fig. 2. Example of convex and concave envelope.

III. Derivation of Convex Restriction

Suppose the equality constraint can be decomposed into a linear combination of basis functions:

$$F(x,u) = M\psi(x,u) \tag{7}$$

where $M \in \mathbb{R}^{n\times r}$ is a sparse constant matrix and $\psi(x,u) \in \mathbb{R}^r$ is a vector of basis functions. In power flow equation in (1), $v_i v_k \cos(\theta_{ik})$ and $v_i v_k \sin(\theta_{ik})$ could be chosen as basis functions as an example. Basis functions involved in $F(x,u)$ may not be unique, and its choice can be exploited. Choosing appropriate basis functions is crucial in this approach as it determines both approximation gap and complexity of the constraints. For power flow equations, nonlinearities are involved with transmission lines and buses, and the number of basis function r grows linearly with respect to number of buses and number of lines. Suppose a convex restriction is constructed around some base point (x^{base}, u^{base}) that has a non-singular Jacobian $J_F^{base} = \frac{\partial F}{\partial x}\big|_{x^{base},u^{base}}$. Let us consider and define residue of basis function at the base point:

$$f(x,u) = \psi(x,u) - J_\psi(x - x^{base}) - \psi(x^{base}, u^{base}). \tag{8}$$

where $J_\psi^{base} = \frac{\partial \psi}{\partial x}\big|_{x^{base},u^{base}}$. Note that $f(x^{base}, u^{base}) = 0$ and $\frac{\partial f}{\partial x}\big|_{x^{base},u^{base}} = 0$. Based on this definition, an equality constraint can be written as

$$F(x,u) = J_F x + M f(x,u) \tag{9}$$

where $J_F^{base} = M J_\psi^{base}$ is a non-singular square Jacobian at the base point, and $Mf(x)$ is a residue of $F(x,u)$ after linearization. the equality constraint can be written in the following fixed point iteration:

$$x = -J_F^{-1} M f(x,u). \tag{10}$$

This fixed point from is the same form used in Newton-Raphson method, which is one of the most popular algorithms for solving steady state power flow equation [25]. This is widely used in practice due to its fast convergence given a good initial guess. For our framework, this fixed point form allows tight bound on the nonlinearity over a range of x, which plays an important role for constructing tight convex restriction to the actual feasible region.

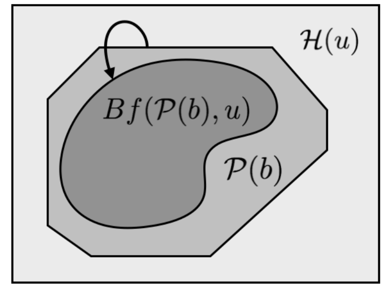

Fig. 3. The self-mapping is illustrated in this figure in the domain of $\mathcal{X}$. Here, $\mathcal{H}(u) = \{x \mid (x,u) \in \mathcal{H}\}$, and existence of the self-mapping set $\mathcal{P}(b)$ ensures solvability and feasibility of u.

Remark. The fixed point condition in Equation (10) is an equivalent condition to the equality condition in (3a), so $\mathcal{F} = \{(x,u) \mid x = -J_x^{-1} M f(x,u)\}$.

Since the fixed point form is not unique, rest of the paper will be developed in a general fixed point form:

$$x = Df(x,u) \tag{11}$$

where $D \in \mathbb{R}^{m\times r}$ is a constant matrix and $f(x,u) : (\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^r$ is a smooth multi-dimensional map. The fixed-point form in (10) is a special case where the constant matrix is $D = -J_x^{-1} M$. Given a fixed point equation, Brouwer's fixed point theorem provides a condition for solvability of the equality constraint.

Theorem 1. (**Brouwer's fixed point theorem**) Let $G : \mathcal{P} \to \mathcal{P}$ be a continuous map where $\mathcal{P}$ is a compact and convex set in $\mathbb{R}^m$. Then the map has a fixed point in $\mathcal{P}$, namely $x = G(x)$ has a solution in $x \in \mathcal{P}$.

Brouwer's Fixed Point Theorem provides a sufficient condition for existence of solution. This Theorem can be directly applied to the fixed point map in equation (11).

Lemma 2. If $Df(x,u) \in \mathcal{P}$ for all $x \in \mathcal{P}$, then the system is solvable and has at least one solution in $x \in \mathcal{P}$ for the controllable variable u.

Proof. Let $G(x) = Df(x,u)$. Then, there exist a solution $x \in \mathcal{P}$ from Brouwer's fixed point theorem. □

The self-mapping condition is illustrated in Figure 3. Lemma 2 provides a sufficient condition for existence of solution in the controllable space $\mathcal{U}$. The objective of convex restriction is to describe $\mathcal{N}_u$ such that the existence of the solution is guaranteed inside the inequality constraint. If the set $\mathcal{P}$ satisfy the self-mapping condition, then the equality condition is solvable. Moreover, if $(u,x) \subseteq \mathcal{H}$ for all $x \in \mathcal{P}$, then the feasibility of solution can be also guaranteed. Brouwer's fixed point theorem states the solution belongs to $\mathcal{P}$, which belongs to the feasibility set. Later the outer approximation of the set $Df(\mathcal{P},u)$ will be used to form a tractable self-mapping condition.

Notice that the self-mapping set is not unique, and the union of all possible sets that satisfy the self-mapping condition can describe the feasibility set. This brings the idea of describing the feasibility set into the lifted space where additional parameter describes the self-mapping set. Suppose $\mathcal{P}(b) \subseteq \mathcal{X}$ is a self-mapping set that is parametrized by b. Then, existence of b such that $Df(\mathcal{P}(b), u) \subseteq \mathcal{P}(b)$ also satisfies the Brouwer's fixed point condition.

Lemma 3. The union of convex and compact self-mapping set parametrized by $b \in \mathbb{R}^s$ can be written in a lifted domain, $(u, b) \in (\mathbb{R}^s, \mathbb{R}^m)$, as

$$\mathcal{W} = \{(u, b) \mid \forall x \in \mathcal{P}(b),\ Df(x, u) \in \mathcal{P}(b),\ (x, u) \in \mathcal{H}\}. \tag{12}$$

Then, $\mathcal{W}_u = \{u \mid \exists\, b, (u, b) \in \mathcal{W}\}$ is restriction of the feasible region, i.e. $\mathcal{W}_u \subseteq \mathcal{N}_u$.

