Passive scalar evolution in peripheral regions

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We consider the evolution of a passive scalar (concentration of pollutants or temperature) in a chaotic (turbulent) flow. A universal asymptotic behavior of the passive scalar decay (homogenization) related to peripheral regions (near walls) is established. The passive scalar moments and their pair correlation function in the peripheral region are analyzed. A special case investigated in our paper is the passive scalar decay along a pipe.

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INTRODUCTION

The problem of mixing attracts great attention both due to its fundamental significance and a variety of applications. Mixing rate of an additive to a fluid is very sensitive to the character of the hydrodynamic flow excited in the fluid. Recently an essential breakthrough was achieved in the theory of the so-called passive scalar (which can be concentration of pollutants or temperature) in chaotic and turbulent flows (see reviews [1,2]). The word "passive" means that a feedback of the scalar field to the flow is negligible. It is correct for dilute solutions of pollutants and for relatively weak fluctuations of temperature. The passive scalar in a fluid is subjected to diffusion (thermodiffusion) and advection, leading to the evolution of its spatial distribution. A final result of the evolution is a homogeneous state of the passive scalar. We are interested in the passive scalar decay, that is, in the laws governing the homogenization of the passive scalar field in chaotic and turbulent flows. We consider the case of the large Schmidt number Sc which is the ratio of the kinematic viscosity of the fluid ν to the diffusion (thermodiffusion) coefficient κ.

As is noted in the paper [3], at large Sc the homogenization of the passive scalar field in the peripheral regions is slower than in the bulk since mixing is suppressed near walls of the vessel. Then the advanced stages of the passive scalar decay are dominated just by the peripheral regions. In the paper [3] main stages related to the passive scalar evolution in the peripheral region were established (see also the paper [4]). Here we develop further the theory of the passive scalar evolution in the peripheral region having in mind two different physical situations. The first case is the passive scalar in the viscous boundary layer of the developed (high-Reynolds-number) turbulence (explanations of its properties can be found in the book of Monin and Yaglom [5]). The second case is the peripheral region of a chaotic flow. A perfect example of such a flow is the so-called elastic turbulence state revealed by Groisman and Steinberg [6] in dilute polymer solutions. Both the velocity in the viscous boundary layer (in the first case) and the chaotic velocity (in the second case) can be treated as smooth. For the elastic turbulence it is explained by the character of the velocity spectrum which decays faster than k^{-3} [6]. Though we consider two physically different cases, their unified description is admitted because of the universal behavior of the fluid flow near a wall.

The smoothness of the random velocity field is one of the key ingredients of our analysis. The theory of the passive scalar in a smooth random flow was pioneered by Batchelor [7] and Kraichnan [8] who considered the cases of velocities long correlated and short correlated in time, respectively. The case of arbitrary velocity correlation time was considered in the paper [9]. In the works the flow was treated as unbounded and the passive scalar was assumed to possess a stationary statistics due to pumping. Properties of the passive scalar decay in an unbounded statistically homogeneous random flow are also known. As was demonstrated in the paper [10], in the inertial range of the developed turbulence the scalar decays in accordance with a power law. Besides, it is interesting to consider the passive scalar decay in the Batchelor region of scales (below the Kolmogorov viscous scale), where the velocity field is smooth (for details see, e.g., the book of Batchelor [11]). The decay law of the passive scalar in the region is exponential, as is demonstrated in the papers [12,13]. The same consideration is applicable to the elastic turbulence. In the paper [3] the combined case was considered, when both the inertial region of scales and the Batchelor region of scales are taken into account. Then the passive scalar decay is dominated by eddies from the inertial interval (excluding, maybe, some initial stage of evolution) and is, consequently, governed by a power law.

As is noted in the paper [3] the theory of the passive scalar in the random flow, developed for the bulk, needs a modification for the peripheral region even for the smooth flow. The reason is that the approximation of the velocity by linear profiles, used for the bulk, fails in the peripheral region. However, near a wall, the velocity possesses a definite dependence on the separation from the wall, explaining the universal properties of the passive scalar decay in the peripheral region. Another property, simplifying an analysis, is the slowness of the passive scalar mixing in the peripheral region, enabling one to treat the velocity as short correlated. The first problem, we consider, is an evolution of the concentration of pollutants, when the boundaries are considered as impenetrable for the pollutants. This approach can be extended to the case of the binary chemical reactions (see the paper [4]). Another problem, which can be treated inside our approach, is the temperature relaxation in the bulk, if the temperature at the walls of the vessel is fixed. Then a heat flow from the boundary to the bulk is forced which is governed just by the velocity fluctuations in the peripheral region. The passive scalar evolution can be examined also in a fluid (where random motion is excited) flowing through a pipe. Then the role of time is played by the coordinate along the pipe. This setup is closely related to the experiment of Groisman and Steinberg [14].

A remarkable property of a turbulent flow is its strong intermittency leading to the so-called anomalous scaling of the velocity correlation functions, which are dependent on the integral scale of turbulence at a power, referred to as the anomalous scaling index (see, e.g., the book of Frisch [15]). Theoretically, an existence of the anomalous scaling of such kind was established for the passive scalar in the framework of the so-called Kraichnan model (see Ref. [16]). One could anticipate that the same phenomenon takes place for the passive scalar decay. We demonstrate that in the peripheral region the passive scalar moments possess, indeed, an anomalous scaling, which is in some sense extreme. Namely, all the moments of the passive scalar damp in accordance with the same power law at increasing the separation from the wall.

The paper is organized as follows. In Sec. I we present general equations, which describe the passive scalar evolution in the peripheral region. In Sec. II we analyze the passive scalar evolution near the wall of the vessel. In Sec. III we consider the passive scalar decay along a pipe. Some general remarks and a short comparison with experiment are presented in the Conclusion.

I. GENERAL RELATIONS

Advection of a passive scalar field θ by a moving fluid (accompanied by the passive scalar diffusion) is described by the equation

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta = \kappa \nabla^2 \theta , \tag{1.1}$$

where $\boldsymbol{v}$ is the flow velocity and κ is the diffusion coefficient. Below, the fluid is assumed to be incompressible (that is, $\nabla \cdot \boldsymbol{v} = 0$). A formal solution of the Cauchy problem for Eq. (1.1) can be written as

$$\theta(t_2) = T \exp\left\{ \int_{t_1}^{t_2} dt \left[-\boldsymbol{v}(t) \cdot \nabla + \kappa \nabla^2 \right] \right\} \theta(t_1), \tag{1.2}$$

where $\exp T$ means a chronologically ordered exponent. Of course, some boundary conditions for the passive scalar θ should be introduced. There could be two different types of the boundary conditions. If θ is temperature (and walls are made of a well heat conducting material) then θ is fixed at the boundary. If θ is the density of pollutants and the wall is impenetrable for the pollutants then the gradient of θ in the direction perpendicular to the boundary is zero (which corresponds to zero pollutant flux to the boundary).

