Solvability regions of affinely parameterized quadratic equations

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***Abstract*—Quadratic systems of equations appear in multiple applications. The results in this paper are motivated by quadratic systems of equations that describe equilibrium behavior of physical infrastructure networks like the power and gas grids. The quadratic systems in infrastructure networks are parameterized - the parameters can represent uncertainty or controllable decision variables. It is then of interest to understand conditions on the parameters under which the quadratic system is guaranteed to have a solution within a specified set. Given nominal values of the parameters at which the quadratic system has a solution, we develop a general framework to construct regions around the nominal parameter value such that the system is guaranteed to have a solution within a given distance of the nominal solution. The regions are described by explicit norm-like constraints on the parameters. We compare the results to previous approaches in the context of power systems.**

***Index Terms*—Stability of nonlinear systems, Optimization, Robust control, Uncertain systems**

I. INTRODUCTION

In this paper, we study systems of affinely parameterized quadratic equations. We are interested in the set of parameters for which the system of equations is guaranteed to have a solution. Due to the nonlinearity of system of equations, this set is nonconvex and difficult to characterize exactly. In this paper, we aim to find *tractable sufficient conditions* on the parameters that guarantee existence of solutions.

This question is motivated by systems of equations that describe the equilibrium behavior of infrastructure systems like the power grid or the gas grid. Network operators need to ensure that the equations (that describe the steady-state behavior of the grid) have a solution within acceptable limits (voltage/pressure limits). Thus, it is of interest to find conditions on parameters that guarantee the existence of a solution to the equations within given limits. The conditions can be used for several applications, including robust feasibility assessment (given an uncertainty set for the parameters, assess whether the solution will remain within limits for all values of parameters in the uncertainty set) and robust optimization (choosing values for controllable parameters so as to ensure robust feasibility with respect to all realizations of uncertain parameters).

A. Literature Review

General approaches from algebra and optimization: Methods from algebra and constraint programming [1], [2], [3], [4] study this problem, but typically rely on a discretization of the state space (simplicial or box decomposition) that grows exponentially with the problem dimension. Algorithms developed in the numerical analysis community [5], [6] verify the existence of a solution for a given value of parameters but do not easily extend to computing inner approximations of the set of parameters for which a solution exists (**the solvability set**). In [7] a general framework is proposed to study feasible sets of polynomial systems with quantifiers. However, the approach can only compute *outer* approximations of the solvability set. Further, the framework requires solution of a large semidefinite programs (SDP) and is not suitable for fast computations required in real-time applications.

Our previous work: In previous work [8], we developed a framework based on polynomial optimization to check whether a given set is contained in the solvability set - however, this framework also requires solving a large SDP to perform the check and is hence computationally challenging to scale. The work in [9] studies the problem of solving quadratic systems of equations for a given value of parameters but does not construct an approximation of the solvability set.

Approaches focused on power systems: In [10], [11], the authors propose approaches to develop inner approximations of quadratically constrained quadratic programs arising in power systems. However, their approach cannot handle nonlinear equality constraints (only inequalities) and hence requires that a generator is present at every node in the power grid. [12], [13], [14] construct explicit convex inner approximations of the solvability set under some conditions. However, the results obtained in these papers can be conservative and cannot handle all the parameters in the power flow equations (they typically only account for power injections, all other parameters like line impedances etc. are treated as fixed quantities). Further, they are restricted to power distribution systems and do not extend to power transmission systems with voltage-controlled buses. Necessary and sufficient conditions for solvability of power flow equations have been established in other work [15], [16], [15] - however these papers consider a simplified form of the power flow equations (completely decoupled active and reactive power) and do not handle the fully coupled AC power flow equations.

B. Contributions of this paper

In this paper, we establish a novel framework that has the following advantages:

(1) We construct *explicit inner approximations* of the solvability set that are stated as a condition on the norm of an affine

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function of the parameters. This is computationally simple as opposed to approaches studied in prior work [8], [4].
(2) We handle arbitrary quadratic equations with affine parameters appearing in the equations in the quadratic, linear and constant terms. In the case of the power grid, this means that we can handle transmission systems with voltage-controlled PV buses and uncertainty in line impedance parameters, which goes beyond what is considered in prior work [13], [17], [14], [12], [15], [16], [16].
(3) In special cases (when the variable dependent terms in the equations do not depend on parameters), we can provide tightness guarantees on our inner approximations (i.e, a measure of how far the inner approximations are from the boundary of the solvability set).
The rest of this paper is organized as follows: In section II, we formulate the problem mathematically and introduce the relevant background. In section III, we present our main results on constructing tractable inner approximations of $\mathcal{U}$ and theoretical results on tightness of the inner approximations. In section IV, we compare our approach theoretically and numerically to previous approaches for the case of the AC power flow equations. In section V, we wrap up with conclusions and directions for future work.

