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NONLINEAR PHYSICS

Dynamic Correlations in a Thermalized System Described by the Burgers Equation

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Abstract—For the field $u(x, t)$ governed by the Burgers equation with a thermal noise, short-time asymptotics of multipoint correlators are obtained. Their exponential parts are independent of the correlator number. This means that they are determined by a single rare fluctuation and exhibit an intermittency phenomenon. *© 2002 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

Values of various observables in a field system in a chaotic state undergo fluctuations in space and time. To describe them, correlation functions of various orders are usually used; moreover, nonsimultaneous correlators carry more detailed information about the system than simultaneous ones. The main contribution to the measured average quantities is made by events of superposition of weakly correlated signals arriving from different points in space. Statistical properties of this kind are exhibited by free fields. For such fields, the statistics of fluctuations in a thermodynamic equilibrium is a Gaussian one; i.e., both the simultaneous and nonsimultaneous correlation functions reduce to a sum of products of second-order correlators. However, it is possible that a set of average quantities of interest is determined by a rare but large (in natural units) fluctuation that is coherent with respect to space. Such a situation is referred to as intermittency. Formally, it manifests itself in that the irreducible correlation functions, which are distinct from zero in any nonlinear system, become much greater then the reducible parts of total correlators. A classical example is hydrodynamic turbulence [1, 2]. Here, the magnitude of total correlators as compared to their reducible parts is characterized by the Reynolds number $\mathrm{Re} = l/r_d$, where l is the scale of either initial perturbations or external sources of energy, and r_d is the scale at which the energy dissipates.

Intermittency also occurs in fields of another type—wave functions Ψ of electrons in a random one-dimensional and two-dimensional potential. Indeed, due to localization effects, all moments of the density $|\Psi|^2$ are inversely proportional to the volume V; therefore,

$$\langle|\Psi|^{2n}\rangle \gg \langle|\Psi|^2\rangle^n \text{ at } V \longrightarrow \infty$$

(see [3, 4]). Here, we face the case when an infinite set of correlation functions is determined by a single rare event.

By itself, a large value of fluctuations does not necessarily lead to intermittency. For example, even at the point of a second-order phase transition, irreducible and total correlators have the same scale dimension and are, therefore, of the same order of magnitude [5]. For this reason, intermittency effects under thermal equilibrium had not been discussed in the literature until [6, 7]. In these studies, the statistics of a vortex field in two-dimensional films was investigated, and it was discovered that nonsimultaneous correlators of various orders are determined by a single fluctuation evolving in time. Hence, the correlators are proportional to the small probability of this fluctuation. However, the reducible parts of higher order correlators include this probability to powers greater than unity; as a result, the values of reducible parts are much less than the values of the total correlation functions.

In [8], it was shown that similar properties are characteristic of nonsimultaneous correlation functions in systems whose evolution obeys the one-dimensional Burgers equation with a thermal noise

$$u_t + uu_x - \nu u_{xx} = \xi. \quad (1)$$

Here, $u(x, t)$ is a function of the spatial coordinate x and time t, the parameter ν plays the role of viscosity, and $\xi(x, t)$ is a random short-correlated (in space and time) force with a zero average satisfying Gaussian statistics.

Equation (1) describes a system of one-dimensional weak shock waves. In this case, the field $u(x, t)$ is proportional to the speed of the medium with respect to a reference frame moving at the speed of sound [1, 9]. This equation also appears in the problem on fluctuations of solution–precipitate type interfaces that grow due to a random inflow of atoms from the solution. In this case, $u = \partial_x h$, where h is the height of the surface, and the equation of evolution of the field $h(x, t)$, which is derived from Eq. (1), is called the (1 + 1)-dimensional Kardar–Parisi–Zhang equation [10]. A problem of the same type arises in the study of statistics of vortex lines in superconductors with impurities [11]. In what follows, we will use hydrodynamic terms and call $u(x, t)$ the velocity field.

We consider the second-order correlator of the random force $\xi(x, t)$ in the form

$$\langle \xi(x,t)\xi(x_1,t_1)\rangle = -\nu\beta^{-1}\delta''(x-x_1)\delta(t-t_1). \quad (2)$$

Then, there exists a time-independent distribution $\mathcal{P}[u]$ of the field u that has the form of the Gibbs distribution

$$\mathcal{P}[u] = \mathcal{N}\exp\{-\mathcal{F}[u]\}, \qquad \mathcal{F}[u] = \beta\int dx u^2(x). \quad (3)$$

Here, $\mathcal{N}$ is a normalization constant. It is seen from (3) that the parameter β plays the role of the inverse temperature, so that the statistics of the velocity field at a given instant of time is also Gaussian with the two-point average

$$\langle u(x,t)u(x',t)\rangle = (2\beta)^{-1}\delta(x-x'), \quad (4)$$

corresponding to the zero correlation length.