We consider a random flow which has to be characterized statistically: via correlation functions. The correlation functions are averages over time, they can be treated also as averages over velocity realizations. The flow is assumed to be statistically homogeneous in time, whereas there is no homogeneity in space (because of the boundary effects). Here we consider the flow in a closed vessel where the average velocity is equal to zero. A generalization to the case when the average velocity is nonzero is trivial (it can be found in Sec. III where a fluid pushed through a pipe is treated). The pair velocity correlation function $\langle v_\alpha(t_1, \boldsymbol{r}_1) v_\beta(t_2, \boldsymbol{r}_2) \rangle$ depends on the time difference $t_1 - t_2$ only (due to the assumed time homogeneity), and on coordinates of both points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

We have in mind two physically different situations. First, it is the viscous boundary layer of the developed high-Reynolds number turbulence (for details see, e.g., the book [5]). Second, it is the peripheral region of a chaotic flow. As a perfect example of the chaotic flow, the elastic turbulence can be noted [6]. In both cases the velocity is smooth in the peripheral region we are interested in. That means that the velocity does not contain components with scales smaller than L where L is the width of the viscous boundary layer (in the first case) and the vessel size (in the second case). Besides, the velocity field possesses some peculiarities, related to zero value of the velocity at the boundary. That makes the passive scalar decay in the peripheral region slow. Since the velocity correlation time is determined by dynamics in the bulk, which is relatively fast, at examining the passive scalar evolution in the peripheral region the velocity can be treated as short correlated in time. It is well known that in this case closed equations for the passive scalar correlation functions can be derived. Below, we demonstrate principal steps of the derivation, based on Eq. (1.2).

Let us examine the passive scalar evolution on a time interval (t_1, t_2) taking the difference $t_2 - t_1$ much larger than the velocity correlation time τ, but much smaller than the characteristic mixing time (the gap between the mixing time and the velocity correlation time exists due to the noted weakness of mixing in the peripheral region related to the inequality $\mathrm{Sc} \gg 1$). The last condition enables one to produce an expansion of the T exponent in Eq. (1.2). One may keep only two first terms of the expansion

$$\theta(t_2) \approx \theta(t_1) + (t_2 - t_1) \kappa \nabla^2 \theta(t_1) - \int_{t_1}^{t_2} dt \boldsymbol{v}(t) \cdot \nabla \theta(t_1) + \int_{t_1}^{t_2} dt \int_{t_1}^{t} dt' \boldsymbol{v}(t) \cdot \nabla [\boldsymbol{v}(t') \cdot \nabla \theta(t_1)]. \tag{1.3}$$

The next step is averaging over the velocity statistics inside the interval (t_1, t_2). This averaging is independent of the velocity profiles at $t < t_1$ and $t > t_2$ due to the condition $t_2 - t_1 \gg \tau$. Averaging the expression (1.3), one obtains a relation for the average value $\langle \theta \rangle$ of the passive scalar

$$\langle \theta(t_2, \boldsymbol{r}) \rangle - \langle \theta(t_1, \boldsymbol{r}) \rangle = (t_2 - t_1) \kappa \nabla^2 \langle \theta(t_1, \boldsymbol{r}) \rangle + (t_2 - t_1) \times \nabla_\alpha [D_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}) \nabla_\beta \langle \theta(t_1, \boldsymbol{r}) \rangle], \tag{1.4}$$

$$D_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int_0^\infty dt \, \langle v_\alpha(t, \boldsymbol{r}_1) v_\beta(0, \boldsymbol{r}_2) \rangle, \tag{1.5}$$

where a fast enough decay of the pair velocity correlation function with t is implied. We used also the incompressibility condition $\nabla \cdot \boldsymbol{v} = 0$. The quantity $\langle\theta\rangle$ means a value averaged over the velocity fluctuations. If $t_2 - t_1$ is smaller than the mixing time, then the right-hand side of Eq. (1.4) is a small correction to $\langle\theta\rangle$. Therefore Eq. (1.4) can be rewritten in the differential form

$$\partial_t \langle\theta\rangle = \nabla_\alpha [D_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}) \nabla_\beta \langle\theta\rangle] + \kappa \nabla^2 \langle\theta\rangle. \tag{1.6}$$

The quantity $D_{\alpha\beta}$, entering Eq. (1.6), can be called the eddy diffusion tensor, since it describes diffusion of the passive scalar related to its chaotic displacements in the random flow (similar to the Brownian motion). This effect can be compared to the turbulent diffusion causing a passive scalar evolution in turbulent flows on scales larger than the integral scale. However, the eddy diffusion tensor $D_{\alpha\beta}$ is connected with a smooth flow, and can be, consequently, used for description of the passive scalar dynamics on small scales. A coordinate dependence of $D_{\alpha\beta}$ is related to the statistical inhomogeneity of the random flow near the boundary (wall).

Analogously, starting from Eq. (1.2), one can derive closed equations for high-order correlation functions of θ. Say, the equation for the pair correlation function F is

$$\begin{aligned} \partial_t F(t,\boldsymbol{r}_1,\boldsymbol{r}_2) = {} & \kappa(\nabla_1^2 + \nabla_2^2)F + \nabla_{1\alpha}[D_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_1)\nabla_{1\beta}F] \\ & + \nabla_{2\alpha}[D_{\alpha\beta}(\boldsymbol{r}_2,\boldsymbol{r}_2)\nabla_{2\beta}F] \\ & + \nabla_{1\alpha}[D_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2)\nabla_{2\beta}F] \\ & + \nabla_{2\alpha}[D_{\alpha\beta}(\boldsymbol{r}_2,\boldsymbol{r}_1)\nabla_{1\beta}F], \end{aligned} \tag{1.7}$$

$$F(t,\boldsymbol{r}_1,\boldsymbol{r}_2) = \langle \theta(t,\boldsymbol{r}_1)\,\theta(t,\boldsymbol{r}_2)\rangle. \tag{1.8}$$

Generally, the equation for the nth order correlation function F_n of the passive scalar is

$$\partial_t F_n = \kappa \sum_{m=1}^{n} \nabla_m^2 F_n + \sum_{m,k=1}^{n} \nabla_{m\alpha}[D_{\alpha\beta}(\boldsymbol{r}_m,\boldsymbol{r}_k)\nabla_{k\beta}F_n], \tag{1.9}$$

$$F_n(t,\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) = \langle \theta(t,\boldsymbol{r}_1) \cdots \theta(t,\boldsymbol{r}_n)\rangle. \tag{1.10}$$

The structure of Eq. (1.9) is transparent: the evolution of the passive scalar correlation function is determined by the molecular diffusion (the first term in the right-hand side) and by the eddy diffusion (the second term in the right-hand side).

Note that if the molecular diffusivity is negligible then it is possible to obtain a closed equation for the moments $\langle\theta^n(\boldsymbol{r})\rangle = F_n(\boldsymbol{r}, \ldots, \boldsymbol{r})$ from Eq. (1.9),

$$\partial_t \langle\theta^n(\boldsymbol{r})\rangle = \nabla_\alpha [D_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}) \nabla_\beta \langle\theta^n(\boldsymbol{r})\rangle], \tag{1.11}$$

which is identical to Eq. (1.6), without the molecular diffusion term. Equation (1.11) is a direct consequence of the relation $\partial_t(\theta^n) = -\boldsymbol{v}\cdot\nabla(\theta^n)$, which follows from Eq. (1.1), if the molecular diffusion is neglected.

The eddy diffusion tensor D can be estimated as $D \sim V^2\tau$, where V is the characteristic value of the velocity fluctuations, and τ is the velocity correlation time, which can be estimated as $\tau \sim L/V_L$. Here L is the size of the viscous boundary layer (Kolmogorov length) in the case of the high-Reynolds number turbulence and the size of the vessel in the case of the chaotic flow (elastic turbulence), and V_L is characteristic value of the velocity fluctuations in the bulk. For the high-Reynolds number turbulence V_L is the friction velocity in the boundary layer. The weakness of the passive scalar decay inside the peripheral region is explained by smallness of the ratio V/V_L there. The weakness makes the passive scalar evolution in the peripheral region slow, thus justifying our approach where velocity is treated as short correlated in time. Note that for the elastic turbulence τ is determined by the polymer relaxation time and the condition $\tau \sim L/V_L$ is no other than the Lumley criterion of strong polymer elongation formulated in Ref. [17] (see also Ref. [18]).

