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Effect of storage characteristics on wind intermittency mitigation effectiveness

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Abstract— **Stochastic variations and unpredictability of wind energy are the major concerns of power industry and hinder the wide scale adoption of wind power. Compensation of short term variability is one of the major challenges that the industry will face in the coming years. We analyze the potential impact of advanced control and storage technologies in reducing the intermittency of wind power. Using the convex optimization techniques we study the theoretical limits on the performance of storage technologies. Specifically we analyze the interplay between the variations of electric power fluctuations and the technical characteristics of storage available in the system. We quantify the trade-off between the reduction in power intermittency, storage capacity, power rating, and overall system efficiency.**

I. INTRODUCTION

Penetration of intermittent and non-dispatchable power generation resources is expected to increase by at least an order of magnitude in the coming decade. Much of this generation will be deployed at the distribution grid, in form of distributed generation. Although these trends will likely increase the overall security of the power system, they will also significantly complicate the problem of local and global power balancing, and more generally maintaining the grid stability. New balancing mechanisms will need to be introduced to mitigate the problem of intermittency [1], [2]. The most promising technologies expected to alleviate the balancing problems are the fast gas-fired generation units capable of quickly adjusting their output, responsive power consumption technologies, and finally stand-alone energy storage units [2], [3]. Rapid advances in storage technologies, as well as relative simplicity of introducing this balancing technology, makes it especially attractive for managing fluctuations of intermittent renewables in autonomous or weakly coupled grids that have to balance the generation and consumption levels locally.

Modern power systems have several mechanisms of balancing the generation and consumption levels. These mechanisms operate on several time-scales and work almost independently of each other. On shortest time scale second to minute time scale the balance is restored via primary and secondary frequency regulation. Most of synchronous generators, and some loads participating in ancillary service program, adjust their power generation and consumption level based on the deviation of the frequency from its nominal level. This way the inertia of the large turbines plays the role of the energy storage buffer. Primary frequency control loop stabilizes the dynamics of frequency in situations where the generation output setting are not balanced with the overall consumption level. The secondary frequency control loop returns the frequency to its nominal level ($60Hz$ in US). On longer time-scales, ranging from tens of minutes to days, the fluctuations are suppressed by real-time market operation and optimal dispatch of generators produced by solving the unit commitment problem. The grid operators find optimal resource allocation using the existing demand/generation forecasts and update the allocation plan when better forecast data becomes available.

Most of the existing power systems compensate for the fluctuations of the uncontrollable renewable sources with the help of fast ramping generation units, that are used to fill the gap between electricity supply and demand [4], [5]. Increasing the penetration of wind power increases the demands on the fast ramping generators. [2] The limited ramping capacity of the generators, the cost of cycling them, and the additional emissions they produce prevents fast ramping generation alone from being the solution to the power intermittency problem. [2], [4]–[7] Other solutions, such as introducing curtailment of the wind power or transmission of extra power to other grids, have a limited capacity for regulating variability as they waste power and rely on other grids being flexible. Electrical energy storage systems can efficiently lessen variability without generating harmful emissions.

Choosing the best storage solution and assessing its effectiveness in frequency regulation or load following is by no means a simple problem. There are at least dozen of multiple energy storage technologies, each characterized by several important constraints, such as power rating, roundtrip efficiency, self-discharge rate, etc [8], [9]. The most cost-effective solution depends also on the temporal statistics of the fluctuations that are mitigated by the distributed storage system. The statistical properties of typical loads are poorly understood, and have not been systematically studied to our knowledge. The statistical properties of the wind fluctuations and associated output of wind turbine are much better understood. Power spectrum of these fluctuations is characterized by the well-known Kolmogorov $5/3$ law [10], [11]. However, the probability distribution of the velocity field is highly non-Gaussian, with high-order correlation functions characterized by anomalous exponents, phenomenon known as intermittency in fluid dynamics community [12]–[14]. Analytical studies attempting to assess the effectiveness of storage in mitigation of wind intermittency have to rely either on the

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time series data or on sophisticated stochastic model that incorporate at least some features of real wind fluctuations.