II. Background and Mathematical formulation

We begin by introducing notation that will be used in this paper: $\mathbb{R}$: Set of real numbers
$\mathbb{C}$: Set of complex numbers
$\overline{x}$: Complex conjugate of x
$\mathrm{Re}\,(x)$: Real part of complex number x
$\mathrm{Im}\,(x)$: Imaginary part of complex number x
$\|\cdot\|$: This denotes a norm, i.e., any function that satisfies

$$\begin{aligned}&\|x+y\| \leq \|x\|+\|y\| \quad \forall x,y\in\mathbb{R}^n \text{ (or } \mathbb{C}^n)\\&\|\lambda x\| = |\lambda|\,\|x\| \quad \forall \lambda\in R, x\in\mathbb{R}^n \text{ (or } \mathbb{C}^n)\\&\|x\|=0 \iff x=0\end{aligned}$$

$|x|$: For $x\in\mathbb{R}^n$, this denotes a vector with i-th entry $|x_i|$ for each $i\in\{1,\ldots,n\}$
$\|x\|_\infty$: For $x\in\mathbb{R}^n$, this denotes the ∞ norm: $\max_i |x_i|$
$\|M\|_\infty$: For $M\in\mathbb{R}^{n\times n}$, this denotes the induced ∞ norm: $\max_{i=1}^n \sum_{j=1}^n |M_{ij}|$
$\mathrm{Int}\,(S)$: $\{x\in S : \exists\epsilon>0, \{x' : \|x'-x\|_\infty \leq \epsilon\}\subseteq S\}$
We study systems of quadratic equations that depend affinely on parameters u:

$$f(x,u) = \underbrace{Q(x,x,u)}_{\text{Quadratic terms}} + \underbrace{L(x,u)}_{\text{Linear terms}} + \underbrace{K(u)}_{\text{Constant terms}} = 0 \quad (1)$$

where
$Q : \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^k \mapsto \mathbb{R}^n$ is symmetric ($Q(x,y,u) = Q(y,x,u)$) and bi-linear in its first two-arguments and affine in its third argument. It represents the quadratic terms in x.
$L : \mathbb{R}^n\times\mathbb{R}^k\mapsto\mathbb{R}^n$ is linear in its first argument and affine in its second argument. It represents the linear terms in x.
$K : \mathbb{R}^n\mapsto\mathbb{R}^n$ is an affine function and represents the constant terms (with respect to x).

We view (1) as a system of quadratic equations in x parameterized by u. The Jacobian of f is defined as:

$$\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix}\in\mathbb{R}^{n\times n}$$

We use the notation $\left.\frac{\partial f}{\partial x}\right|_{x,u}$ to denote the Jacobian evaluated at the point (x,u). Let $(x^\star,u^\star)$ satisfy (1). Define:

$$J(x,u) = \left.\frac{\partial f}{\partial x}\right|_{x,u}, J_\star(u) = \left.\frac{\partial f}{\partial x}\right|_{x^\star}, J_{\star\star} = \left.\frac{\partial f}{\partial x}\right|_{x^\star,u^\star} \quad (2)$$

We now consider the specific cases of the abstract model (1):
Power flow: The AC power flow equations characterize the steady state of the power grid and can be written as

$$\mathrm{Re}\left(\sum_{k=1}^n V_i(\overline{Y_{ik}V_k}+\overline{Y_{i0}V_0})\right) = p_i \quad \forall i\in\mathrm{PQ} \quad (3a)$$

$$\mathrm{Im}\left(\sum_{k=1}^n V_i(\overline{Y_{ik}V_k}+\overline{Y_{i0}V_0})\right) = q_i \quad \forall i\in\mathrm{PQ} \quad (3b)$$

$$\mathrm{Re}\left(\sum_{k=1}^n V_i(\overline{Y_{ik}V_k}+\overline{Y_{i0}V_0})\right) = p_i \quad \forall i\in\mathrm{PV} \quad (3c)$$

$$|V_i|^2 = v_i, i\in\mathrm{PV} \quad \forall i\in\mathrm{PV} \quad (3d)$$

where V_i denotes complex voltage phasor at node i, p_i,q_i denote the active and reactive power injection at node i, Y denotes the admittance matrix, PV denotes the set of PV nodes (typically generators), PQ denotes the set of PQ buses and v_i denotes the squared voltage magnitude setpoints at the PV buses. We can rewrite (3) in the form (1) with

$$\begin{aligned}&x = \begin{pmatrix}\mathrm{Re}\,(V_1) & \ldots & \mathrm{Re}\,(V_n) & \mathrm{Im}\,(V_1) & \ldots & \mathrm{Im}\,(V_n)\end{pmatrix}\\&u = \begin{pmatrix}p_1\ldots p_n & q_1\ldots q_n & v_1\ldots v_n\end{pmatrix}\end{aligned}$$