Let us explain the physical meaning of the passive scalar correlation functions. They are averages of the passive scalar fluctuations over the velocity statistics. Therefore, to obtain the correlation functions experimentally or in numerics, one has to measure the passive scalar decay many times (for many realizations of the velocity field) and then to average the result over the attempts. Initial conditions for the passive scalar field are implied to be fixed at the averaging procedure. If we consider the case of a fluid pushed through a pipe, then the passive scalar correlation functions are stationary. Then they can be treated as averages over long time.

Below we consider a passive scalar evolution in the peripheral region. We assume that mixing already produced the homogeneous distribution of the passive scalar in the bulk (recall that the process in the bulk is much faster than in periphery). And we subtract from θ a constant, corresponding to the bulk value of θ (this redefinition does not change the equations describing the passive scalar evolution). In other words, the value of θ tends to zero when we go away from the boundary. Analyzing the evolution of the passive scalar after the homogenization in the bulk, we treat its initial value θ_0 as dependent mainly on the separation from the wall and practically independent of coordinates along the wall.

II. DECAY IN A VESSEL

Here we consider the passive scalar decay in a closed vessel. If the walls of the vessel are smooth and their curvature is of the order of the vessel size, then at considering the peripheral region the boundary can be treated as flat in the main approximation. Then it is possible to introduce the orthogonal reference system, where the coordinate q measures separation from the boundary and r_ζ are coordinates along the wall. In the reference frame the incompressibility condition is written as $\partial_q v_q + \partial v_1/\partial r_1 + \partial v_2/\partial r_2 = 0$, where v_q is the velocity component perpendicular to the wall and $v_{1,2}$ are the components along the wall. Since $v_{1,2}$ tends to zero as $q \to 0$ then $v_{1,2} \propto q$ near the boundary, and the incompressibility leads to the proportionality law $v_q \propto q^2$. This is the main feature of the velocity profile in the peripheral region.

For the flat wall, it is natural to assume homogeneity of the velocity correlation functions along the wall and also their isotropy in the planes parallel to the wall. Then we obtain a general expression for the components of the eddy diffusion tensor (1.5),

$$D_{\zeta\sigma}(\boldsymbol{r}_1,\boldsymbol{r}_2)=H_1(\varrho)\delta_{\zeta\sigma}q_1q_2+H_2(\varrho)\varrho_\zeta\varrho_\sigma q_1q_2,$$

$$D_{q\sigma}(\boldsymbol{r}_1,\boldsymbol{r}_2)=-\frac{1}{2}H_1'(\varrho)\frac{\varrho_\sigma}{\varrho}q_1^2q_2-\frac{1}{2}[\varrho H_2'(\varrho)+3H_2(\varrho)]\varrho_\sigma q_1^2q_2,$$

$$D_{\zeta q}(\boldsymbol{r}_1,\boldsymbol{r}_2)=\frac{1}{2}H_1'(\varrho)\frac{\varrho_\zeta}{\varrho}q_1q_2^2+\frac{1}{2}[\varrho H_2'(\varrho)+3H_2(\varrho)]\varrho_\zeta q_1q_2^2,$$

$$D_{qq}(\boldsymbol{r}_1,\boldsymbol{r}_2)=-\frac{1}{4}\left[\frac{H_1'(\varrho)}{\varrho}+H_1''(\varrho)\right]q_1^2q_2^2-\frac{1}{4}[6H_2(\varrho)+6\varrho H_2'(\varrho)+\varrho^2H_2''(\varrho)]q_1^2q_2^2, \tag{2.1}$$

where $\varrho_\zeta=r_{1\zeta}-r_{2\zeta}$, H_1 and H_2 are some functions of ϱ, and the subscripts ζ, σ are running over 1 and 2. The structure of the eddy diffusion tensor (2.1) is explained by the q dependence of the velocity near the wall (stated above). The dependence leads to the proportionality laws $D_{q\alpha}\propto q_1^2$, $D_{\zeta\alpha}\propto q_1$, and analogously for the second point. Then the ϱ dependence of the tensor components is established using the conditions $\nabla_{1\alpha}D_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2)=0=\nabla_{2\beta}D_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_2)$, following from the incompressibility.

Since we consider the region where the velocity field is smooth, then both H_1 and H_2 have regular expansion in ϱ (containing even powers) at small ϱ, that is, at $\varrho\ll L$ (recall that L is the thickness of the peripheral region which is the thickness of the viscous boundary layer in the case of the developed turbulence and is of the order of the vessel size for the elastic turbulence). In this limit the main contribution to the eddy diffusion tensor is determined by first terms of the expansion

$$H_1\approx H_{10}-(\mu+3H_{20}/2)\varrho^2,\quad H_2\approx H_{20}, \tag{2.2}$$

$$D_{\zeta\sigma}\approx H_{10}\delta_{\zeta\sigma}q_1q_2,\quad D_{qq}\approx\mu q_1^2q_2^2. \tag{2.3}$$

The quantity μ characterizes the flow intensity near the boundary. It can be estimated as $\mu\sim V_L/L^3$, where, as previously, V_L is the characteristic velocity fluctuation in the bulk. In the framework of the Karman-Prandtl theory of the viscous boundary layer (for details see, e.g., the book [5]) the width of the layer is estimated as $L\sim\nu/V_L$ (where ν is kinematic viscosity of the fluid) and one finds $\mu\sim V_L^4\nu^{-3}$. For the elastic turbulence L is the vessel size and $\mu\sim V_L^4\nu^{-3}\mathrm{Re}^{-3}$, where $\mathrm{Re}=V_LL/\nu$ is the Reynolds number.

Comparing the eddy diffusion term for the motion perpendicular to the wall, which can be estimated as $\mu q^4\partial_q^2$, and the molecular diffusion term $\kappa\partial_q^2$, one finds the width of the boundary diffusion layer

$$r_{bl}=(\kappa/\mu)^{1/4}. \tag{2.4}$$

We assume that $r_{bl}\ll L$. The relation can be rewritten as $\mathrm{Pe}\gg 1$ where Pe is the Peclet number: $\mathrm{Pe}=V_LL/\kappa$. As follows from Eq. (2.4), $L/r_{bl}\sim\mathrm{Pe}^{1/4}$. For the case of the viscous boundary layer, one obtains Pe∼Sc, where $\mathrm{Sc}=\nu/\kappa$ is the Schmidt number. For the elastic turbulence, one finds a slightly different estimate Pe∼Sc×Re.

A. Average scalar

Here we consider the simplest possible correlation function: the average value of the passive scalar $\langle\theta\rangle$. The quantity essentially depends on the separation from the wall q and slowly depends on the coordinates along the wall. We ignore the last dependence, assuming that its characteristic length is the vessel size. Then we find from Eqs. (1.6) and (2.3) the following equation for the object:

$$\partial_t\langle\theta\rangle=\mu\partial_q(q^4\partial_q\langle\theta\rangle)+\kappa\partial_q^2\langle\theta\rangle. \tag{2.5}$$

The advection term (with μ) in Eq. (2.5) dominates in $q\gg r_{bl}$ and the diffusion term (with κ) dominates at $q\ll r_{bl}$, where r_{bl} is the thickness of the boundary diffusion layer defined by Eq. (2.4). As we explained in the last paragraph of Sec. I, at solving the problem we should assume $\lim_{q\to\infty}\langle\theta\rangle=0$, since large q corresponds to the bulk, where θ is assumed to tend to zero. The condition is implied below.

We examine the passive scalar evolution in the peripheral region, which begins after its homogenization in the bulk is finished. Then the initial distribution of the passive scalar has the characteristic length L. The subsequent evolution is divided into two stages. At the first stage the thickness δ of the layer, where θ is concentrated, diminishes as $\delta=(\mu t)^{-1/2}$. When δ reaches r_{bl}, the second stage starts, which is characterized by the fixed spatial scale r_{bl}.