In this paper, we study the asymptotic behavior of the correlation functions

$$\mathcal{K}(X,t) = \langle u(0,0)u(X,t)\rangle,$$
$$T(X,Y,t) = \langle u(0,0)u(Y,0)u(X,t)\rangle, \quad (5)$$
$$S(X,Y,\Delta,t) = \langle u(0,0)u(Y,0)u(X,t)u(X+\Delta,t)\rangle$$

at a small time t and a large distance X. It is assumed that the viscosity is very small ($\nu \longrightarrow 0$). However, it is crucial for the theory that it be distinct from zero (see below). It is natural to assume that the influence of the noise ξ can be neglected at small time intervals t (below, we consider this assumption in more detail). Then, averaging is performed over distribution (3) of the field values $u_0(x) = u(x, 0)$. In [8], principal exponential asymptotics terms of correlators (5) were found. The derivation used the fact that, for $\nu \longrightarrow 0$, the dynamics of the correlation of velocities at spatially distant points is a Lagrangian transition. Therefore, the correlator $\mathcal{K}(X, t)$ is determined by the probability of an initial fluctuation $u_0(x)$ such that a Lagrangian particle travels from 0 to the point X in time t. It was shown in [8] that the optimal profile is linear:

$$u_0^*(x) = (X-x)/T, \quad 0<x<X, \qquad u_0(x) = 0, \quad x<0, \quad x>X. \quad (6)$$

The exponential asymptotics part of the function $\mathcal{K}(X, t)$ is written as

$$\mathcal{P}[u_0^*] = \exp(-\beta X^3/3t^2). \quad (7)$$

Notice that the same fluctuation "brings" all particles from the interior of the interval $(0, X)$ to the point $x = X$ in time t. Then, we conclude that, for $\Delta \ll X$ and $0 < Y < X$, we have

$$\mathcal{K}(X,t) \sim T(X,Y,t) \sim S(X,Y,\Delta,t) \sim \exp\left(-\frac{\beta X^3}{3t^2}\right). \quad (8)$$

The condition $\Delta \ll x$ ensures a small difference of the optimal profile from the linear profile (6). The asymptotics under consideration corresponds to the inequality

$$\beta X^3/t^2 \gg 1,$$

and relations (8) are interpreted as intermittency.

In the reasoning above, we assumed that correlation functions of form (5) tend to a finite limit as $\nu \longrightarrow 0$. This assumption was made in [12]. It was also mentioned in that study that there exists only a single dimensionless combination of the parameters $\beta X^3/t^2$ in this limit, which is usually formulated in terms of dispersion as

$$\omega_k \propto k^{3/2}.$$

The smallness of ν implies the inequality

$$x^2(\nu t)^{-1} \gg \beta X^3 t^{-2}.$$

The absence of divergences as $\nu \longrightarrow 0$ can be verified as follows. If there is a divergence, then the main contribution to the averages is made by short-wave fluctuations. For these fluctuations, the term proportional νu_{xx} dominates in the equation of motion (1). Then, perturbation theory could be applied to the nonlinear terms. An analysis confirming the existence of a limit as $\nu \longrightarrow 0$ based on Wild's diagram technique was carried out in [13]. Note that, in the case under consideration, the convergence of the renormalized diagrams of perturbation theory at $\nu \longrightarrow 0$ implies that this theory is inapplicable. However, it must be stressed that ν cannot be immediately set to zero. Indeed, shock waves, which can be correctly described only with regard for the dissipative term in Eq. (1), play a key role in the dynamics of finite fluctuations. The approach used in [8] was based on Lagrangian trajectories. For this reason, an independent confirmation of the basic result of [8] is of interest. Moreover, the approach used in [8] does not yield preexponential coefficients in the expressions for correlators. In the same paper, it was noted that the integral over fluctuations at the background of the optimal profile (6) is not Gaussian; thus, the problem is not reduced to the applications of the standard saddle point method to functional integrals.