Proof. Condition $\{\forall x \in \mathcal{P}(b),\ Df(x, u) \in \mathcal{P}(b)\}$ ensure the self-mapping under the map $x \to Df(x, u)$, and thus there exists a solution u for the equality constraint in equation (3a) with $x \in \mathcal{P}(b)$ by Brower's fixed point Theorem. Condition $\{\forall x \in \mathcal{P}(b),\ (x, u) \in \mathcal{H}\}$ ensures $\mathcal{P}(b)$ belongs to the feasible set for inequality constraint in equation (3b). $u \in \mathcal{W}_u$ satisfies both constraint in (3), and thus belongs to the feasibility set. □

The set in equation (12) lifts the domain to interval space characterized by b. It was shown in Lemma 1 that convex restriction is simple for inequality constraints, but it is not obvious for nonlinear equality constraints. In this paper, it will be shown that the lifting of variable space to interval space allows convex restriction of nonlinear equality constraints in a similar way to convex restriction of inequality constraints.

A. Self-mapping with a Polytope Set

While the self-mapping set can be any convex and compact set, a polytope will be considered in this paper. There is a significant computational advantage of using polytope because nonlinear functions only goes through a linear transformation. Let us consider a non-empty compact polytope set $\mathcal{P}$,

$$Ax \le b, \tag{13}$$

where $A \in \mathbb{R}^{c \times n}$ and $b \in \mathbb{R}^c$ are constant matrix and vector that sets the operational constraint over the internal state space. This map will be denoted as $\mathcal{P}(b) = \{x \mid Ax \le b\}$. One of the advantages of using constant A is that the variables are involved as a linear combination. Convexity is preserved if all the entries of A is positive, and it will be used to enforce convexity in our approach [26].

Lemma 4. There exists a solution $x \in \mathcal{P}(b) = \{x \mid Ax \le b\}$ that satisfies the equality constraint in equation (3a) for u if

$$\max_{x \in \mathcal{P}(b)} ADf(x, u) \le b. \tag{14}$$

Proof. The above condition is equivalent to $\forall \{x \mid Ax \le b\},\ ADf(x, u) \le b$, which is shows self-mapping of $\mathcal{P}(b)$. Then from Lemma 2, there exists a solution $x \in \mathcal{P}(b)$. □

The nonlinearity is still contained in $f(x, u)$, so using the polytope as the self-mapping set does not introduce extra complexity.

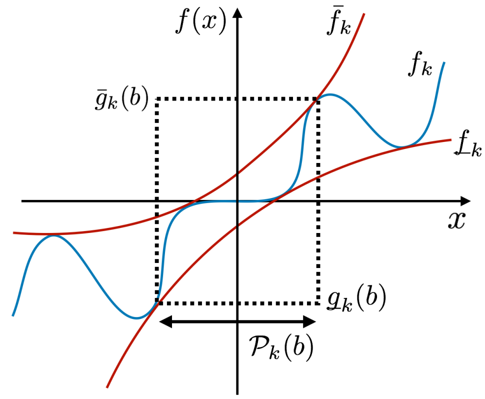

Fig. 4. Variable $\overline{g}_k(b)$ and $\underline{g}_k(b)$ are illustrated on this figure. In the case of scalar x, the self-mapping set $\mathcal{P}_k(b)$ forms a range. $\overline{g}_k(b)$ and $\underline{g}_k(b)$ defines maximum and minimum bound of f_k over this range of x, and their convexity is enforced by putting concave envelope over f_k.

B. Enclosure of Concave Envelope

Consider over and under estimators of $f(x, u)$, denoted by $\overline{f}(x, u)$ and $\underline{f}(x, u)$ establishing bounds on individual components:

$$\underline{f}_k(x, u) \le f_k(x, u) \le \overline{f}_k(x, u). \tag{15}$$

For a system with both equality and inequality constraint, the envelope needs to be valid only over the feasible set for the inequality constraint, $x \in \mathcal{H}_x$. This may be exploited to develop tighter envelope. The idea of *concave envelope* of the nonlinear function extends naturally to the equality constraints in our framework. Consider an equality constraint $F_j(x, u) = 0$. Convex relaxation replaces the equality constraint with concave over-estimator and convex under-estimator. This is a necessary condition for the equality constraint since $\underline{F}_j(x, u) \ge 0$ and $\overline{F}_j(x, u) \le 0$ needs to be satisfied in order to satisfy the equality constraint. The same trick does not work in convex restriction because the slack cannot be afforded for deriving a sufficient condition. However, note that constructing the envelope in an opposite way made the restriction convex for inequality condition as stated on Lemma 1. Similarly, let us consider *concave envelope*, an over-estimator, $\overline{f}_k(x, u)$, and under estimator, $\underline{f}_k(x, u)$, that are constructed in a way that they are convex and concave in (x, u) respectively. This envelope again is illustrated on Figure 4.

Existence of such bound allows us to establish the following enclosures of the original nonlinear functions over the self-mapping set:

$$\overline{g}_k(u, b) = \max_{x \in \mathcal{P}(b)} \overline{f}_k(x, u), \tag{16a}$$

$$\underline{g}_k(u, b) = \min_{x \in \mathcal{P}(b)} \underline{f}_k(x, u), \tag{16b}$$

Note that these definitions results in non-convex optimizations, however for sparse nonlinearities $f_k(x, u)$, they can be carried out easily as the number of vertices of the polytope $\mathcal{P}_k(b)$ is small.

Lemma 5. Functions $\underline{g}_k(u,b)$ and $\overline{g}_k(u,b)$ are respectively concave and convex in (u,b) and are given by

$$\overline{g}_k(u,b) = \max_{v\in\partial\mathcal{P}_k(b)} \overline{f}_k(v,u), \tag{17a}$$

$$\underline{g}_k(u,b) = \min_{v\in\partial\mathcal{P}_k(b)} \underline{f}_k(v,u) \tag{17b}$$

where $\overline{f}_k(v,u)$ and $\underline{f}_k(v,u)$ are convex and concave functions, respectively, and $\partial\mathcal{P}_k(b)$ denotes the vertices of polytope $\mathcal{P}_k(b)$.