At treating the first stage one can omit the diffusion term (with κ) in Eq. (2.5), which leads to the equation

$$\partial_t\langle\theta\rangle=\mu\partial_q(q^4\partial_q\langle\theta\rangle). \tag{2.6}$$

Looking for a solution $\langle\theta\rangle=\exp(-st)\varphi_s(q)$, one obtains from Eq. (2.6),

$$\varphi_s=\sqrt{\frac{\pi}{2}}\left(\frac{s}{\mu q^2}\right)^{3/4}J_{3/2}(\sqrt{s/\mu}\,q^{-1}). \tag{2.7}$$

Using the orthogonality relation

$$\int_0^\infty dq\,\varphi_s(q)\varphi_\sigma(q)=\frac{\pi s^{3/2}}{\mu^{1/2}}\delta(s-\sigma),$$

one finds a general solution of Eq. (2.6),

$$\langle\theta(t,q)\rangle=\int_0^\infty ds\,\varphi_s(q)\frac{\sqrt{\mu}\,e^{-st}}{\pi s^{3/2}}\int_0^\infty dq'\,\varphi_s(q')\theta_0(q') \tag{2.8}$$

in terms of the initial passive scalar distribution. The approximation (2.8) is correct provided $\delta\gg r_{bl}$ (then it is possible to neglect the diffusion boundary layer where diffusion is relevant).

We start from a distribution of θ with the characteristic length L. Therefore at times, when $\delta\ll L$ and at $q\ll L$, one can substitute $\theta(t=0)$ in Eq. (2.8) by $\vartheta_0=\theta(t=0,q=0)$. Taking then the integrals in Eq. (2.8), one derives

$$\langle\theta(t,q)\rangle=\frac{\vartheta_0}{\pi}\int_0^\infty\frac{ds}{s}e^{-st}\varphi_s(q)$$
$$=\vartheta_0\left[\mathrm{erf}\left(\frac{\delta}{2q}\right)-\frac{\delta}{\sqrt{\pi}\,q}\exp\left(-\frac{\delta^2}{4q^2}\right)\right]. \quad (2.9)$$

This profile has a universal form insensitive to details of the initial distribution of θ. Let us stress that the expression (2.9) implies contraction of the region occupied by the passive scalar since $\delta=(\mu t)^{-1/2}$. If $q\gg\delta$ then one obtains from Eq. (2.9),

$$\langle\theta\rangle\approx\frac{\vartheta_0\delta^3}{6\sqrt{\pi}q^3}. \quad (2.10)$$

If $q\ll\delta$ then we find $\langle\theta\rangle=\vartheta_0$. So, the value of θ is practically unchanged inside the layer $q<\delta$.

Note that though Eq. (2.6) has form of the conservation law for $\langle\theta\rangle$, the total amount of the passive scalar in the peripheral region $\int dq\,\langle\theta\rangle$ appears to be time dependent, if the expression (2.9) is integrated. The reason is that the considered solution corresponds to nonzero passive scalar flux directed to large q, i.e., to the bulk, which can be treated as a big reservoir. This flux $\mu q^4\partial_q\langle\theta\rangle$ can be obtained directly from the asymptotic expression (2.10). Note also that the passive scalar evolution at the first stage is insensitive to the boundary conditions, and therefore it is described identically for the concentration of pollutants and temperature.

Now we analyze the passive scalar behavior at the second stage, then the diffusion term cannot be ignored. Let us first examine the case when the passive scalar represents the concentration of pollutants. Then Eq. (2.5) has to be supplemented by the boundary condition $\partial_q\langle\theta\rangle=0$ at $q=0$, which means zero flux of pollutants to the boundary. At long times, only the contribution related to the minimal (by its absolute value) eigenvalue of the operator in the right-hand side of Eq. (2.5) is left. That leads to the exponential decay $\langle\theta\rangle\propto\exp(-\gamma t)$. The value of γ is $\gamma=c_E\sqrt{\kappa\mu}$, where the factor c_E can be found numerically, it is $c_E\approx1.81$. The asymptotic behavior of $\langle\theta\rangle$ can be related to the initial value of the passive scalar ϑ_0 near the wall: $\langle\theta(q=0)\rangle=c_0\vartheta_0\exp\times(-\gamma t)$, where $c_0\approx1.55$. The total amount of the scalar near the boundary behaves as $\int dq\,\langle\theta\rangle=c_1\vartheta_0 r_{bl}\exp(-\gamma t)$, where $c_1\approx1.55$.

Let us now consider the case when the passive scalar θ represents temperature, assuming that it is fixed at the boundary $\theta(q=0)=\vartheta_0$. Then after the first stage a quasistationary distribution of $\langle\theta\rangle$ is formed, since the bulk can be treated as a big reservoir having a constant temperature. This quasistationary distribution can be found directly from Eq. (2.5) where the term with the time derivative has to be omitted,

$$\langle\theta\rangle=\frac{2\sqrt{2}}{\pi}\kappa^{3/4}\mu^{1/4}\vartheta_0\int_q^\infty\frac{dq_1}{\mu q_1^4+\kappa}. \quad (2.11)$$

At $q\gg r_{bl}$ we find, again, $\langle\theta\rangle\propto q^{-3}$. That corresponds to a nonzero passive scalar flux (heat flux) to the bulk. This flux is time independent in the case considered.

Now we can justify the time separation leading to Eq. (1.9). For the intermediate evolution characterized by the profile (2.9), the characteristic time is $t\sim(\mu\delta^2)^{-1}$. Thus, the time ratio t/τ (where τ is the velocity correlation time) can be estimated as $t/\tau\sim(L/\delta)^2\gg1$. In the case when the passive scalar evolution is determined by the interplay of the viscosity and advection, the length δ has to be substituted by r_{bl}. Then using the estimates formulated earlier we find that $t/\tau\sim\mathrm{Pe}^{1/2}\gg1$.

B. High moments

As we already noted, at the first stage the diffusive term in the equation for the passive scalar correlation functions can be omitted. Then the closed equation (1.11) for high moments of the passive scalar is correct. The high moments (similar to the first one) depend mainly on q and we ignore their dependence on the coordinates along the wall. Then, substituting into the equation the expressions (2.3), one finds

$$\partial_t\langle\theta^n\rangle=\mu\partial_q(q^4\partial_q\langle\theta^n\rangle), \quad (2.12)$$

which is a generalization of Eq. (2.6). Its solution can be written as the expression (2.8),

$$\langle\theta^n\rangle=\int_0^\infty ds\;\varphi_s(q)\frac{\sqrt{\mu}\,e^{-st}}{\pi s^{3/2}}\int_0^\infty dq'\;\varphi_s(q')\theta_0^n(q'). \quad (2.13)$$

The expression (2.13) implies that $\delta\gg r_{bl}$ since only at this condition it is possible to neglect the region $q\sim r_{bl}$ where the diffusion is relevant. Since the initial distribution of θ has the characteristic length L, at the condition $\delta\ll L$ we obtain, again, a universal expression

$$\langle\theta^n\rangle=\vartheta_0^n\left[\mathrm{erf}\left(\frac{\delta}{2q}\right)-\frac{\delta}{\sqrt{\pi}\,q}\exp\left(-\frac{\delta^2}{4q^2}\right)\right], \quad (2.14)$$

which is a generalization of Eq. (2.9). The expression shows that in the region $q\gg\delta$, $\langle\theta^n\rangle\approx\vartheta_0^n\delta^3/(6\sqrt{\pi}\,q^3)$.