In this paper, we calculate complete asymptotic expressions for the correlation functions (5) at $\beta X^3/3t^2 \gg 1$ in the framework of the approximation whereby the effect of noise on the interval t is neglected. We immediately note that the asymptotic expressions obtained are in agreement with estimate (8); they are also in complete agreement with the optimal fluctuation if it is assigned the following measure in the functional space:

$$\Omega = \left(\frac{\beta X^3}{t^2}\right)^{-1/3}(2\mathrm{Ai}'(-a_0))^{-1}\exp\left(-a_0\left(\frac{\beta X^3}{t^2}\right)^{1/3}\right). \quad (9)$$

Here, $-a_0$ is the first zero of the Airy function:

$$\mathrm{Ai}(-a_0) = 0.$$

The measure is defined in the same way as in the calculus of instantons in quantum field theory [14]: the average of the product of the velocity field values is calculated by substituting the optimal profile (6) into it and by multiplying the result by the Gibbs weight (7) and by the measure Ω. It is seen from the expression for Ω that fluctuations are small relative to (6); indeed, Ω decreases with growing X^3/t^2. However, the dependence of Ω on the saddle point parameter $(\beta X^3/t^2)^{1/2}$ is substantially nonanalytic. Such a behavior of the measure implies that it cannot be calculated using the quadratic (in fluctuations) expansion of the effective action.

The turbulence phenomenon for the Burgers equation with the initial distribution (3) for the case of correlator decomposition was studied in [15, 16]. However, only the simultaneous statistics, which does not exhibit intermittency for the velocity field, was considered in [15]. In the paper [16], the case $t \longrightarrow \infty$, which is opposite to the case considered here, was studied. Moreover, nosimultaneous and multipoint averages were not considered in [16]. We note that the intermittency of simultaneous structure functions of the velocity field caused by discontinuous fluctuations was correctly described in [15, 16]. It is also characteristic of the general statement of the problem on the Burgers turbulence [2, 17–19], which assumes that the length L of the external noise correlation or the initial distribution of velocities is considerably greater than the dissipation scale r_d. Moreover, the distances r for which the correlations are analyzed are assumed to belong to the inertia interval; i.e., $r_d \ll r \ll L$.

2. CORRELATORS AT A SMALL DIFFERENCE OF TIMES

We begin with the calculation of asymptotic expressions for the function $\mathcal{K}(X, t)$ at $\beta X^3/t^2 \gg 1$. Since the averaging over the initial condition $u_0(x)$ is carried out with the Gaussian weight (3), the correlator $\mathcal{K}$ can be written as

$$\mathcal{K}(X, t) = \frac{1}{2\beta}\left\langle\frac{\delta u(X, t)}{\delta u_0(0)}\right\rangle. \quad (10)$$

When calculating the variational derivative in (10), we consider $u(X, t)$ as a functional of the field $u_0(x)$. The solution to the Cauchy problem for Eq. (1) is obtained with the help of the well-known Cole–Hopf substitution such that

$$u(X, t) = -2\nu\partial_X \ln\left\{\int_{-l}^{l_1} dz F(z, X)\right\},$$
$$F(z, X) = \exp\left(-\frac{(z-X)^2}{4t\nu} - \frac{1}{2\nu}\int_{-l}^{z} u_0(y)dy\right). \quad (11)$$

Here, $-l$ and l_1 are the coordinates of the remote endpoints of the interval occupied by the medium. Formula (11) is valid for $l, l_1 \longrightarrow \infty$. In this limit, l and l_1 do not appear in the final expressions for the correlators, although it is convenient to retain them in the process of intermediate manipulations. The independence of $\mathcal{K}(X, t)$ of l and l_1 makes it possible to shift the interval endpoints as follows:

$$(-l, l_1) \longrightarrow (-l + X, l_1 + X).$$

Using the translational invariance of the energy $\mathcal{F}$, we replace $u_0(x)$ by $u_0(x - X)$. After these transformations, taking the variational derivative, and differentiation with respect to X in (10) and (11), the correlator takes the simple form

$$\mathcal{K}(X, t) = \frac{1}{2\beta}\left\langle F(-X, 0)\left\{\int_{-l}^{l_1} dz F(z, 0)\right\}^{-1}\right\rangle$$
$$= \frac{1}{2\beta}\int_0^{\infty} d\lambda\left\langle\exp\left(-\int_{-l}^{l_1} dz \exp\left(-\frac{\Phi(z)}{2\nu}\right)\right)\right\rangle. \quad (12)$$

Here,

$$\Phi(z) = -2\nu\ln\lambda + \frac{z^2 - X^2}{2t} + \int_{-X}^{z} d\tau u_0(\tau). \quad (13)$$

The averaging $\langle\ldots\rangle$ is performed as the functional integration with respect to $\mathcal{D}u_0$. Now, using formula (13), we change the integration variable $u_0(z)$ for $\Phi(z)$. By construction, $\Phi(z)$ satisfies the condition

$$\Phi(-X) = -2\nu\ln\lambda. \quad (14)$$

Expressing $\mathcal{F}$ in terms of Φ as

$$\mathcal{F}[\Phi]$$
$$= \mathcal{S}[\Phi] - \frac{2\beta}{t}(l_1\Phi(l_1) + l\Phi(-l)) + \frac{\beta}{3t^2}(l_1^3 + l^3), \quad (15)$$
$$\mathcal{S}[\Phi] = \int_{-l}^{l_1} dx\left\{\left(\frac{d\Phi}{dx}\right)^2 + \frac{2}{t}\Phi\right\},$$

and taking into account Eq. (12), we reduce the calculation of $\mathcal{K}(X, t)$ to the Feynman–Kac path integral