Proof. We restrict our proof only to the first inequality for $\overline{g}_k(u,b)$. Any point inside the polytope $\mathcal{P}_k(b)$ can be represented as $x = \sum_n \lambda_n \boldsymbol{v}_n(b)$ where $\boldsymbol{v}_n(b) \in \partial\mathcal{P}_k(b)$ is the n-th vertex of the polytope $\mathcal{P}_k(b)$, while $\lambda_n \geq 0$ and $\sum_n \lambda_n = 1$. Therefore, due to convexity, $\overline{f}(x,u) \geq \sum_n \lambda_n \overline{f}(\boldsymbol{v}_n(b),u)$ and thus $\overline{g}(u,b) = \max_n \overline{f}(\boldsymbol{v}_n(b),u)$. To prove convexity with respect to b, note that every vertex is an intersection of some subset of all the polytope faces characterized by a projection operator P_k and thus solves the equation $P_n A\boldsymbol{v}_n(b) = P_n b$, so $\boldsymbol{v}_n$ is linear in b, which proves convexity of $\overline{g}(u,b)$. ☐

The following bound is established where the bound on $f_k(x,u)$ is established based on the domain of the self-mapping set:

$$\underline{g}_k(b,u) \leq f_k(x,u) \leq \overline{g}_k(b,u),\ \forall\, x\in\mathcal{P}(b). \tag{18}$$

This forms a compact region that contains the nonlinearity in the given domain, $\mathcal{P}$, which is illustrated in Figure 4. The bound illustrated here are also studied in interval analysis to provide enclosures of the function [27].

C. Enforcing Convexity in Restriction Set

Based on the concave envelope proposed in the previous section, convexity can be enforced for the restriction of feasibility set. First, positive and negative parts of matrix $A\in\mathbb{R}^{c\times r}$ are defined as $[A]^{\pm}\in\mathbb{R}^{c\times r}$ with

$$[A]^+_{ij} = \begin{cases} A_{ij} \text{ if } A_{ij} > 0 \\ 0 \text{ otherwise}\end{cases} \qquad [A]^-_{ij} = \begin{cases} A_{ij} \text{ if } A_{ij} < 0 \\ 0 \text{ otherwise}\end{cases} \tag{19}$$

where $[A]^+_{ij}$ refer to i^{th} row and j^{th} column of matrix $[A]^+$. So $A = [A]^+ + [A]^-$ and $\pm[A]^{\pm}_{ij} \geq 0$.

Lemma 6. For matrix $AD\in\mathbb{R}^{c\times r}$, there exists a nonlinear map $w(u,b) : (\mathbb{R}^s,\mathbb{R}^m)\to\mathbb{R}^c$ with every entry $w_i(u,b)$ is a convex function with respect to (b,u) and

$$\max_{x\in\mathcal{P}(b)} ADf(x,u) \leq w(u,b). \tag{20}$$

This function is given by

$$w(u,b) = [AD]^+\overline{g}(u,b) + [AD]^-\underline{g}(u,b), \tag{21}$$

where $[AD]^{\pm}\in\mathbb{R}^{c\times r}$ refer to positive and negative parts of the matrix defined in equation 19.

Proof. The functions $\overline{g}(u,b)$ and $\underline{g}(u,b)$ establish bounds on $f(x,u)$. For all x in $\mathcal{P}(b)$, $AD\overline{f}(x,u) \leq [AD]^+\overline{g}(u,b) + [AD]^-\underline{g}(u,b)$. Since $\overline{g}(x,u)$ and $-\underline{g}(x,u)$ are convex functions from Lemma 5 and $[AD]^+$ and $-[AD]^-$ have nonnegative entries, convexity is preserved to $w(u,b)$. ☐

Construction of these convex maps allows us to establish convex solvability regions of the original equation (3). Let us first consider the solvability set of equation (3a).

Theorem 2. Suppose $\widehat{\mathcal{W}}^{\mathcal{F}}_u = \left\{u \mid \exists\, b, (u,b)\in\widehat{\mathcal{W}}^{\mathcal{F}}\right\}$ and

$$\widehat{\mathcal{W}}^{\mathcal{F}} = \left\{(u,b) \mid \left[AD\right]^+ \overline{g}(u,b) + \left[AD\right]^- \underline{g}(u,b) \leq b\right\}. \tag{22}$$

If $u\in\widehat{\mathcal{W}}^{\mathcal{F}}_u$, then there exists a solution $x^\star\in\mathcal{F}_x$ to equation (3a). $\widehat{\mathcal{W}}^{\mathcal{F}}_u$ is a convex restriction of solvability set $\mathcal{F}_x$.

Proof. $\left[AD\right]^+\overline{g}(u,b)+\left[AD\right]^-\underline{g}(u,b)\leq b$ is an upper bound of $\max_{x\in\mathcal{P}(b)} ADf(x,u)$ from Lemma 6. Thus the condition in 22 is a sufficient condition for $\max_{x\in\mathcal{P}(b)} ADf(x,u)\leq b$, which ensures solvability according to Lemma 4. ☐

Let us define convex bound on $h(x,u)$ given given a convex over-estimator $\overline{h}(x,u)$ using Lemma 5:

$$\overline{\nu}_k(u,b) = \max_{v\in\partial\mathcal{P}_k(b)} \overline{h}_k(v,u). \tag{23}$$

$\overline{\nu}_k(u,b)\leq 0$ is a sufficient condition for $h_k(x,u)\leq 0$ for all $x\in\mathcal{P}_k(b)$. This ensures the self-mapping set is contained in the feasible set for inequality constraint, $\forall\, x\in\mathcal{P}_k(b)$, $(x,u)\in\mathcal{H}$. Based on this condition, the convex restriction of feasibility set can be established.

Theorem 3. Suppose $\widehat{\mathcal{W}}_u = \left\{u \mid \exists\, b, (u,b)\in\widehat{\mathcal{W}}\right\}$ and

$$\begin{aligned}\widehat{\mathcal{W}} = \big\{(u,b) \mid \left[AD\right]^+\overline{g}(u,b) + \left[AD\right]^-\underline{g}(u,b) \leq b \\ \overline{\nu}_k(u,b)\leq 0,\ k=1,...,r\big\}.\end{aligned} \tag{24}$$

If $u\in\widehat{\mathcal{W}}_u$, then there exists a solution $x^\star$ that satisfies equation (3). $\widehat{\mathcal{W}}_u$ is a convex restriction of feasibility set $\mathcal{N}_x$.