Actually, the expression (2.13) is correct for any averaged local function of the scalar θ. We can use it to find the local scalar PDF $P(t,q,\theta)=\langle\delta[\theta-\theta(t,q)]\rangle$. Let us first rewrite the expression like Eq. (2.13) as

$$P(t,q)=\frac{1}{q^{3/2}}\int_0^\infty dk\int_0^\infty\frac{dq'}{(q')^{3/2}}\,k\exp(-\mu k^2t)$$
$$\times J_{3/2}(k/q)J_{3/2}(k/q')P(0,q'). \quad (2.15)$$

One can take the integral over k in Eq. (2.15) to obtain

$$P=\frac{1}{\sqrt{\pi\mu t}}\int_0^\infty \frac{dq'}{qq'}\exp\left[-\frac{1}{4q^2\mu t}-\frac{1}{4q'^2\mu t}\right]P(0,q')\times\left\{\cosh\left[\frac{1}{2qq'\mu t}\right]-2\mu tqq'\sinh\left[\frac{1}{2qq'\mu t}\right]\right\}. \tag{2.16}$$

Let us assume that the initial scalar distribution is a monotonic function $\theta_0(q)$, which is equal to zero at $q\to\infty$ and reaches a maximum value at $q\to 0$. Then $P(t=0,q,\theta)=-[1/\theta_0'(q)]\delta[q-q_0(\theta)]$, where the function $q_0(\theta)$ is determined from the relation $\theta_0(q_0)=\theta$. Calculating the integral (2.16), one finds

$$P(t,q,\theta)=\frac{1}{qq_0|\theta_0'(q_0)|}\left[g\left(\frac{1}{q}-\frac{1}{q_0}\right)(1-2\mu tqq_0)+g\left(\frac{1}{q}+\frac{1}{q_0}\right)(1+2\mu tqq_0)\right],$$

$$g(x)=\frac{1}{2\sqrt{\pi\mu t}}\exp\left(-\frac{x^2}{4\mu t}\right). \tag{2.17}$$

When t grows, the boundary layer, determined by δ, shrinks, that is, the characteristic value of q decreases. Besides, q_0 is fixed at a given θ. Therefore one can expand the expression in the square brackets in Eq. (2.17) in $1/q_0$ to obtain a universal probability distribution

$$P=\frac{1}{12q^2\sqrt{\pi}(\mu t)^{5/2}q_0^3|\theta_0'(q_0)|}\exp\left(-\frac{1}{4q^2\mu t}\right). \tag{2.18}$$

Unfortunately, expression (2.18) cannot be utilized for calculation of the moments of θ, since the integrals $\int d\theta\,\theta^n P(\theta)$ diverge near the maximum value of θ, ϑ_0. The reason is that the expression (2.18) is correct only if $q_0(\theta)\gg q$, which is violated at small q_0, corresponding to θ close to ϑ_0.

At the second stage diffusion starts to be relevant, and it is impossible to obtain closed equations for the moments $\langle\theta^n(t,q)\rangle$. To find the moments one has to solve the complete equations (1.9) for the passive scalar correlation functions, which is a complicated problem. One can say only that due to linearity of the problem, the asymptotic in time behavior of the correlation functions (and, consequently, moments) is determined by the minimal eigenvalue of the operator in the right-hand side of Eq. (1.9). Therefore for the concentration of pollutants $\langle\theta^n(t,q)\rangle\propto\exp(-\gamma_n t)$ where $\gamma_n\sim\sqrt{\kappa\mu}$. The estimate can be obtained by equating the time derivative, the eddy diffusion term, and the molecular diffusion term in Eq. (1.9). If we consider the region $q\gg r_{bl}$ then it is possible to neglect the molecular diffusion term in Eq. (1.9) and we return to the closed equation (2.12). There the time derivative ∂_t can be substituted by $-\gamma_n$ and we conclude that at the same condition $q\gg r_{bl}$ it is possible to neglect the term with the time derivative in Eq. (2.12). Therefore we obtain, again, the asymptotic behavior

$$\langle\theta^n(t,q)\rangle\propto q^{-3}, \tag{2.19}$$

at $q\gg r_{bl}$. The situation here is analogous to one for the average passive scalar, since Eq. (2.12) has the form of conservation law for the high moments. The laws $\langle\theta^n(t,q)\rangle\propto q^{-3}$ correspond to nonzero fluxes of high degrees of the passive scalar.

Now we can turn to the case when the passive scalar is temperature, fixed at the boundary. If the temperature in the bulk is different from that at the boundary then a heat flow is produced from the boundary to the bulk. Since the bulk is a big reservoir then in the main approximation its temperature can be treated as time independent. In this case statistics of θ becomes quasistationary at the second stage (all the correlation functions are independent of time). The equations for the passive scalar correlation functions are determined by Eq. (1.9) where the time derivative can be omitted. Particularly, at $q\gg r_{bl}$ one obtains the closed equation for the moments (2.12) (where time derivative has to be omitted) leading to the same law (2.19). Inside the diffusion boundary layer $\theta\sim\vartheta_0$, where ϑ_0 is the temperature at the boundary (recall that we assume $\theta=0$ in the bulk). Then, as it follows from Eq. (1.11), $\langle\theta^n\rangle\sim\vartheta_0^n(r_{bl}/q)^3$.

We established that if $q\gg\delta$ at the first stage or if $q\gg r_{bl}$ at the second stage then the law (2.19) is valid. Moreover, it is possible to formulate the estimates $\langle\theta^n\rangle\sim\langle\theta\rangle^n(q/\delta)^{3(n-1)}$ for the first stage and the inequalities $\langle\theta^n\rangle\gtrsim\langle\theta\rangle^n(q/r_{bl})^{3(n-1)}$ for the second stage. The estimates show that the high moments of the passive scalar are much larger than their Gaussian evaluation. This property implies strong intermittency in the system. The expression (2.19) can be treated as a manifestation of anomalous scaling. It is extreme since the exponent characterizing q dependence of the moments is independent of n.

C. Pair correlation function

Here we examine the pair correlation function $F(\boldsymbol{r}_1,\boldsymbol{r}_2)$. One could anticipate its nontrivial scaling behavior in the region $q_1,q_2\gg\delta$ (for the first stage) or $q_1,q_2\gg r_{bl}$ (for the second stage). If $q_1\gg q_2$ then only the term $\nabla_{1\alpha}[D_{\alpha\beta}(\boldsymbol{r}_1,\boldsymbol{r}_1)\nabla_{1\beta}F]$ in Eq. (1.7) should be kept, and we come to the conclusion that $F\propto q_1^{-3}$. Therefore we obtain the behavior, which is similar to the passive scalar moments in this asymptotic case.

Now we consider the case of close points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, assuming small value of $\varrho_\zeta=r_{1\zeta}-r_{2\zeta}$, $\varrho\ll L$, admitting the expressions (2.3) and $q\ll Q$, where $Q=(q_1+q_2)/2$ and $q=q_1-q_2$. Then one obtains from Eq. (1.7),

$$\begin{aligned}\partial_t F=&\mu\partial_Q(Q^4\partial_Q F)+(\kappa/2)\partial_Q^2F+2\kappa\partial_q^2F+\{4\mu Q^3q\partial_Q\partial_q\\&-2\mu Q^3\varrho\partial_Q\partial_\varrho+[H_{10}q^2/\varrho-2(2\mu\\&+3H_{20})Q^2\varrho]\partial_\varrho\}F+\{4\mu Q^2q^2\partial_q^2-2H_{30}Q^4\varrho^2\partial_q^2\\&-2\mu Q^2\varrho q\partial_q\partial_\varrho+(2\mu+H_{20})Q^2\varrho^2\partial_\varrho^2+H_{10}q^2\partial_\varrho^2\}F.\end{aligned} \tag{2.20}$$

Details of the derivation of Eq. (2.20) can be found in the Appendix. It is natural to write $F=\langle\theta^2(Q)\rangle(1+\varsigma)$. Here unity represents the main contribution to the pair correlation function, related to the second-order moment, and ς is a small correction tending to zero at $\boldsymbol{r}_1\to\boldsymbol{r}_2$.