Finally, the cost of an actual storage system will include the cost of sensing devices to create wind forecasts, automatic control systems employed by the local grid operators and the storage system itself. Many possible approaches have been proposed in the literature that address all the above problems, which make the problem of storage effectiveness analysis even more complicated.

In this study we attempt to answer some of the questions raised above. In order to obtain results that are universal in the sense of their applicability to a maximal broad range of systems we focus on the analysis of the fundamental limits that can be achieved by the energy management system with perfect forecast of the wind fluctuations. In this setting the problem can be analyzed in the setting of infinite horizon model predictive control and solved easily with convex optimization approaches. The resulting solutions allow us to relate the technical characteristics of the storage systems with various performance objectives, such as efficiency of the system, variability reduction, etc. This approach also decouples the problem of forecasting from the control problem, and permits independent assessment of the forecast information value. Although not addressed in this paper, this problem is of utter importance and is actively studied by the authors of this manuscript.

II. Energy storage and capture systems

In our work we focus on modeling and analysis of a simple system consisting of a single turbine attached to storage device and external power grid. This setting, although simplistic, captures the essential features of large wind farms. It can be also directly applied to the important problem of stabilizing micro-grids and, more generally, future systems with strong penetration of distributed renewable generation. In our model we assume that the energy captured to the wind can be transferred either directly to external grid or routed to storage system and transfered to the grid at later times. To model this simple setting we use the following set of equations defined on the discrete time interval set where each unit of time corresponds to $dt = 1$ minute:

$$P_g(t) = P_w(t) + \eta P_d(t) - P_c(t) - P_l(t) \tag{1}$$

$$E(t+1) = E(t) + (\eta P_c(t) - P_d(t))dt \tag{2}$$

Here $P_g(t)$ represents the average power delivered to grid during the time interval $[t, t+dt]$, whereas $P_w(t)$ is the average power produced by the wind turbine on the same time interval. The process of storage charging is represented by the average charging power $P_c(t)$ and discharging power $P_d(t)$ and the roundtrip efficiency represented by coefficient $0 < \eta^2 < 1$. In our model we ignore the self-discharge rate, assuming that the energy lost to self discharge is small compared to the amount of energy flowing through our system. Finally, we introduce the effect of curtailment in the system through the variable $P_l(t)$ which represents the average power that has been "lost" during the time interval t. This curtailment can happen either at wind turbine control level, by angling the blades to intentionally reduce their performance, or at the point of interconnection of the grid in consideration with the outer power system.

In order to model the intermittent wind power $P_w(t)$ we used the real wind speed data from the National Weather Service's ASOS data sets [15]. These data sets are time-series of real wind speed averaged over one minute time intervals, measured with one knot ($.514$ m/s) resolution. Better time resolution of data is not required because the inertia of the blades smooths out fluctuations that occur on the seconds timescale. We have processed these data sets, and analyzed only the intervals where the 10 minute average wind speed exceeded the cut in speed of the turbine. The maximum power available from the wind from the turbine's energy conversion process was modeled by [16]

$$P_w(t) = \frac{1}{2}\rho C_p A v(t)^3 \tag{3}$$

where ρ is the density of air, A is the swept area of the turbine blades, C_p is the coefficient of power and $v(t)$ is the speed of the wind, capped at the turbine's maximum power rating. The turbine parameters are shown in Table I and were taken from [17].

The ASOS wind data represented several US locations and months. Intervals ranging from 15-4000 minutes of sustained windspeed were considered separately in the optimization procedure. Between consecutive intervals, the initial charge on the storage system was set to be the final charge state of the previous interval. Otherwise, the initial charge was set to zero, the worst possible case.