Proof. Constraint $\left[AD\right]^+\overline{g}(u,b)+\left[AD\right]^-\underline{g}(u,b)\leq b$ ensures the existence of solution according to Theorem 2. $\overline{\nu}_k(u,b)\leq 0,\ k=1,...,r$ ensure that the polytope $\mathcal{P}(b)$ lies within the feasible region of inequality constraint, i.e. $(u,x)\subseteq\mathcal{H},\ \forall x\in\mathcal{P}(b)$. Therefore, this is a sufficient constraint for solvability of equation (3a) and feasibility of equation (3b). ☐

Remark. Suppose we are given a base point $(x^{base},u^{base})\in\mathcal{N}$. If $\overline{f}_i(x^{base},u^{base}) = \underline{f}_i(x^{base},u^{base})$, and $\underline{h}_i(x^{base},u^{base}) = h(x^{base},u^{base}) \leq 0$ (i.e. concave envelopes are tight and feasible at the base point), then the convex restriction in Equation 24 is non-empty and contains the base point.

Proof. Since $\mathcal{P}(b) = \{x \mid Ax\leq b\}$ is closed, there exists b such that $\mathcal{P}(b) = \{x^{base}\}$. Since the concave envelopes are tight at the base point, $\overline{g}_i(u^{base},b^{base}) = \underline{g}_i(u^{base},b^{base}) = 0$ and $\overline{\nu}_i(u^{base},b^{base}) = h(x^{base},u^{base})$. Given the base point is feasible, the condition in Theorem 3 is always satisfied at the base point, and thus the convex restriction contains the base point and is non-empty. ☐

From the above remark, a non-empty convex restriction can be always constructed around a feasible base point. The current or planned operating point can be naturally used as the base point for power flow feasibility set. The change of base point can be also explored to construct convex restriction around the desired region.

IV. Convex Restriction of Power Flow Feasibility Set

In this section, convex restriction is constructed for AC power flow equation in polar coordinate. The fixed point form of the power flow equation is not unique, but the polar representation of power flow equation is chosen. The polar representation includes the voltage magnitude explicitly in the equation so it is easier to ensure feasibility of voltage magnitude limits. AC power flow equation in equation (1) can be written in complex plane as follows:

$$p_i + jq_i = \sum_k Y_{ik}^H v_i v_k e^{-j\theta_{ik}}, \quad i \in \mathcal{V}, \tag{25}$$

where $Y_{ik} = G_{ik} + jB_{ik}$, and Y_{ik}^H is a conjugate of Y_{ik}. Suppose the feasible base point if provided by θ^{base} and v^{base}. Let $\Delta\theta = \theta - \theta^{base}$ and $\Delta v = v - v^{base}$. Let the deviation of phase difference be $\theta_{ik} = \theta_{ik}^{base} + \Delta\theta_{ik}$, then

$$p_i + jq_i = \sum_k \left(Y_{ik}^H e^{-j\theta_{ik}^{base}}\right) v_i v_k e^{-j\Delta\theta_{ik}}, \quad i \in \mathcal{V}, \tag{26}$$

where the base point phase is combined with the admittance matrix. This can be expressed with trigonometric functions for $i \in \mathcal{V}$,

$$\begin{aligned} p_i &= \sum_{k\in\mathcal{V}\backslash i} v_i v_k (\widetilde{G}_{ik}\cos\Delta\vartheta_e + \widetilde{B}_{ik}\sin\Delta\vartheta_e) + \widetilde{G}_{ii}v_i^2 \\ q_i &= \sum_{k\in\mathcal{V}\backslash i} v_i v_k (\widetilde{G}_{ik}\sin\Delta\vartheta_e - \widetilde{B}_{ik}\cos\Delta\vartheta_e) - \widetilde{B}_{ii}v_i^2, \end{aligned} \tag{27}$$

where $\widetilde{G}_{ik} + j\widetilde{B}_{ik} = Y_{ik}^H e^{-j\theta_{ik}^{base}}$ and $\Delta\vartheta_e = \Delta\theta_i - \Delta\theta_k,\ e = (i,k)$. The advantage of using equation (27) over (1) is that the concave envelope over trigonometric function is tighter and easier to derive when the base point is at the zero of the trigonometric functions. From the power flow equation, basis functions can be recognized by identifying sparse nonlinearities,

$$\psi(x,u) = \begin{bmatrix} p_{\mathcal{V}} \\ q_{\mathcal{L}} \\ v_{fr}v_{to}\cos\Delta\vartheta \\ v_{fr}v_{to}\sin\Delta\vartheta \\ v_{\mathcal{L}}^2 \\ v_{\mathcal{G}}^2 \end{bmatrix}. \tag{28}$$

where v_{fr} and v_{to} are voltages at the *from* and *to* end of transmission line respectively. The equality condition over active power balance at every bus and reactive power balance at PQ buses will be enforced. Let the uncontrollable variables be the deviation of voltage magnitude and phase deviation, $x = \begin{bmatrix}\Delta\theta_{\mathcal{V}}^T & \Delta v_{\mathcal{L}}^T\end{bmatrix}$. The phase at the slack bus is fixed to zero and the voltage magnitude at PV buses are also fixed. The reactive power balance at PV buses will be placed as the inequality condition to represent reactive power limit. With the basis function in (28), the equality constraint is $F(x,u) = M\psi(x,u) = 0$ where

$$M = \begin{bmatrix} I_{|\mathcal{V}|} & \mathbb{0} & -\widetilde{G}_{\mathcal{V}}^{\mathcal{E}+} & -\widetilde{B}_{\mathcal{V}}^{\mathcal{E}-} & -\widetilde{G}_{\mathcal{L}}^{d} & -\widetilde{G}_{\mathcal{G}}^{d} \\ \mathbb{0} & I_{|\mathcal{L}|} & \widetilde{B}_{\mathcal{L}}^{\mathcal{E}+} & -\widetilde{G}_{\mathcal{L}}^{\mathcal{E}-} & \widetilde{B}_{\mathcal{L}}^{d} & \widetilde{B}_{\mathcal{G}}^{d} \end{bmatrix} \tag{29}$$

where $I_{|\mathcal{V}|} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ is an identity matrix and $\mathbb{0}$ is a matrix with zero entries of appropriate size. Moreover,

$$\widetilde{G}_{le}^{\mathcal{E}+} = \begin{cases} \widetilde{G}_{ik} & \text{if } l = i \\ \widetilde{G}_{ik} & \text{if } l = k \\ 0 & \text{otherwise} \end{cases} \quad \widetilde{G}_{le}^{\mathcal{E}-} = \begin{cases} \widetilde{G}_{ik} & \text{if } l = i \\ -\widetilde{G}_{ik} & \text{if } l = k \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