In the region $Q\gg\delta$ for the first stage or in the region $Q\gg r_{bl}$ for the second stage the time derivative in Eq. (2.20) can be omitted in comparison with the eddy diffusion term, which is the first one in the right-hand side of Eq. (2.20). If the molecular diffusion terms in Eq. (2.20) are also neglected then one finds

$$\{(2H_{30}\varrho^2-7x^2)\partial_x^2-11x\partial_x+3+[(2+H_{20})\varrho^2+H_{10}x^2]\partial_\varrho^2 + [2(1-3H_{20})\varrho+H_{10}x^2/\varrho]\partial_\varrho\}\varsigma=0, \tag{2.21}$$

where $x=q/Q$ and we have taken into account that $\langle\theta^2(Q)\rangle\propto Q^{-3}$. A solution of Eq. (2.21) can be written as a sum of $\varsigma_b=(qQ)^b f_b(x,\varrho)$ where f_b satisfy Eq. (2.21) (since qQ is zero mode of the operator figuring in this equation). It is clear that the main contribution is related to $b=0$ (since negative b are forbidden). Since the operator in Eq. (2.21) has definite scaling properties, solutions of the equation can be written in a simple self-similar form $\varsigma=(q/Q)^a\Psi(Q\varrho/q)$. The principal contribution to ς is associated with the smallest a. Unfortunately, the exponent a is nonuniversal, being dependent on the coefficients H_{ij}. The molecular diffusion smoothes the function ς at $q\sim r_{bl}^2/Q$.

The above expressions demonstrate the following natural behavior of the pair correlation function. The main anomalous dependence of the function is related to the second moment of the passive scalar. As to the dependence on the relative separation between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, it is described in terms of the function ς possessing a nontrivial scaling behavior with a self-similar factor depending on the combination $Q\varrho/q$.

III. DECAY ALONG A PIPE

Here we discuss the case when the passive scalar decays along a pipe in a statistically homogeneous flow, which is assumed to be chaotic and has an average velocity u along the pipe. Such a setup was used by Groisman and Steinberg in their experiments [14]. In this case the chaotic flow (elastic turbulence) was excited in a polymer solution pushed through a curvilinear pipe. The scalar dynamics is then governed by the equation

$$\partial_t\theta+u\partial_z\theta+v_\alpha\nabla_\alpha\theta=\kappa\nabla^2\theta, \tag{3.1}$$

where $\boldsymbol{v}$ is fluctuating part of the velocity (with zero mean) and z is coordinate along the pipe. Below, the average velocity u is assumed to be much larger than $\boldsymbol{v}$ (which corresponds to the experimental situation).

If the pressure difference, pushing the flow, is constant, the flow is statistically stationary and homogeneous along the pipe (which is assumed to have a constant cross section). Therefore u is independent of z and the velocity correlation functions do depend on the time differences and coordinate differences along the pipe. Thus, for a problem with a stationary scalar injection to the pipe the scalar statistics is time independent and the coordinate z along the pipe plays the role of time in the decay problem. Following the procedure described in the first section one obtains from Eq. (3.1) the following equations for the scalar correlation functions:

$$u\partial_z F_n=\kappa\sum_{m=1}^{n}\nabla_m^2F_n+\sum_{m,k=1}^{n}\nabla_{m\alpha}[D_{\alpha\beta}(\boldsymbol{r}_m,\boldsymbol{r}_k)\nabla_{k\beta}]F_n\,. \tag{3.2}$$

The only difference in comparison with Eq. (1.9) is that the time derivative is substituted by the advection term along the pipe.

As previously, we introduce the coordinate q measuring a separation of a given point from the wall. The average velocity u is a function of q, tending to zero as $u\propto q$ near the boundary. That leads to the main difference between this situation and the one discussed in the preceding section. The equation for the average scalar turns to

$$s_0q\partial_z\langle\theta\rangle=[\mu\partial_q q^4\partial_q+\kappa\partial_q^2]\langle\theta\rangle. \tag{3.3}$$

Despite the additional factor q in the left-hand side of the equation in comparison with Eq. (2.5), the qualitative picture of scalar evolution remains the same. At the first stage the scalar is mostly situated in the layer of the width $\delta=s_0/(\mu z)$, and the molecular diffusion can be neglected. The scalar decay at this stage is algebraic with the longitudinal coordinate z. When δ reaches the boundary layer width r_{bl}, the molecular diffusion becomes relevant, and the scalar decay starts to be exponential.

As in the previous case, it is possible to obtain complete statistical properties of the scalar at the first stage. In this case one can omit the molecular diffusion term. Hence, averages of any single-point functions $\Phi(\theta)$, such as the moments $\langle\theta^n\rangle$ or the scalar PDF $P(\theta,q,z)=\langle\delta[\theta-\theta(q,z)]\rangle$, are described by the same equation

$$\partial_z\langle\Phi\rangle=\frac{\mu}{s_0q}\partial_q q^4\partial_q\langle\Phi\rangle\equiv\hat{H}\langle\Phi\rangle. \tag{3.4}$$

This is a linear equation, and it can be solved using the Green function formalism. Namely, the solution of Eq. (3.4) can be written as

$$\langle\Phi(z,q)\rangle=\int_0^\infty dq'\,G(z,q,q')\Phi(z=0,q'), \tag{3.5}$$

where G is the Green function. In order to obtain an explicit equation for the Green function one should first solve the corresponding eigenvalue problems. Since the operator $\hat{H}$ is not Hermitian, then one should find both its right and left eigenfunctions. They are solutions of the equations $\hat{H} f_\lambda + \lambda f_\lambda = 0$ and $\hat{H}^+ g_\lambda + \lambda g_\lambda = 0$, respectively, where $H^+ = (s_0/\mu)\partial_q q^4 \partial_q q^{-1}$. We find

$$f_\lambda = x^3 J_3(x), \quad g_\lambda = x J_3(x), \quad x = 2\sqrt{\lambda/q}. \tag{3.6}$$

Using the orthogonality relation

$$\int_0^\infty dq g_\lambda(q) f_{\lambda'}(q) = 16\lambda^2 \delta(\lambda - \lambda'),$$

one derives

$$G(z,q,q') = \int_0^\infty \frac{d\lambda}{16\lambda^2} \exp(-\lambda \delta^{-1}) f_\lambda(q) g_\lambda(q') = \frac{\delta}{q^{3/2}(q')^{1/2}} \exp(-\delta/q - \delta/q') I_3(2\delta/\sqrt{qq'}). \tag{3.7}$$

Since we assumed that $\theta = \theta_0(q)$ initially, then for the nth order scalar moment $\langle \theta^n(q,z) \rangle$ the initial condition is $\theta_0^n(q)$. If $\delta \ll L$ then one can substitute $\langle \theta^n(0,z) \rangle$ by ϑ_0^n, where $\vartheta_0 = \theta_0(0,0)$. In this case an explicit integration in Eq. (3.5) leads to the expression

$$\langle \theta^n(z,q) \rangle = \frac{\vartheta_0^n \delta^3}{6q^3} \exp(-\delta/q) \; {}_1F_1(1,4,\delta/q). \tag{3.8}$$