TABLE I
Wind turbine parameters

Parameter	Value
Maximum rated power	2 MW
Blade radius	35 m
Coefficient of performance C_p	0.48
Wind cut in speed v	3.5 m/s
Air density ρ	1.225 kg/m3

We have incorporated several characteristics of storage technologies in our model via constraints on the functions entering in the main equations (1,2). The most important one is the energy capacity E_{max} which restricts the total amount of energy that can be stored in the storage system at any moment of time: $0 \leq E(t) \leq E_{max}$. Next, we have used the realistic power ratings of the storage technologies, that limited the amounts of power that could be transferred to and from the storage at any moment of time: $0 \leq P_c(t), P_d(t) \leq P_{max}$. Excessive cycling of storage technologies causes them to degrade and eventually become useless. The total cycled power was limited to ensure the storage technologies could achieve a reasonable operating lifetime: $dt \sum_{t=1}^{T} [P_d(t) + P_c(t)] < 2\bar{C} E_{max} T$ with T as the simulation time interval and $\bar{C}$ as the allowed full charge/discharge cycles per unit time. Finally, we assume that no power can be drawn from the grid, which was formally accounted for with the inequalities $P_g(t), P_l(t) \geq 0$. We have also constrained the overall levels of system efficiency: $\sum_{t=1}^{T}[P_l(t) + (1-\eta)(P_d(t) + P_c(t))] \leq \kappa \sum_{t=1}^{T} P_w(t)$ where κ is the wind to grid efficiency that satisfies $0 \leq \kappa \leq 1$. The overall system efficiency enforces that the wind turbine system is supposed to deliver a set amount of the available power from the wind to the grid. This forces the turbine to operate efficiently, preventing solutions where the majority of the power is curtailed to make a smooth output. Because a turbine can delivery a constant power output if it only operates at its minimum generation level, $1-10\%$ of its capacity, κ forced the system to mitigate the wind fluctuations without wasting available power. An additional set of simulations removed the cycling and wind to grid efficiency constraints and instead made them costs in the objective function to see what pricing conditions made storage use or curtailment effective in mitigating fluctuations to show why the constraint was needed.

Performance of real distributed generation and storage systems depends on multiple factors, including among others the accuracy of the wind forecast systems and the characteristic of the control system employed by the turbine/storage operator. In order to separate the effect of the storage characteristics from the characteristics of control and forecast systems we have analyzed the idealistic control system. This system implemented open loop control that was based on the perfect forecast of the incoming wind fluctuations. The control actions represented by $P_c(t), P_d(t)$, and $P_l(t)$ were found via global optimization over the full time interval.

In order to characterize the effectiveness of the storage in mitigation of the short term intermittency we have chosen the timestep variations of the power output, $\sum_{t=1}^{T}(P_g(t+1) - P_g(t))^2$, to be the objective function of the optimization. This choice assumes penalizes the high frequency component of the fluctuations implying that those are the most difficult to compensate using existing technologies, for example because of insufficient spinning reserves or lack of fast ramping generators. On the other hand the objective function does not penalize slow variations of power, which are easier to compensate with market and unit commitment type mechanisms. Although our objective function is chosen somewhat arbitrarily, we believe that it reflects the major properties of real cost of the system operations. Its quadratic structure is related to the fact that the system always operates close to the optimum and the penalization of high frequencies is motivated by the assumption of relatively high costs of ancillary services and/or lack of fast generators.

This objective function enforces minimization of short term power output fluctuations that are especially difficult to balance with traditional grid control approaches. Although this setting is rather idealistic, the results of optimization and consequent analysis provide valuable information about the fundamental limitation of storage systems associated with technological constraints. The results provide guidance in terms of what storage characteristics are most important for balancing purposes and what is theoretically possible to achieve with given amount of storage. The results also provide a benchmark for future analysis of realistic control and forecast techniques.