for $l \in \mathcal{V}$ and $e = (i,k) \in \mathcal{E}$. $\widetilde{B}^{\mathcal{E}\pm}$ are defined in the same way by replacing $\widetilde{G}_{ik}$ with $\widetilde{B}_{ik}$. $\widetilde{G}^d$ and $\widetilde{B}^d$ are diagonal matrices with its diagonal elements equal to diagonals of $\widetilde{G}$ and $\widetilde{B}$, respectively. $\widetilde{G}_{\mathcal{V}}^{\mathcal{E}+}$ denotes selecting only rows $i \in \mathcal{V}$ from $\widetilde{G}^{\mathcal{E}+}$. Residues of basis functions are computed using equation (8),

$$f(x,u) = \begin{bmatrix} p_{\mathcal{V}} - p_{\mathcal{V}}^{base} \\ q_{\mathcal{L}} - p_{\mathcal{V}}^{base} \\ v_{fr}v_{to}(\cos\Delta\vartheta - 1) - v_{fr}^{base}\Delta v_{to} - \Delta v_{fr}v_{to}^{base} \\ v_{fr}v_{to}\sin\Delta\vartheta - v_{fr}^{base}v_{to}^{base}\Delta\underline{\vartheta} \\ v_{\mathcal{L}}^2 - 2v_{\mathcal{L}}^{base}\Delta v_{\mathcal{L}} - \left(v_{\mathcal{L}}^{base}\right)^2 \\ \mathbb{0}_{|\mathcal{G}|} \end{bmatrix}, \tag{31}$$

where a product notation is overloaded to element-wise product. For example, $v_{fr}v_{to}\cos\Delta\vartheta$ is element-wise product of v_{fr}, v_{to}, and $\cos\Delta\vartheta$. The operational constraint on the voltage magnitude and phase can be written as $Ax \le b^{max}$ where

$$A = \begin{bmatrix} E_{\mathcal{V}}^T & \mathbb{0} \\ \mathbb{0} & I_{|\mathcal{L}|} \\ -E_{\mathcal{V}}^T & \mathbb{0} \\ \mathbb{0} & -I_{|\mathcal{L}|} \end{bmatrix} \text{ and } b^{max} = \begin{bmatrix} \Delta\vartheta_{\mathcal{V}}^{max} \\ \Delta v_{\mathcal{L}}^{max} \\ -\Delta\vartheta_{\mathcal{V}}^{min} \\ -\Delta v_{\mathcal{L}}^{min} \end{bmatrix} \tag{32}$$

where $\Delta\vartheta^{min/max} = \vartheta^{min/max} - \vartheta^{base}$ and $\Delta v^{min/max} = v^{min/max} - v^{base}$. $E_{\mathcal{V}}$ is the incidence matrix with columns chosen for only non-slack buses. The reactive power limit constraint on PV buses can be written as $Cf(x,u) \le d$ where

$$C = \begin{bmatrix} \mathbb{0} & \mathbb{0} & -\widetilde{B}_{\mathcal{G}}^{\mathcal{E}+} & \widetilde{G}_{\mathcal{G}}^{\mathcal{E}-} & -\widetilde{B}_{\mathcal{L}}^{d} & -\widetilde{B}_{\mathcal{G}}^{d} \\ \mathbb{0} & \mathbb{0} & \widetilde{B}_{\mathcal{G}}^{\mathcal{E}+} & -\widetilde{G}_{\mathcal{G}}^{\mathcal{E}-} & \widetilde{B}_{\mathcal{L}}^{d} & \widetilde{B}_{\mathcal{G}}^{d} \end{bmatrix} \tag{33}$$

and $d = \begin{bmatrix}(\Delta q_{\mathcal{G}}^{max})^T & -(\Delta q_{\mathcal{G}}^{min})^T\end{bmatrix}^T$. Operational constraint over the active power at every bus and reactive power at load buses will not be considered. These constraints only add upper and lower bound over u, which does not add any interesting features. The inequality constrained set is then $\mathcal{H} = \{(x,u) \mid Ax \le b^{max}, Cf(x,u) \le d\}$. The matrix A will be also used for the self-mapping set, $\mathcal{P} = \{x \mid Ax \le b\}$. Therefore, $\mathcal{P} \subseteq \mathcal{H}$ if $b \le b^{max}$ and $[C]^+\overline{g}(u,b) + [C]^-\underline{g}(u,b) \le d$. The trigonometric terms and its product with voltage magnitudes are bounded effectively with the phase angle differences and voltage magnitudes, and tight concave envelopes can be derived with such self-mapping set. Quadratic concave envelopes will be derived for bilinear and trigonometric functions, and the convex restriction will be constructed as a second order cone.

A. Quadratic concave envelope

The main nonlinearities involved in the power flow equation in polar coordinates are the bilinear, trilinear and trigonometric functions. The concave envelopes of these functions are developed as building blocks.

Corollary 1. Bilinear function can be bounded by the following concave envelopes with $\rho_1,\, \rho_2 > 0$ and $x = x^{base} + \Delta x$ and $y = y^{base} + \Delta y$. :

$$\begin{aligned} xy &\geq -\frac{1}{4}(\rho_1 \Delta x - \frac{1}{\rho_1}\Delta y)^2 + x^{base}\Delta y + \Delta x y^{base} + x^{base} y^{base} \\ xy &\leq \frac{1}{4}(\rho_2 \Delta x + \frac{1}{\rho_2}\Delta y)^2 + x^{base}\Delta y + \Delta x y^{base} + x^{base} y^{base}. \end{aligned} \tag{34}$$

The over-estimator is tight along $\rho_2\Delta x - \frac{1}{\rho_2}\Delta y = 0$, and the under-estimator is tight along $\rho_2\Delta x + \frac{1}{\rho_2}\Delta y = 0$. Both over and under-estimators are tight at the base point.

A scalar quadratic function can be bounded using the bilinear envelope as well. Choose $\rho_1 = \rho_2 = 1$, then $x^2 \geq 0$ is a concave under estimator and x^2 itself is a convex over estimator of x^2.