[20]. It follows from (14) that the integration with respect to $d\lambda$ removes the boundary condition imposed on $\Phi(x)$ at the point $x = -X$, and the correlator $\mathcal{K}(X, t)$ is expressed in terms of the matrix element of the Euclidean quantum mechanics as

$$\mathcal{K}(X,t) = \frac{\exp\left\{-\frac{\beta}{3t^2}(l_1^3 + l^3)\right\}}{4\nu\beta}\left\langle \exp\left(\frac{2\beta l_1}{t}\Phi\right)\right| \times \exp(-(l_1 - X)\hat{H})\exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(l+X)\hat{H}) \times \left|\exp\left(\frac{2\beta l}{t}\Phi\right)\right\rangle \quad (16)$$

with the Hamiltonian

$$\hat{H} = -\frac{1}{4\beta}\frac{d^2}{d\Phi^2} + U(\Phi), \qquad U(\Phi) = \frac{2\beta}{t}\Phi + \exp\left(-\frac{\Phi}{2\nu}\right). \quad (17)$$

Unfortunately, we failed to find the basis diagonalizing the operator $\hat{H}$ in a closed form. However, problem (16), (17) admits an analytical solution for $\nu \longrightarrow 0$ and $\beta X^3/t^2 \gg 1$.

Indeed, for small ν, the axis Φ can be decomposed into two domains such that one of the terms in $U(\Phi)$ is dominant in each domain. In the leading approximation with respect to ν, the eigenfunctions of $\hat{H}$ are easily found:

$$\hat{H}\psi_n(\Phi) = \left(\frac{\beta}{t^2}\right)^{1/3} a_n\psi_n(\Phi), \quad (18)$$

$$\psi_n(\Phi) = \begin{cases} 4\nu\left(\frac{8\beta^2}{t}\right)^{1/2} K_0\left(8\nu\sqrt{\beta}\exp\left(-\frac{\Phi}{4\nu}\right)\right), \\ \Phi < \nu\ln\frac{1}{\nu}, \\ \frac{1}{\mathrm{Ai}'(-a_n)}\left(\frac{8\beta^2}{t}\right)^{1/6}\mathrm{Ai}\left(\left(\frac{8\beta^2}{t}\right)^{1/3}\Phi - a_n\right), \\ \Phi > \nu\ln\frac{1}{\nu}. \end{cases} \quad (19)$$

Here, K_0 is the elliptic Macdonald function. In (19), $-a_n$ such that $a_n > 0$ and $a_{n+1} > a_n$ ($n = 0, 1, \ldots$) form the sequence of zeros of the Airy function: $\mathrm{Ai}(-a_n) = 0$. When calculating the average in Eq. (16), we have to find the result of the action of the "evolution" operator $\exp(-T\hat{H})$ on the initial state $\exp(\alpha\Phi)$. This state cannot be normalized. Hence, an expansion in basis (19) is either possible or not, depending on the relationship between T and α. The values of α corresponding to the boundary wave functions in Eq. (16) are large:

$$\alpha^3 t/\beta^2 \gg 1.$$

Then, the scalar product $\langle\psi_n|\exp(\alpha\Phi)\rangle$ has the following form up to terms that vanish as $\nu \longrightarrow 0$:

$$c_n(\alpha) = \langle\psi_n|\exp(\alpha\Phi)\rangle = \frac{1}{\mathrm{Ai}'(-a_n)}\left(\frac{t}{8\beta^2}\right)^{1/6} \times \exp\left(\alpha a_n\left(\frac{t}{8\beta^2}\right)^{1/3} + \alpha^3\frac{t}{8\beta^2}\right). \quad (20)$$

The following series for the function

$$f(\Phi, X) = \exp\left(-\frac{\beta l^3}{3t^2}\right)\exp(-(l+X)\hat{H}) \times \exp\left(\frac{2\beta l}{t}\Phi\right) \quad (21)$$

is convergent:

$$f(\Phi, X) = \sum_n c_n\left(\frac{2\beta l}{t}\right) \times \exp\left(-\frac{\beta l^3}{3t^2} - a_n\left(\frac{\beta}{t^2}\right)^{1/3}(l+X)\right)\psi_n(\Phi) \quad (22)$$
$$= \sum_n \frac{1}{\mathrm{Ai}'(-a_n)}\exp\left(-a_n\left(\frac{\beta}{t^2}\right)^{1/3}X\right)\psi_n(\Phi).$$