The main technical advantage of the proposed approach is that the associated optimization problem is convex and quadratic, and can be solved very efficiently with standard software tools. The above discussion can be summarized in the following optimization problem performed on an entire interval of length T:

$$\min_{P_c, P_d, P_l} \sum_{t=1}^{T} [P_g(t+1) - P_g(t)]^2 \tag{4}$$

$$\text{subject to :} \tag{5}$$

$$\begin{aligned} P_g(t) = P_w(t) + \eta P_d(t) - P_c(t) - P_l(t), \\ E(t+1) = E(t) + (\eta P_c(t) - P_d(t))dt, \\ 0 \leq E(t) \leq E_{max}, \\ 0 \leq P_c(t), P_d(t) \leq P_{max}, \\ dt \sum_{t=1}^{T}[P_d(t) + P_c(t)] < 2\bar{C}E_{max}T \\ P_g(t), P_l(t) \geq 0 \\ \sum_{t=1}^{T}[P_l(t) + (1-\eta)(P_d(t) + P_c(t))] \leq \kappa \sum_{t=1}^{T} P_w(t) \end{aligned}$$

We have used the CVX convex optimization matlab toolbox [18] to explore the set of solutions and the dependencies between storage characteristics and the performance indicators of the system. CVX found the optimal sequence of charging, discharging and curtailment actions to achieve the lowest step by step variability. General storage properties were chosen to measure the effects of each property of the storage system. The performance of the general system is compared to a system with the properties of current storage technologies.

Additionally, the effectiveness of storage capacity was measured when the wind to grid efficiency and cycling constraints were changed to costs in the objective function, $\min_{P_c, P_d, P_l} \sqrt{\sum_{t=1}^{T}(P_g(t+1) - P_g(t))^2} + \sum_{t=1}^{T} A * (P_c(t) + P_d(t)) + B * ((1-\eta)(P_c(t) + P_d(t)) + P_l(t))$, where A represents the relative cost of cycling 1 watt of power compared to a step of 1W in the power output to the grid and B represents the relative cost of losing 1W of power compared to the step of 1W of power output. The loss of power from the storage conversion efficiency as well as curtailment contribute to power not delivered to the grid. A comparison of the total capture of wind power and largest contributing factor to the cost with respect to the factors A and B are shown in table II. It was found that without the constraint forcing efficient delivery of power to the grid, it was cheapest to curtail $70-90\%$ of the power available from the wind. This indicates that the best way to best way to remove variability in wind is to not use turbines to their full capacity. Because this result says that the best way to use wind power is to operate well below turbine ratings, the wind to grid efficiency was constrained to κ in the following analysis to show how the power grid must compensate for regulations that demand wind power be used.

TABLE II

TOTAL POWER OUTPUT AND MAJOR COST FACTORS

B	A: .0001	.001	.01	.1
.01	25% capture, conversion	25% capture, conversion	0.5% capture, curtailment	13% capture, curtailment
.1	29% capture, curtailment	29% capture, curtailment	29% capture, conversion	8% capture, curtailment
1	19% capture, P_g mismatch	19% capture, P_g mismatch	19% capture, P_g mismatch	16% capture, P_g mismatch
10	15% capture, P_g mismatch	15% capture, P_g mismatch	15% capture, P_g mismatch	15% capture, P_g mismatch

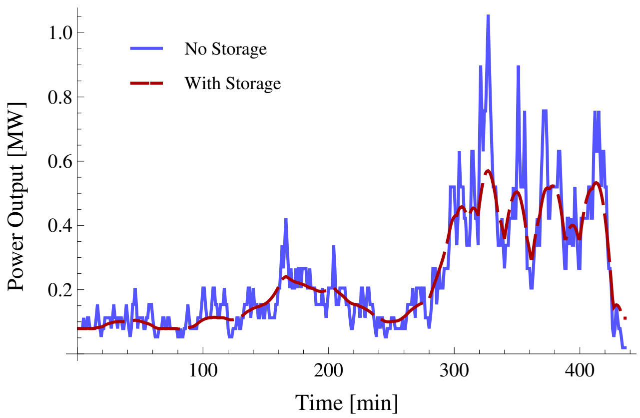

Fig. 1. Variation reduction for a storage system with 33kWh capacity, charging rate of 100kW, 500 charge cycles/month and 92% efficiency. 95% of the wind power was delivered to the grid and 20% variation reduction was achieved.