Corollary 2. Trigonometric functions can be bounded by the following quadratic concave envelopes:

$$\begin{aligned} \sin\theta &\geq \theta + \left(\frac{\sin\theta^{max} - \theta^{max}}{(\theta^{max})^2}\right)\theta^2,\ \ \theta < \theta^{max} \\ \sin\theta &\leq \theta + \left(\frac{\sin\theta^{min} - \theta^{min}}{(\theta^{min})^2}\right)\theta^2,\ \ \theta > \theta^{min} \end{aligned} \tag{35}$$

where $\theta^{max} \in [0,\, \pi]$ and $\theta^{min} \in [-\pi,\, 0]$, and

$$\begin{aligned} \cos\theta &\geq 1 - \frac{1}{2}\theta^2 \\ \cos\theta &\leq 1 \end{aligned} \tag{36}$$

for all θ.

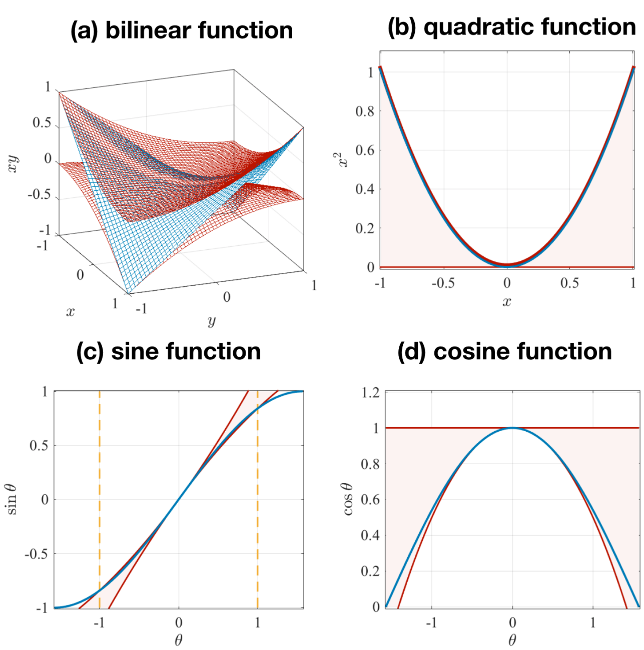

Fig. 5. Illustration of concave envelopes in Corollary 1 and 2. In (c), θ^{max} and θ^{min} is drawn with yellow dashed line.

Envelopes for bilinear and trigonometric functions are illustrated in Figure 5. Given the concave envelopes, the bound over intervals can be defined and given in the Appendix. These are special cases of Lemma 5 for the given function. Trilinear functions are bounded by cascading bilinear concave envelope. Essentially, $v_{fr}v_{to}\Delta\cos\vartheta$ was bounded by defining intermediate variable $vv = v_{fr}v_{to}$, and the bilinear envelope was applied to vv and $vv\Delta\cos\vartheta$. It is important to note that the vertices still need to be aligned in the overall function, not only within the intermediate function. In the following Lemma, we finally state the analytical expression in the form of second order cone for the convex restriction of power flow feasibility set.

Corollary 3. (**Convex Restriction of Power Flow Feasibility Set with Second Order Cone**) The following second order cone constraint provide a convex restriction of power flow feasibility set described in equation (1) and (2). This provides a sufficient condition for existence of a feasible solution in controllable variable domain, $u = \left[(p_{\mathcal{V}} - p_{\mathcal{V}}^{base})^T \quad (q_{\mathcal{L}} - q_{\mathcal{L}}^{base})^T\right]^T$.

$$\begin{aligned} & [AD]^+\,\overline{g} + [AD]^-\,\underline{g} \leq b \\ & [C]^+\overline{g} + [C]^-\underline{g} \leq d,\ \ b \leq b^{max} \\ & b = \begin{bmatrix} -\Delta\underline{\vartheta}^T & -\Delta\underline{v}^T & \Delta\overline{\vartheta}^T & \Delta\overline{v}^T \end{bmatrix}^T \\ & \overline{g} = \begin{bmatrix} p_{\mathcal{V}}^T & q_{\mathcal{L}}^T & \overline{g}_3^T & \overline{g}_4^T & \overline{g}_5^T & \overline{g}_6^T \end{bmatrix}^T \\ & \Delta\underline{\vartheta} \leq \Delta\overline{\vartheta},\ \ \Delta\underline{v} \leq \Delta\overline{v} \\ & \forall\ e = (i,k) \in \mathcal{E} \\ & \quad \overline{g}_{3,e} \geq \overline{g}_{1,e}^{vv\cos} - v_i^{base}\Delta\overline{v}_k - \Delta\overline{v}_i v_k^{base} - v_{ik}^{base} \\ & \quad \overline{g}_{3,e} \geq \overline{g}_{2,e}^{vv\cos} - v_i^{base}\Delta\underline{v}_k - \Delta\underline{v}_i v_k^{base} - v_{ik}^{base} \\ & \quad \underline{g}_{3,e} \leq \underline{g}_{1,e}^{vv\cos} - v_i^{base}\Delta\underline{v}_k - \Delta\overline{v}_i v_k^{base} - v_{ik}^{base} \\ & \quad \underline{g}_{3,e} \leq \underline{g}_{2,e}^{vv\cos} - v_i^{base}\Delta\overline{v}_k - \Delta\underline{v}_i v_k^{base} - v_{ik}^{base} \\ & \quad \overline{g}_{4,e} \geq \overline{g}_{1,e}^{vv\sin} - v_{ik}^{base}\Delta\overline{\vartheta} \\ & \quad \overline{g}_{4,e} \geq \overline{g}_{2,e}^{vv\sin} - v_{ik}^{base}\Delta\underline{\vartheta} \\ & \quad \underline{g}_{4,e} \leq \underline{g}_{1,e}^{vv\sin} - v_{ik}^{base}\Delta\overline{\vartheta} \\ & \quad \underline{g}_{4,e} \leq \underline{g}_{2,e}^{vv\sin} - v_{ik}^{base}\Delta\underline{\vartheta} \\ & \forall\ k \in \mathcal{G},\ \ \underline{g}_{5,k} \geq 0,\ \ \underline{g}_{5,k} \leq 0 \\ & \forall\ k \in \mathcal{L},\ \ \overline{g}_{6,k} \geq \Delta\overline{v}_k^2,\ \ \overline{g}_{6,k} \geq \Delta\underline{v}_k^2,\ \ \underline{g}_{6,k} \leq 0 \end{aligned} \tag{37}$$

where $v_{ik}^{base} = v_i^{base}v_k^{base}$. $\overline{g}_e^{vv\cos}/\underline{g}_e^{vv\cos}$ and $\overline{g}_e^{vv\sin}/\underline{g}_e^{vv\sin}$ denote the interval bound of $v_i v_k\cos\vartheta_e$ and $v_i v_k \sin\vartheta_e$. These bounds are provided in the Appendix.