At $\beta X^3/t^2 \gg 1$, the sum in (22) is determined by the contribution of the ground state. For the further considerations, only the domain $\Phi < \nu\ln(1/\nu)$, in which

$$f(\Phi, X) \approx \frac{4\nu}{\mathrm{Ai}'(-a_0)}\left(\frac{8\beta^2}{t}\right)^{1/3} \times K_0\left(8\nu\sqrt{\beta}\exp\left(-\frac{\Phi}{4\nu}\right)\right)\exp\left(-a_0\left(\frac{\beta}{t^2}\right)^{1/3}X\right), \quad (23)$$

is of importance.

A series for the state

$$g(\Phi, X) = \exp\left(-\frac{\beta l_1^3}{3t^2}\right)\exp(-(l_1 - X)\hat{H}) \times \exp\left(\frac{2\beta l_1}{t}\Phi\right), \quad (24)$$

which is similar to (22), is divergent. However, for $\beta X^3/t^2 \gg 1$, we can use the fact that the main part of the function $g(\Phi, \tau)$ for $\tau > X$ belongs to the domain where the linear term in $U(\Phi)$ is dominating. The corresponding evolution problem is easily solved:

$$g_0(\Phi, X) = \exp\left(\frac{2\beta X}{t}\Phi - \frac{\beta X^3}{3t^2}\right). \quad (25)$$

The behavior of $g(\Phi, X)$ for small and negative Φ can be found using the substitution

$$g(\Phi, \tau) = g_0(\Phi, \tau)G(\Phi, \tau),$$
$$G(\Phi \longrightarrow +\infty, \tau) \longrightarrow 1, \quad G(\Phi \longrightarrow -\infty, \tau) \longrightarrow 0. \quad (26)$$

The solution to the equation

$$\partial_\tau G = \left(\frac{1}{4\beta}\partial_\Phi^2 + \frac{\tau}{t}\partial_\Phi - \exp\left(-\frac{\Phi}{2\nu}\right)\right)G \quad (27)$$

for G with the boundary conditions (26) can be found, for large τ, using the adiabatic expansion. In the leading approximation, the right-hand side of equality (27) vanishes, and an explicit expression for G is easily found. Then, the total function $g(\Phi, X)$ has the following form in the principal order with respect to $\beta X^3/t^2$:

$$g(\Phi, X) = 8\beta\nu\frac{X}{t}$$
$$\times K_{8\beta\nu X/t}\left(8\sqrt{\beta}\nu\exp\left(-\frac{\Phi}{4\nu}\right)\right)\exp\left(-\frac{\beta X^3}{3t^2}\right). \quad (28)$$

In the domain $\Phi < \nu\ln(1/\nu)$, we may neglect the fact that the index of the Macdonald function is different from zero. The substitution of (28) and (23) into (16) yields the final expression for the short-time asymptotics of the second-order correlator for the velocity field:

$$\mathcal{K}(X, t) \approx \frac{(X/t)^2}{2\mathrm{Ai}'(-a_0)}\left(\frac{\beta X^3}{t^2}\right)^{-1/3}$$
$$\times \exp\left(-\frac{\beta X^3}{3t^2} - a_0\left(\frac{\beta}{t^2}\right)^{1/3}X\right). \quad (29)$$

We see that the correlator $\mathcal{K}(X, t)$ is indeed finite at $\nu \longrightarrow 0$. Furthermore, it follows from (29) that the integral over fluctuations in the vicinity of the optimal profile (6) depends on the parameter of the theory $\beta X^3/t^2$; more precisely, it is proportional to $((\beta X^3/t^2)^{-1/3}\exp(-a_0(\beta X^3/t^2)^{1/3})$ and cannot be obtained by a Gaussian integration.

In the approximation that takes into account the decomposition of correlators, the third-order correlation function $T(X, Y, t)$ (see (5)) is also expressed in terms of the variational derivatives of the solution (11) with respect to the initial field:

$$T(X, Y, t) = \frac{1}{4\beta^2}\left\langle\frac{\delta^2 u(X, t)}{\delta u_0(0)\delta u_0(Y)}\right\rangle. \quad (30)$$