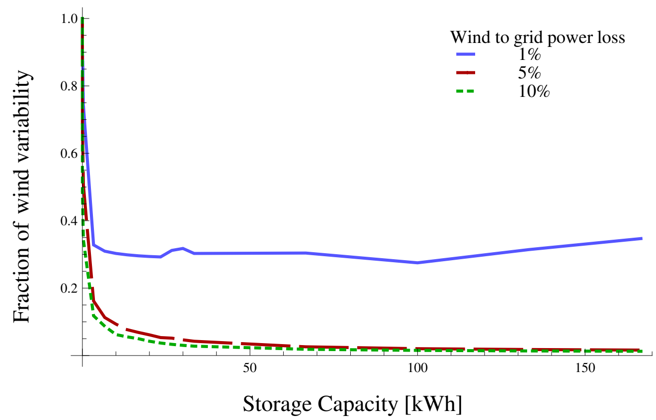

Fig. 2. Diminishing returns of storage capacity with a 92% efficient storage system.

III. SIMULATION RESULTS AND DISCUSSION

Simulations showed which parameters imposed the greatest constraints on the storage system's performance. They also sized the storage system properties required to suppress the fluctuations of a single wind turbine. While the setting of a single wind turbine and storage device is considered uneconomical, the same methods could be used to size storage for a wind farm or agregate of wind farms. A single wind turbine was simulated because of the lack of data on the minute timescale for wind farms. Figure 1 shows a typical comparison of the fluctuations with and without storage. The effectiveness of a storage system was measured by the fraction of wind variability, measured as $\sqrt{\sum_{t=1}^{T}(P_g(t+1)-P_g(t))^2/\sum_{t=1}^{T}(P_w(t+1)-P_w(t))^2}$.

A. Diminishing returns of storage capacity

We first constrained storage capacity independently of the other storage parameters. With no restrictions on the magnitude of charge rate and number of cycles, the effectiveness of the storage alone was measured. Since constraints on rate and lifetime only make the storage system less effective, we found the required energy capacity first and then matched the rate and cycle constraints to that capacity. To avoid trivial solutions where all the power was curtailed, giving a constant power output, the efficiency of power conversion and total wasted power were still constrained in the model via the wind to grid efficiency. . The effectiveness of storage showed a point of diminishing returns, shown in Figure 2. Introducing storage greatly reduced the measure of variability, but after about 33 kWh (one minute of the turbine operating at full power), there was almost no additional variability reduction with more added storage. The amount of required storage does not change with the amount of curtailed or wasted power allowed as long as limits on the wasted power do not interfere with the charging rate and cycling constraint. In the 5%, and 10% curtailment/waste allowed scenarios, enough waste is allowed so that the storage is only being constrained by the storage capacity. However, in the 1% case, the waste constraint becomes the strictest constraint, and does not allow the storage to use its full cycling and charging capacity. This means that the effect of the energy capacity alone is not seen. The imposed charging and cycling constraints lower the storage capacity of diminishing returns but also diminish the overall effectiveness of the storage system. Limiting the charging rate and cycling does not let the storage system fully utilize it's charging capacity.

B. Diminishing returns of charging rate and cycle lifetime

Once the optimal capacity was set, we next constrained the charging rate while allowing unlimited cycling. A point of diminishing returns was found, shown in Figure 3, indicating an optimal charging rate of 200 kW. This charging rate allows for full charge or discharge in 10 minutes.