Remark. The number of constraints grows linearly with respect to the number of bus and number of lines. The number of equality and inequality constraints involved in Corollary 3 is $11|\mathcal{V}| + 38|\mathcal{E}| + |\mathcal{L}|$ where $|\mathcal{V}|$, $|\mathcal{E}|$ and $|\mathcal{L}|$ are number of buses, lines and PQ buses, respectively.

B. Visualization

Here, we provide visualization of the convex restriction in 2 dimensional space. The plots are drawn for a cross section of the region crossing the base point along two chosen plot variables. The actual feasible set was solved using Newton Raphson over a mesh grid in the plot region. MATPOWER package was used for solving Newton-Raphson and the same data set was used for convex restriction [28]. The second order cone convex restriction was plotted by testing feasibility of the constraint over the same mesh grid. The results are shown for IEEE 9 bus, 39 bus and 118 bus system in Figure 6, 7, and 8. The convex restriction appears to be very tight along some of the boundaries. The blue lines in the figure plots individual voltage magnitude and reactive power limits. Individual blue line indicates the boundary of violating a single operational constraint.

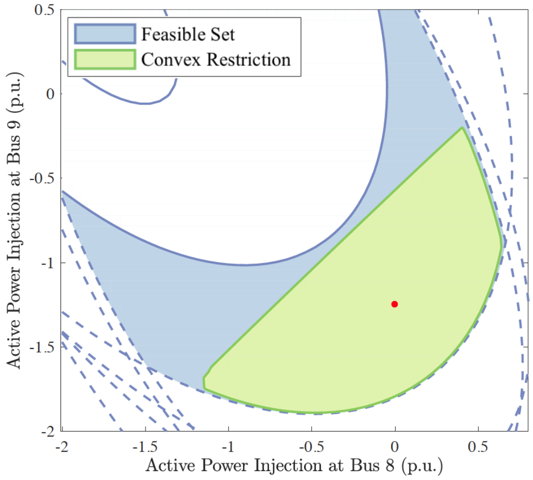

Fig. 6. Convex restriction of feasible active power injection set in 9 bus system with voltage limit of 1% deviation from the base operating point.

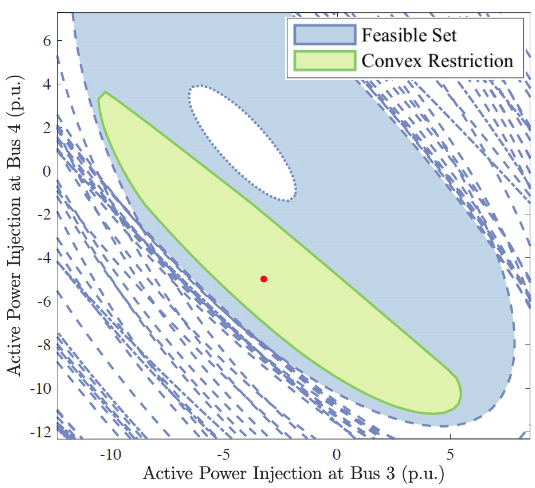

Fig. 7. Convex restriction of feasible active power injection set in 39 bus system with voltage limit of 1% deviation from the base operating point.

V. Conclusion

This paper proposed a convex restriction of a general feasibility set and presented its application to power flow equation with operational constraints. These results suggest a new understanding of power flow feasibility set as a counterpart to the convex relaxation. The convex restriction of power flow feasibility set was constructed in a closed-form expression as a second order cone constraint. Reliability of the power

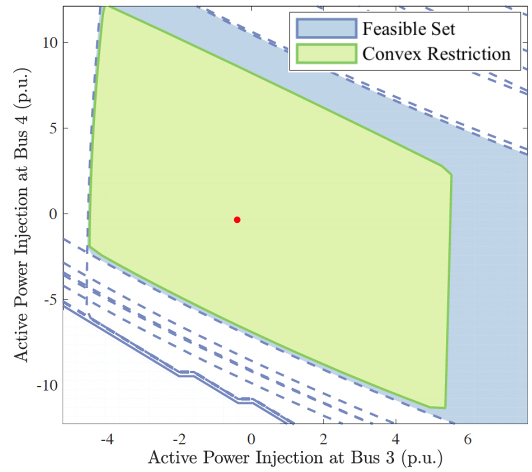

Fig. 8. Convex restriction of feasible active power injection set in 118 bus system with voltage limit of 5% deviation from the base operating point.

grid is a top priority in the operation and analysis, and the convex restriction gives a guarantee for existence of steady-state solution that respects the operational constraint. Plots of the region shows that our construction is very close to the actual feasible region along some of the boundaries. For future work, our closed-form expression can replace power flow equations to design tractable algorithms in emergency control and security assessment.

Appendix

The bound over the interval used in the convex restriction of power flow feasibility set are listed here. The intervals are given by $\vartheta_e \in [\Delta\underline{\vartheta}_e, \Delta\overline{\vartheta}_e]$ and $v_e \in [\Delta\underline{v}_e, \Delta\overline{v}_e]$. These are results directly from Lemma 5 with envelopes presented in Corollary 1 and 2. $\rho_1 = \rho_2 = 1$ was used for bounding bilinear function. These bounds are established for the following indices: $\forall\ e = (i,k) \in \mathcal{E}$.