Again, we obtain a universal profile. If $q \gg \delta$ then both the exponent and ${}_1F_1$ can be substituted by unity to obtain $\langle \theta^n(z,q) \rangle = \vartheta_0^n \delta^3/(6q^3)$. If $q \ll \delta$ then $\langle \theta^n(z,q) \rangle = \vartheta_0^n$. In order to obtain the scalar PDF, one should use the initial distribution $P(z=0,q,\theta) = \delta(\theta - \theta_0(q)) = -[1/\theta_0'(q)]\delta(q - q_0(\theta))$. Here $q_0(\theta)$ is defined, as previously, by the relation $\theta_0(q_0) = \theta$. The convolution with the Green function is now

$$P(z,q,\theta) = \int dq' G(z,q,q') P(0,q,\theta) = -\frac{1}{q^{3/2} q_0^{1/2}} \frac{\delta}{\theta_0'(q_0)} \exp\left(-\frac{\delta}{q} - \frac{\delta}{q_0} \right) I_3\left(2\frac{\delta}{\sqrt{q q_0}} \right). \tag{3.9}$$

At large distances, $q \gg \delta^2/L$ (and also $L \gg \delta$), one can use the following approximation:

$$P(z,q,\theta) \approx -\frac{\delta^4}{6q^3 q_0^2 \theta_0'(q_0)} \exp(-\delta/q). \tag{3.10}$$

Again, this expression cannot be exploited for calculation of moments of the passive scalar because of the divergency near the maximum value of θ, ϑ_0.

For the pipe, the characteristic length where the diffusion becomes relevant is determined by the same expression (2.4). When δ diminishes down to r_{bl} another regime comes. For the density of pollutants the regime is characterized by an exponential decay of the passive scalar moments along the pipe. For the first moment the decrement of the decay can be extracted from Eq. (3.3). It is determined by the smallest eigenvalue of the operator in the right-hand side of the equation. Thus, for large enough z the decay law $\langle \theta \rangle \propto \exp \times(-\alpha z)$ is correct where $\alpha \sim \kappa^{1/4} \mu^{3/4} s_0^{-1}$. The eigenvalue problem, corresponding to Eq. (3.3) can be solved numerically, then one obtains $\alpha \approx 3.72 \kappa^{1/4} \mu^{3/4} s_0^{-1}$.

For the temperature, fixed at the boundary, we obtain a quasistationary distribution if the fixed value is z independent. The average temperature is described by Eq. (3.3) where z derivative is dropped. Then we return to the same equation as previously (for the decay in the vessel), with the solution (2.11). Higher moments of the temperature are slightly different than for the decay in the vessel. However, qualitatively their behavior is the same. Namely, $\langle \theta^n \rangle \approx \vartheta_0^n$ if $q \ll r_{bl}$, and $\langle \theta^n \rangle \sim \vartheta_0^n (r_{bl}/q)^3$, if $q \gg r_{bl}$.

One can examine the pair correlation function F of the passive scalar. Qualitatively, it is the same, as for the decay in the vessel. Namely, the main anomalous dependence of the function is related to the second moment of the passive scalar. The function is practically undistinguished from the second moment if $q_1, q_2 \ll r_{bl}$. As to the dependence on the relative separation between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ for $q_1, q_2 \gg r_{bl}$, it is described in terms of a function ζ, $\langle \theta^2(Q) \rangle (1 + \varsigma)$, possessing a nontrivial scaling behavior with a self-similar factor depending on the combination $Q\varrho/q$.

Now we can justify the time separation leading to Eq. (1.9) in the case of the decay along the pipe. For the intermediate evolution characterized by the profile (3.8), the characteristic time is $t \sim (\mu \delta^2)^{-1}$. Thus, the time ratio t/τ (where τ is the velocity correlation time) can be estimated as $t/\tau \sim (L/\delta)^2 \gg 1$. In the case when the passive scalar evolution is determined by the interplay of the viscosity and advection the length δ has to be substituted by r_{bl}. Then using the estimates formulated earlier we find that $t/\tau \sim \mathrm{Pe}^{1/2} \gg 1$.

IV. CONCLUSION

We have investigated the passive scalar (concentration of pollutants or temperature) evolution in the chaotic and turbulent flows near the boundary (wall), which dominates (at large Sc) the advanced stages of the passive scalar homogenization. There are some universal features of the decay related to the universal velocity dependence on the coordinate perpendicular to the wall. We assumed that the Schmidt number Sc is large enough to guarantee the condition $L \gg r_{bl}$ where L is the width of the peripheral region and r_{bl} is the thickness of the diffusive boundary layer (which is determined by equating the molecular diffusivity and the eddy diffusivity). The width of the peripheral region is the thickness of the viscous boundary layer for the developed high-Reynolds-number turbulence. For the elastic turbulence L is the vessel size (or the pipe radius for the decay along the pipe).

We have considered the evolution beginning after that homogenization of the passive scalar is finished in the bulk. Then the initial passive scalar distribution is characterized by the length L, which is much larger than r_{bl}. The first stage of the passive scalar decay in the peripheral region is insensitive to diffusion. The characteristic length of the passive scalar distribution at this stage δ satisfies the inequalities $L \gg \delta \gg r_{bl}$. For the decay in a closed vessel the length behaves as $\delta \propto t^{-1/2}$, whereas for the decay along pipe $\delta \propto z^{-1}$ (where z is coordinate along the pipe). The passive scalar inhomogeneity is kept mainly for separations q from the boundary, satisfying $q < \delta$ whereas for larger q the advection effectively homogenizes the passive scalar (the mechanism is in stretching producing small-scale fluctuations of the passive scalar, which are effectively removed by diffusion).

When δ achieves the thickness of the diffusion boundary layer, r_{bl}, the decay of the concentration of pollutants becomes exponential. This character of the decay is explained by the space distribution of the passive scalar inhomogeneities, which are concentrated mainly in the diffusive boundary layer, and decay due to the flow to the bulk, which is proportional to the level of the inhomogeneities. The decrements of the decay can be expressed in terms of the Peclet number which is $\mathrm{Pe} = V_L L/\kappa$ (where V_L is the amplitude of the velocity fluctuations in the bulk and κ is the molecular diffusivity). For the viscous boundary layer the Peclet number coincides with the Schmidt number, whereas for the elastic turbulence it differs from the Schmidt number by the factor which is the Reynolds number. For the time decay the decrement is proportional to $\mathrm{Pe}^{-1/2}$, whereas for the decay along the pipe the decrement is proportional to $\mathrm{Pe}^{-1/4}$. For the temperature fixed at the boundary the second stage is characterized by a quasistationary distribution, since the bulk serves as a big reservoir keeping its temperature constant and absorbing heat, transferred from the boundary.

We neglected an inhomogeneity of the passive scalar along the wall. It is motivated by the faster decay of the passive scalar fluctuations, related to such inhomogeneity, both in time and in space (in the region $q \gg \delta$ or $q \gg r_{bl}$, depending on the stage). However, in some cases the inhomogeneity can play an essential role. It is a subject of separate investigation.

Our theoretical predictions can be compared with the experimental data presented in the papers [14,19], obtained for the dye concentration decay (homogenization) along the curvilinear pipe, where the chaotic flow (elastic turbulence) is excited in a dilute polymer solution. The most interesting comparison concerns the Pe dependence of different quantities, which was extracted from the experiment [19] by varying the diffusion coefficient κ by two orders of magnitude (it was done by changing the molecular weight of the dye carriers). The experimental data (corresponding to the second stage in our theoretical scheme) are in a good agreement with the law $\mathrm{Pe}^{-1/4}$ both for the thickness of the diffusion boundary layer, r_{bl} (which is determined experimentally by a maximum in the passive scalar fluctuations), and for the decrements of the passive scalar decay along the pipe.