Cycle lifetime was the last constraint to be sized. Cycling is the strictest constraint on the performance of the system because it determines the total amount of power that can be shifted. Storage systems with a low cycle lifetime burn out and must be replaced, so storage with long cycle lifetime is

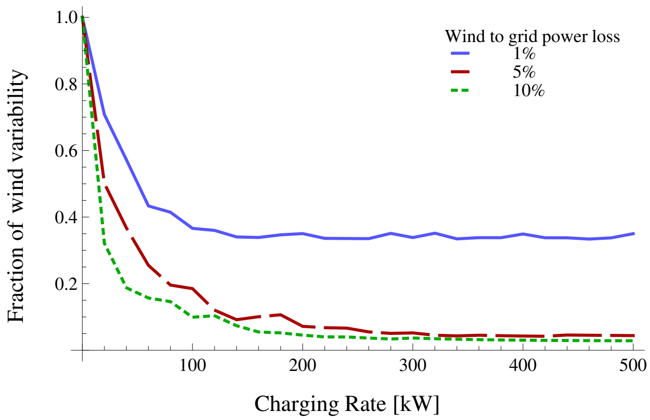

Fig. 3. The minimum required charge/discharge rate of storage is 200kW, or 10% of capacity per minute, when capacity is fixed at 33 kWh. Faster rates are unnecessary as they do not lead to better performance.

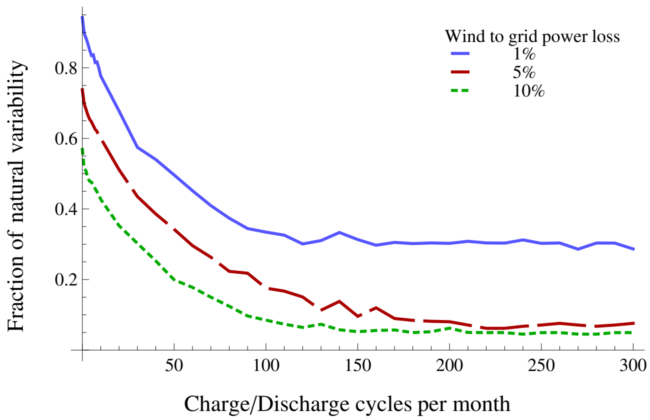

Fig. 4. Cycling shows diminishing returns.

desired. Figure 4 shows 170 cycles per month is sufficiently effective.

C. Efficiency limits for variation reduction

With the capacity set at 33 kWh and the charging rate at 200 kW, and 170 cycles allowed per month, the tradeoffs between storage efficiency and wind to grid conversion are observed in Figure 5. The efficiency of the storage system is critical if a high wind to grid efficiency is required, with 98% efficient systems able to achieve as low as 1% waste with an 85% reduction of variations. However, if wastes are allowed to be greater than 10%, the variation reduction is the same for all efficiencies of conversion to storage. It is impossible to reduce variations to zero even if large amounts of waste are allowed because the limits on energy capacity, charging rate and cycling were imposed.

D. Technology comparison

While this study sought to find the storage requirements necessary for creating an acceptable amount of variation reduction, it is necessary to evaluate technologies that are currently available. Batteries, flywheels and pumped hydro

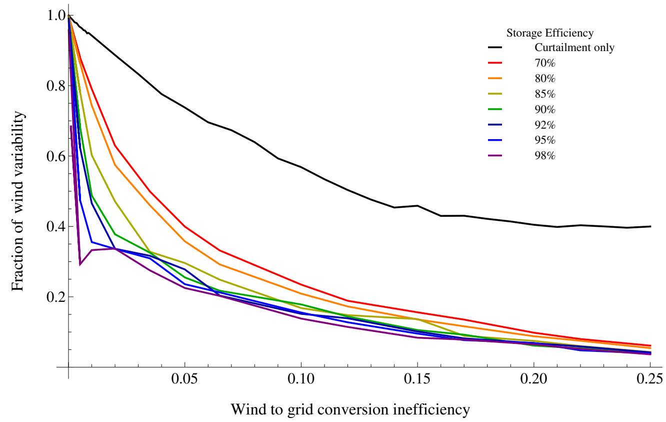

Fig. 5. Storage efficiency and the wind to grid conversion efficiency both determine how much variation reduction is possible.