A. Cosine Bound over Interval

$\cos\Delta\vartheta_e - 1$ over $\Delta\vartheta_e \in [\Delta\underline{\vartheta}_e, \Delta\overline{\vartheta}_e]$ is bounded by the following inequalities:

$$\overline{g}_e^{\Delta\cos} \geq 0 \quad \underline{g}_e^{\Delta\cos} \leq -\frac{\Delta\overline{\vartheta}_e^2}{2}, \quad \underline{g}_e^{\Delta\cos} \leq -\frac{\Delta\underline{\vartheta}_e^2}{2}. \tag{38}$$

B. Sine Bound over Interval

$\sin\Delta\vartheta_e$ over $\Delta\vartheta_e \in [\Delta\underline{\vartheta}_e, \Delta\overline{\vartheta}_e]$ is bounded by the following inequalities:

$$\begin{aligned}
&\overline{g}_{1,e}^{\Delta\sin} \geq \Delta\overline{\vartheta}_e + \left(\frac{\sin\overline{\vartheta}_e^{min} - \overline{\vartheta}_e^{min}}{(\overline{\vartheta}_e^{min})^2}\right)\Delta\overline{\vartheta}_e^2\\
&\overline{g}_{2,e}^{\Delta\sin} \geq \Delta\underline{\vartheta}_e + \left(\frac{\sin\overline{\vartheta}_e^{min} - \overline{\vartheta}_e^{min}}{(\overline{\vartheta}_e^{min})^2}\right)\Delta\underline{\vartheta}_e^2\\
&\underline{g}_{1,e}^{\Delta\sin} \leq \Delta\overline{\vartheta}_e + \left(\frac{\sin\underline{\vartheta}_e^{max} - \underline{\vartheta}_e^{max}}{(\underline{\vartheta}_e^{max})^2}\right)\Delta\overline{\vartheta}_e^2\\
&\underline{g}_{2,e}^{\Delta\sin} \leq \Delta\underline{\vartheta}_e + \left(\frac{\sin\underline{\vartheta}_e^{max} - \underline{\vartheta}_e^{max}}{(\underline{\vartheta}_e^{max})^2}\right)\Delta\underline{\vartheta}_e^2\\
&\Delta\underline{\vartheta}_e \leq \underline{\vartheta}_e^{max}, \ \Delta\overline{\vartheta}_e \geq \overline{\vartheta}_e^{min}.
\end{aligned} \tag{39}$$

C. Bilinear Bound over Interval

$\Delta v_i \Delta v_k - v_{ik}^{base}$ over $\Delta v_i \in [\Delta \underline{v}_i,\, \Delta \overline{v}_i]$ is bounded by the following inequalities:

$$\begin{aligned}
&\overline{g}_{1,e}^{\Delta vv} \geq \frac{1}{4}(\Delta \overline{v}_i + \Delta \overline{v}_k)^2 + v_i^{base}\Delta \overline{v}_k + \Delta \overline{v}_i v_k^{base} \\
&\overline{g}_{2,e}^{\Delta vv} \geq \frac{1}{4}(\Delta \underline{v}_i + \Delta \underline{v}_k)^2 + v_i^{base}\Delta \underline{v}_k + \Delta \underline{v}_i v_k^{base} \\
&\underline{g}_{1,e}^{\Delta vv} \leq -\frac{1}{4}(\Delta \overline{v}_i - \Delta \underline{v}_k)^2 + v_i^{base}\Delta \underline{v}_k + \Delta \overline{v}_i v_k^{base} \\
&\underline{g}_{2,e}^{\Delta vv} \leq -\frac{1}{4}(\Delta \underline{v}_i - \Delta \overline{v}_k)^2 + v_i^{base}\Delta \overline{v}_k + \Delta \underline{v}_i v_k^{base} \\
&\underline{g}_{e}^{\Delta vv} \leq \underline{g}_{1,e}^{\Delta vv},\ \underline{g}_{e}^{\Delta vv} \leq \underline{g}_{2,e}^{\Delta vv} \\
&\overline{g}_{e}^{\Delta vv} \geq \overline{g}_{1,e}^{\Delta vv},\ \overline{g}_{e}^{\Delta vv} \geq \overline{g}_{2,e}^{\Delta vv}
\end{aligned} \tag{40}$$

where $v_{ik}^{base} = v_i^{base} v_k^{base}$.

D. $v_i v_k \cos\theta_{ik}$ Bound over Interval

$\Delta v_i \Delta v_k \cos\theta_{ik} - v_{ik}^{base}$ over $\Delta v_i \in [\Delta \underline{v}_i,\, \Delta \overline{v}_i]$ and $\Delta v_i \in [\Delta \underline{v}_i,\, \Delta \overline{v}_i]$ is bounded by the following inequalities:

$$\begin{aligned}
&\overline{g}_{1,e}^{vv\cos} \geq \overline{g}_{1,e}^{\Delta vv},\ \overline{g}_{2,e}^{vv\cos} \geq \overline{g}_{2,e}^{\Delta vv} \\
&\underline{g}_{1,e}^{vv\cos} \leq -\frac{1}{4}(\underline{g}_{e,1}^{\Delta vv} - \underline{g}_{e}^{\Delta\cos})^2 + v_{ik}^{base}\underline{g}_{e}^{\Delta\cos} + \underline{g}_{e,1}^{\Delta vv} + v_{ik}^{base} \\
&\underline{g}_{2,e}^{vv\cos} \leq -\frac{1}{4}(\underline{g}_{e,2}^{\Delta vv} - \underline{g}_{e}^{\Delta\cos})^2 + v_{ik}^{base}\underline{g}_{e}^{\Delta\cos} + \underline{g}_{e,2}^{\Delta vv} + v_{ik}^{base}.
\end{aligned} \tag{41}$$

E. $v_i v_k \sin\theta_{ik}$ Bound over Interval

$\Delta v_i \Delta v_k \sin\theta_{ik}$ over $\Delta v_i \in [\Delta \underline{v}_i,\, \Delta \overline{v}_i]$ and $\Delta v_i \in [\Delta \underline{v}_i,\, \Delta \overline{v}_i]$ is bounded by the following inequalities:

$$\begin{aligned}
&\overline{g}_{1,e}^{vv\sin} \geq \frac{1}{4}(\overline{g}_{e}^{\Delta vv} + \overline{g}_{1,e}^{\Delta\sin})^2 + v_{ik}^{base}\overline{g}_{1,e}^{\Delta\sin} \\
&\overline{g}_{2,e}^{vv\sin} \geq \frac{1}{4}(\overline{g}_{e}^{\Delta vv} + \overline{g}_{2,e}^{\Delta\sin})^2 + v_{ik}^{base}\overline{g}_{2,e}^{\Delta\sin} \\
&\underline{g}_{1,e}^{vv\sin} \leq -\frac{1}{4}(\underline{g}_{e}^{\Delta vv} - \underline{g}_{1,e}^{\Delta\sin})^2 + v_{ik}^{base}\underline{g}_{1,e}^{\Delta\sin} \\
&\underline{g}_{2,e}^{vv\sin} \leq -\frac{1}{4}(\underline{g}_{e}^{\Delta vv} - \underline{g}_{2,e}^{\Delta\sin})^2 + v_{ik}^{base}\underline{g}_{2,e}^{\Delta\sin}.
\end{aligned} \tag{42}$$

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