The peripheral region serves as a source supplying the passive scalar to the bulk. Since the passive scalar decay in the peripheral region is slow (comparing to the bulk), it can be considered as a quasistationary source of the passive scalar for the bulk, where the passive scalar correlation functions adjust adiabatically to the level of the supply. Therefore the situation is close to one characteristic of the passive scalar, supplied by pumping, in a smooth random velocity field. Statistical properties of the passive scalar in this case were examined in Refs. [7–9]. Particularly, correlation functions of the passive scalar in the bulk should possess a behavior close to the logarithmic one. This prediction is also in agreement with the experimental data [14,19].

One of the remarkable predictions of our theory is strong intermittency of the passive scalar and extreme anomalous scaling of its moments for the dependence on the separation from the boundary q: all the moments scale identically as q^{-3} for $L \gg q \gg \delta$ or $L \gg q \gg r_{bl}$, depending on the stage. Physically that means that a system of filaments (or sheets) of the passive scalar is formed, which contract as q increases. Their dynamics is diffusionless and therefore the passive scalar θ in the filaments is of the order of its boundary value. At the condition $q \gg \delta$ or $q \gg r_{bl}$ the filaments occupy a small fraction of the volume, which explains the strong intermittency. As to the correlation functions of the passive scalar, our theory predicts some definite self-similar and scaling behaviors, which are complex due to the space inhomogeneity characteristic of the problem. It would be interesting to check the predictions experimentally.

The final remark concerns an extension of our results to other problems. As it was noted in the paper [4], the scheme developed for the passive scalar decay can be without serious modifications applied to fast binary chemical reactions. We believe that minor modifications of the scheme can make it applicable for more complicated chemical reactions. The other problem, which can be posed for the peripheral region, is dynamics of polymers (say, for the case of the elastic turbulence). Then one should go beyond the scope of the passive approach.

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APPENDIX: PAIR CORRELATION FUNCTION

The eddy diffusion operator $\hat{A}$ is given by the last terms of Eq. (1.7). It can be rewritten as

$$\hat{A} = D^{ij}\frac{\partial}{\partial X^i}\frac{\partial}{\partial X^j} + \left(\frac{\partial D^{ij}}{\partial X^i}\right)\frac{\partial}{\partial X^j}. \tag{A1}$$

Here X^i is an extended set of variables (coordinates), $\{q_1, r_{11}, r_{12}, q_2, r_{21}, r_{22}\}$, and the tensor D is

$$D=\begin{pmatrix} \mu q_1^4 & 0 & 0 & H_3 q_1^2 q_2^2 & \mu q_1^2 q_2(r_{11}-r_{21}) & \mu q_1^2 q_2(r_{12}-r_{22}) \\ 0 & H_{10} q_1^2 & 0 & \mu q_1 q_2^2(r_{21}-r_{11}) & H_1 q_1 q_2 + d^{11} & d^{12} \\ 0 & 0 & H_{10} q_1^2 & \mu q_1 q_2^2(r_{22}-r_{12}) & d^{21} & H_1 q_1 q_2 + d^{22} \\ H_3 q_1^2 q_2^2 & \mu q_1 q_2^2(r_{22}-r_{12}) & \mu q_1 q_2^2(r_{21}-r_{11}) & \mu q_2^4 & 0 & 0 \\ \mu q_1^2 q_2(r_{11}-r_{21}) & H_1 q_1 q_2 + d^{11} & d^{12} & 0 & H_{10} q_2^2 & 0 \\ \mu q_1^2 q_2(r_{12}-r_{22}) & d^{21} & H_1 q_1 q_2 + d^{22} & 0 & 0 & H_{10} q_2^2 \end{pmatrix}, \tag{A2}$$

$$H_1 = H_{10} - (\mu + 3H_{20}/2)[(r_{11}-r_{21})^2 + (r_{12}-r_{22})^2], \tag{A3}$$

$$H_3 = \mu + H_{30}[(r_{11}-r_{21})^2 + (r_{12}-r_{22})^2], \tag{A4}$$

$$d^{ij} = H_{20} q_1 q_2 (r_{1i}-r_{2i})(r_{1j}-r_{2j}). \tag{A5}$$

Now we turn to the new set of variables $Y^i = \{Q, q, R_1, R_2, \varrho, \phi\}$, where $Q=(q_1+q_2)/2, q=q_1-q_2, R_i = (r_{1i}+r_{2i})/2, \varrho \cos\phi = r_{11}-r_{21}, \varrho \sin\phi = r_{12}-r_{22}$. Then the eddy diffusion operator $\hat{A}$ is transformed to

$$\hat{A} = D^{ij} T_i^k T_j^l \frac{\partial}{\partial Y^k}\frac{\partial}{\partial Y^l} + D^{ij} T_i^k \left(\frac{\partial T_j^l}{\partial Y^k}\right)\frac{\partial}{\partial Y^l} + \left(\frac{\partial D^{ij}}{\partial X^i}\right) T_j^k \frac{\partial}{\partial Y^k}$$
$$\equiv \tilde{D}^{ij}\frac{\partial}{\partial Y^i}\frac{\partial}{\partial Y^j} + f^i \frac{\partial}{\partial Y^i}, \tag{A6}$$

where $T_j^i = \partial Y^i / \partial X^j$ is the Jacoby matrix,

$$T=\begin{pmatrix} 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & \cos\phi & \sin\phi & 0 & -\cos\phi & -\sin\phi \\ 0 & -\sin(\phi)/\varrho & \cos(\phi)/\varrho & 0 & \sin(\phi)/\varrho & -\cos(\phi)/\varrho \end{pmatrix}. \tag{A7}$$

After some algebraic manipulations one can easily obtain the new tensor $\tilde{D}^{ij}$ and the first order term f^i. The exact expressions are rather bulky. However, if the initial distribution of scalar depends solely on q, the only relevant terms are those preceding $\partial_Q, \partial_q, \partial_\varrho$. Therefore one can reduce the final variables set to $\{Q, q, \varrho\}$. In the limit $Q \gg q, \zeta \to 0$ one obtains

$$\tilde{D}=\begin{pmatrix} Q^4(\mu + H_{30}\varrho^2/2) & 2\mu Q^3 q & -\mu Q^3 \varrho \\ 2\mu Q^3 q & 4\mu q^2 Q^2 - 2H_{30} Q^4 \varrho^2 & -\mu Q^2 q \varrho \\ -\mu Q^3 \varrho & -\mu Q^2 q \varrho & (2\mu + H_{20})Q^2 + H_{10} q^2 \end{pmatrix}, \tag{A8}$$

$$f^Q = 4Q^3(\mu + H_{30}\zeta^2/2), \quad f^q = 0,$$

$$f^\varrho = \frac{H_{10} q^2}{\varrho} - 4(\mu + 3H_{20}/2) Q^2 \varrho. \tag{A9}$$

Then we obtain the following expression for the operator $\hat{A}$:

$$\hat{A} = (\mu + H_{30}\varrho^2/2)\partial_Q Q^4 \partial_Q + 4\mu Q^3 q \partial_Q \partial_q - 2\mu Q^3 \varrho \partial_Q \partial_\varrho + (4\mu Q^2 q^2 - 2H_{30} Q^4 \varrho^2)\partial_q^2 - 2\mu Q^2 q \varrho \partial_q \partial_\varrho + [(2\mu + H_{20})Q^2 \varrho^2 + H_{10} q^2]\partial_\varrho^2 + 4Q^3(\mu + H_{30}\rho^2/2)\partial_Q + \left[\frac{H_{10} q^2}{\varrho} - 4(\mu + 3H_{20}/2)Q^2 \rho\right]\partial_\varrho. \tag{A10}$$

This leads to the expression (2.20).

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