of the same capacity are considered. All of the technologies are capable of charging at faster than the required 200 kW. Batteries have a slightly higher charging efficiency than flywheels and both have a huge efficiency gain over pumped hydro. More importantly, flywheels and pumped hydro are able to sustain many more cycles per lifetime than batteries. Approximations of the appropriate parameters for each technology were made from [11]. Parameters for the systems are shown in Table III and their effect of the variability is show in Figure 6. When only very small amounts of power are allowed to be wasted, the three technologies perform almost equally and outperform curtailment only. However, because batteries have a greater limitation on how many cycles they can handle, their effectiveness is limited, because they have a stricter cycling constraint in the optimization. Flywheels and pumped hydro have a greater cycle lifetime, allowing more cycles per month than the required amount found in III-B, so their performance is approximately the same as the general storage system shown in Figure 5. Pumped hydro offers the cheapest solution, however, the large scale and geographic requirements of pumped hydro systems may make them unfeasible to use even for large aggregations of wind turbines.

TABLE III

STORAGE PARAMETERS WITH 33KWHR CAPACITY

Parameter	Batteries	Flywheels	Pumped Hydro
Charging rate [MW]	0.1	1	1000
Cycles per lifetime	5000	35000	35000
Efficiency	0.95	0.92	0.78
Cost of cycling	$132k	$162k	$16k

IV. CONCLUSIONS

We have presented a computational methodology for assessment of the role of storage technology characteristics on the effectiveness of wind intermittency mitigation process. The methodology is based on conventional convex optimization algorithms and provides a way of identifying the fundamental limitations of distributed storage systems.

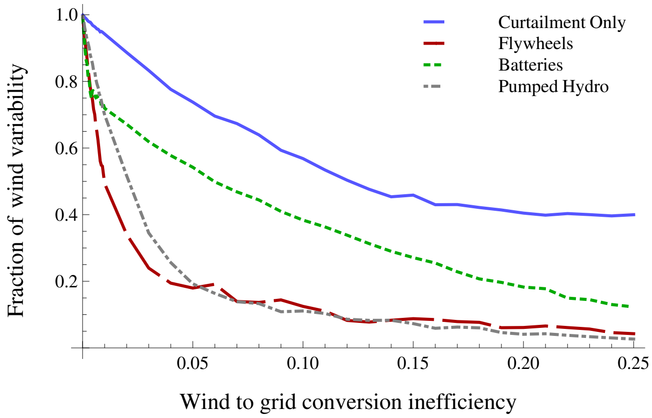

Fig. 6. Variation reduction possible with different storage technologies. The long cycle lifetime of flywheels and pumped hydro explains their good performance.

In this work we have focused on the effectiveness of short term fluctuation suppression that happen on the timescales of order minute.

A relatively small storage capacity is required to smooth the variability of a single wind turbine system, representing only one minute of the maximum rated turbine output. The charging rate does not need to be high either, allowing 10% of the capacity to charge or discharge per minute. With these properties, storage systems are able to achieve substantial reduction, up to 95%, of variability in wind power output. The cycle lifetime of a storage technology is the most important feature for a storage technology - it must be enough absorb and deliver the energy that must be displaced to have a smooth power output.

We have also observed that the regions of non-trivial trade-offs in the parameter space are relatively narrow and have well pronounced optimal conditions at which the maximum variation reduction is achieved. When properties are increased beyond this spot, no appreciable gains in performance are made, so only this minimal value is required to size a storage system. Of current technologies, flywheels look the most promising for suppressing variations. Pumped hydro offers a cheaper solution, however, the large power capacity and typical charge and discharge times of these systems make them more suitable for long term load following.

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