Manipulations similar to (12)–(15) yield the following operator representation for $T(X, Y, t)$:

$$T(X, Y, t) = -\frac{1}{16\nu^2\beta^2}$$
$$\times \exp\left\{-\frac{\beta}{3t^2}(l_1^3 + l^3)\right\}\partial_X\int_{-l}^{-X} dz_1 \int_{Y-X}^{l_1} dz_2$$
$$\times\left\langle \exp\left(\frac{2\beta l_1}{t}\Phi\right)\right|\exp(-(l_1 + z_1)\hat{H}) \quad (31)$$
$$\times \exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(z_2 - z_1)\hat{H})$$
$$\times \exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(l - z_2)\hat{H})\left|\exp\left(\frac{2\beta l}{t}\Phi\right)\right\rangle.$$

For the functions f and g defined in (21) and (24), representation (31) takes the form

$$T(X, Y, t) = -\frac{\partial_X}{16\nu^2\beta^2}\int_X^l dz_1 \int_{-l_1}^{-Y+X} dz_2$$
$$\times\left\langle g(\Phi, z_1)\right|\exp\left(-\frac{\Phi}{2\nu}\right) \quad (32)$$
$$\times \exp(-(z_1 - z_2)\hat{H})\exp\left(-\frac{\Phi}{2\nu}\right)\left|f(\Phi, z_2)\right\rangle.$$

It follows from (28) that, for $\beta X^3/t^2 \gg 1$, the main contribution to the integral with respect to dz_1 is made by a small neighborhood near the upper limit $z_1 = -X$. Note that the intermittency effect (8) manifests itself (at this stage of the calculations) in that the function $g(\Phi, X)$ cannot be resolved in the complete set of eigenfunctions (19); for this reason, $g(\Phi, X)$ has the form (28). If $X - Y \sim X \sim Y$, the matrix element in (32) is determined by the contribution of the intermediate ground state ψ_0 (the product of $g(\Phi, X)$ by $\exp(-\Phi/2\nu)$ can be resolved in the basis (19)). In the integral with respect to dz_2, the domain $-X + Y < z_2 < 0$ makes the major contribution. Inside this domain, the dependence of the average in (32) on z_2 can be neglected. As a result, we obtain the expression

$$T(X, Y, t) \approx \frac{(X/t)^2}{2\mathrm{Ai}'(-a_0)}\left(\frac{\beta X^3}{t^2}\right)^{-1/3}\frac{X - Y}{t}$$
$$\times \exp\left(-\frac{\beta X^3}{3t^2} - a_0\left(\frac{\beta}{t^2}\right)^{1/3}X\right) \approx \frac{X - Y}{t}\mathcal{K}(X, t), \quad (33)$$

which is in agreement with the fact that the initial profile (6) is dominating.

The scheme for the calculation of the fourth-order correlation function

$$S(X, Y, \Delta, t) = S^{(1)}(X, X_1) + S^{(2)}(X, X_1) + S^{(1)}(X_1, X) + S^{(2)}(X_1, X), \tag{34}$$

$$S^{(1)}(X, X_1) = \frac{1}{4\beta^2}\left\langle \frac{\delta^2 u(X, t)}{\delta u_0(0)\delta u_0(y)} u(X_1, t)\right\rangle,$$
$$S^{(2)}(X, X_1) = \frac{1}{4\beta^2}\left\langle \frac{\delta u(X, t)}{\delta u_0(0)} \frac{\delta u(X_1 t)}{\delta u_0(y)}\right\rangle, \tag{35}$$

where $X_1 = X + \Delta$, is somewhat different from the cases of the second- and third-order correlators. This is explained by the fact that the expression to be averaged includes both the combinations $F(z, X_1)$ and $F(z, X)$. The averages are reduced to path integrals by the change

$$\Phi(z) = -2\nu\ln\lambda - 2\nu\ln n_F(z-\mu) + \frac{z^2 - X^2}{2t} + \int_{-X}^{z} d\tau u_0(\tau),$$

where $n_F(z)$ is the Fermi distribution

$$n_F(z) = \left[1 + \exp\left(\frac{\Delta}{2\nu t}z\right)\right]^{-1}. \tag{36}$$

Unfortunately, closed expressions can be obtained only for small $\Delta \ll X$. However, Δ remains much greater than the dissipative scale: $\Delta^2(\nu t)^{-1} \gg 1$. Formally, the smallness of Δ allows us to replace the vertex $\exp(2\beta\Delta/t)$ by 1 in the operator representation of $S^{(1)}$ and $S^{(2)}$. As a result, for the average $S^{(1)}(X, X_1)$, we obtain

$$S^{(1)} = \partial_X \frac{\Delta}{32\beta^2\nu^3 t^2}\hat{\mathcal{L}}$$
$$\times\left\langle f(\Phi, \zeta_1)\right| \exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(\zeta_1 - \zeta_2)\hat{H}) \times \exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(\zeta_2 - \zeta_3)\hat{H}) \times \exp\left(-\frac{\Phi}{2\nu}\right)\left| g(\Phi, \zeta_3)\right\rangle, \tag{37}$$

where

$$\hat{\mathcal{L}} = \int_{Y-X}^{l_1} dz_1 \int_{-l}^{-X} dz_2 \int_{-l}^{l_1} dz_3 (z_3 - \Delta) \times \int_{-l}^{l_1} d\mu n_F(\mu - z_1) n_F(\mu - z_2) n_F(z_3 - \mu), \tag{38}$$

and $\{\zeta_n\}$ is a permutation of the integration variables $\{z_n\}$ such that $\zeta_k < \zeta_{k+1}$. (In effect, (37) is the matrix element of the chronologically arranged product of operators.)

Note that the integrand in (37) has a singularity as a function of $\zeta_3 - \zeta_2$ as $\nu \longrightarrow 0$. Indeed, the matrix elements of the operator

$$\exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(\zeta_3 - \zeta_2)\hat{H})\exp\left(-\frac{\Phi}{2\nu}\right)$$

at $\zeta_3 - \zeta_2 \sim \beta\nu^2/t$ are about ν^{-3} times greater than the values at the separation $\zeta_3 - \zeta_2 \sim X$. This singularity can be singled out using the rule

$$\frac{1}{\nu}\exp\left(-\frac{\Phi}{2\nu}\right)\exp(-(\zeta_3 - \zeta_2)\hat{H})\exp\left(-\frac{\Phi}{2\nu}\right) \longrightarrow \frac{2\beta}{t}\delta(\zeta_3 - \zeta_2), \quad \nu \longrightarrow 0. \tag{39}$$

The factor of order $1/\nu$ is compensated in (37) by the result of the integration over the domain $\mu > z_3$, which has the order $\nu t/\Delta$. The contribution of the domain $z_{1,2} < \mu < z_3$, which has no singularities determined by formula (39), is of order Δ and is, therefore, insignificant. Finally, we have, for $\Delta \ll X$,

$$S^{(1)}(X, X_1) = \frac{(X-Y)^2}{2t^2}\mathcal{K}(X, t). \tag{40}$$

The average $S^{(2)}(X, X_1)$ is calculated similarly to the third-order correlator $T(X, Y, t)$; indeed, the identity

$$\partial_X \frac{\int_0^{l_1} F(z, X)dz}{\int_{-l}^{l_1} F(z, X)dz} = -\partial_X \frac{\int_{-l}^{0} F(z, X)dz}{\int_{-l}^{l_1} F(z, X)dz}$$

allows us to eliminate singularities of type (39):

$$S^{(2)}(X, X_1) = \frac{X^2 - Y^2}{2t^2}\mathcal{K}(X, t). \tag{41}$$

$S^{(1)}(X_1, X)$ and $S^{(2)}(X_1, X)$ are calculated according to the same scheme with the replacement $\Delta \longrightarrow -\Delta$. The result is such that these terms in (34) can be neglected at $\nu \longrightarrow 0$. The total expression for the four-point correlator is

$$S(X, Y, \Delta, t) \approx \frac{X(X-Y)}{t^2}\mathcal{K}(X, t), \tag{42}$$

which confirms, along with (29) and (33), that the initial fluctuation in (6) is a determining factor. It is also in agreement with formula (9) for the measure of the fluctuation.

3. CONCLUSIONS

The main results obtained in this paper are expressions (29), (33), and (42). The exponential smallness of the values of these correlation functions means that the fluctuation that determines these functions occurs rarely. The fact that the exponential parts of those expressions coincide implies the uniqueness of this fluctuation for various averages, i.e., the intermittency phenomenon.

Strictly speaking, formulas (29), (33), and (42) pertain to the initial distribution (3), which corresponds to the case when the correlators decompose. Returning to problem (1), (2), we should estimate the role of the noise ξ. For example, for the second-order correlator, the base expression is

$$\mathcal{K}(X, t) = \langle u(0, 0)\langle u(X, t)\rangle_{\xi}\rangle, \qquad (43)$$

where the outer averaging is carried out over the initial ensemble (3) and $\langle\ldots\rangle_{\xi}$ denotes the average over the noise ξ that acted on the time interval $(0, t)$. If the noise is taken into account in the framework of perturbation theory against the background of the evolving profile (6), then the correction in $\langle u(X, t)\rangle_{\xi}$ appears in the second order and can be easily estimated from dimensional considerations—the correlator $\langle\xi\xi\rangle$ is proportional to the temperature β^{-1}, which is reduced (without ν) to the dimensionless form in a unique fashion. The additional factor $(\beta X^3/t^2)^{-1}$ thus obtained confirms the applicability of expressions (29), (33), and (42) to solving the Burgers equation with the Langevin-type pumping (2). The viscosity that appears as a result of averaging over the noise ξ is compensated by large values of the gradients of the field $u(x, t)$ at the instant of shock formation of the optimal fluctuation.

In conclusion, we make some remarks concerning the conventional turbulent problem statement for the Burgers equation. The main results obtained up to the present time pertain to various asymptotic expressions for simultaneous distribution functions of the gradient of the velocity field u_x and to the difference of velocities $u(r, t) - u(0, t)$ [21–25]. In this case, the problem is reduced to finding an optimal fluctuation that determines the desired far [19, 21–23, 26] or intermediate [18, 24, 25] asymptotics. The decisive role is played by a large value of the argument of the distribution function or, which is the same, the number of the moment to be calculated (see [27–29] for the discussion in a more general context). The asymptotics of correlators (5) studied in this paper are determined by a large (with respect to the parameter $\beta X^3/t^2$) initial fluctuation of the velocity field. For this reason, it would be interesting to analyze this problem in the framework of the direct instanton approach [19, 27–29].

It has already been mentioned that the intermittency phenomenon is, in a certain sense, an inverse limiting case with respect to the problems that are correctly described by perturbation theory; hence, this phenomenon is important for statistical physics as a whole. However, there are only few systems for which a consistent analytical description of this phenomenon can be given. In addition to the studies mentioned above, we also cite the papers [30, 31] in which the problem of transport of a passive scalar by a turbulent flow is solved for certain limiting cases. From this point of view, explicit formulas for the asymptotics of correlation functions (5) and measure (9) seem to be very illustrative.

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REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 6: *Fluid Mechanics* (Nauka, Moscow, 1986; Pergamon, New York, 1987).
2. U. Frisch, *Turbulence. The Legacy of A. N. Kolmogorov* (Cambridge Univ. Press, Cambridge, 1995).
3. I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems* (Nauka, Moscow, 1982; Wiley, New York, 1988).
4. K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge Univ. Press, Cambridge, 1997).
5. A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, *Fluctuation Theory of Phase Transitions* (Nauka, Moscow, 1982; Pergamon, Oxford, 1979).
6. V. V. Lebedev, Pis'ma Zh. Éksp. Teor. Fiz. **70**, 675 (1999) [JETP Lett. **70**, 691 (1999)].
7. V. V. Lebedev, Phys. Rev. E **62**, 1002 (2000).
8. I. V. Kolokolov, Pis'ma Zh. Éksp. Teor. Fiz. **71**, 20 (2000) [JETP Lett. **71**, 12 (2000)].
9. J. M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Dordrecht, 1974).
10. M. Kardar, G. Parisi, and Y. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
11. G. Blatter, M. V. Feigelman, V. B. Geshkenbein, *et al.*, Rev. Mod. Phys. **66**, 1125 (1994).
12. D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
13. V. V. Lebedev and V. S. L'vov, Pis'ma Zh. Éksp. Teor. Fiz. **58**, 301 (1993) [JETP Lett. **58**, 310 (1993)].
14. A. M. Polyakov, *Gage Fields and Strings* (Udmurtskiĭ Univ., Izhevsk, 1999).
15. L. Frachenbourg and Ph. A. Martin, cond-mat/9905056.
16. R. Tribe and O. Zaboronski, Comm. Math. Phys. **101**, 415 (2000).
17. V. Yakhot and A. Chekhlov, Phys. Rev. Lett. **77**, 3118 (1996).

18. E. Weinan, K. Khanin, A. Mazel, and Y. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).
19. E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **78**, 1452 (1997).
20. R. P. Feynmann and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
21. M. V. Feĭgel'man, Zh. Éksp. Teor. Fiz. **79**, 1095 (1980) [Sov. Phys. JETP **52**, 555 (1980)].
22. V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).
23. S. A. Boldyrev, Phys. Rev. E **55**, 6907 (1997).
24. T. Gotoh and R. H. Kraichnan, Phys. Fluids **10**, 2859 (1998).
25. J. Bec and U. Frisch, Phys. Rev. E **61**, 1395 (2000).
26. A. I. Chernykh and M. G. Stepanov, Phys. Rev. E **64**, 026306 (2001).
27. G. Falkovich, I. Kolokolov, V. Lebedev, and A. Migdal, Phys. Rev. E **54**, 4896 (1996).
28. M. Chertkov, Phys. Rev. E **55**, 2722 (1997).
29. E. Balkovsky and V. Lebedev, Phys. Rev. E **58**, 5776 (1998).
30. M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
31. K. Gawedzki and A. Kupianen, Phys. Rev. Lett. **75**, 3834 (1995).

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