Y can also be added to u as well if the admittance parameters also also uncertain. We seek conditions on u that guarantee existence of a solution x to to (3).
Gas flow: The steady-state gas-flow equations can be written as follows [18]:

$$q_i = \sum_k \phi_{ik}, \psi_{ik}^2 = \phi_{ik}^2, \psi\geq 0, \pi\geq 0, \quad (4a)$$

$$\alpha_{ik}\left(\pi_i - r_{ij}\lambda_{ij}\psi_{ij}\right) = \pi_j + (1-r_{ij})\lambda_{ij}\phi_{ij}\psi_{ij} \quad (4b)$$

where q_i denotes the amount of gas produced or consumed at node i, π_i the pressure at node i, ϕ_{ik} the flow of gas on pipeline (i,k), $\psi=|\phi|$ the absolute value of flow, r_{ij},λ_{ij} are parameters of the pipeline and α denotes the compressor gain of a compressor on the pipeline. This can also be put into the form (1) with $x=(\pi,\phi,\psi)$ and $u=(q,\alpha)$.

A. Assumptions and problem definition

In this section, we state and discuss the assumptions we make in the problem formulation:
Existence of a nominal solution: We will assume that a *nominal solution* $x^\star, u^\star$ satisfying (1) is known. This is motivated

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by applications in infrastructure systems like the power grid and gas grid where $u^\star$ represents a forecast for uncertain parameters (like solar or wind generation) and $x^\star$ denotes the state of the grid computed based on the forecasted $u^\star$. However, since $u^\star$ is simply a forecast, the predicted state $x^\star$ can deviate from the actual state of the grid and hence it is of interest to ensure that the predicted state $x^\star$ is a accurate to within a certain error bound (given an error bound on the deviation of u from $u^\star$). A similar assumption has been made in recent papers appearing in the power systems literature [13], [14]. However, if a nominal solution is not available, one can simply choose any value of $x^\star$ (say $x^\star = 1$) and solve the linear system $f(x^\star, u) = 0$ for u to obtain $u^\star$.

Constraints on x: We are interested in finding a solution within some constraints on x (for example voltage upper and lower limits in the power grid, or pressure limits in the gas grid). In this paper, we assume that the constraint on x can be stated as a norm constraint $\|x - x^\star\| \leq r$. Note that this is not a very limiting assumption and we can model any constraint in x of the form $|\beta(x) - \Theta| \leq r$ where $\beta : \mathbb{R}^n \mapsto \mathbb{R}^p$ is a quadratic vector valued function of x and $\Theta \in \mathbb{R}^p$ (for example, limits on voltage magnitudes and flows in the power grid are of this form). We assume that the set $\{x : |\beta(x) - \Theta| \leq r\}$ is bounded, i.e., $|\beta(x) - \Theta| \leq r \implies \|x\| \leq r'$ for some $r' > 0$. Then, we can introduce additional variables $y \in \mathbb{R}^p$ and define an augmented set of variables, norm and system of equations as follows: $\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \tilde{f}(\tilde{x}, u) = \begin{pmatrix} f(x,u) \\ y - \beta(x) + \Theta \end{pmatrix}$, $\|\tilde{x}\| = \max\left(\frac{\|x\|_\infty}{r'}, \frac{\|y\|_\infty}{r}\right)$. Solving $\tilde{f}(\tilde{x}, u) = 0, \|\tilde{x}\| \leq 1$ is equivalent to solving $f(x,u) = 0, |\beta(x) - \Theta| \leq r$ and hence the quadratic constraint $|\beta(x) - \Theta| \leq r$ can be modeled in our framework (with $\tilde{x}^\star = 0$).

Non-singularity: We will assume that the Jacobian of the system $\frac{\partial f}{\partial x}$ is non-singular at the nominal solution $(x^\star, u^\star)$. If this assumption is invalid, there exist infinitesimally small perturbations to u from $u^\star$ such that *no solution exists to* $f(x,u) = 0$ [2] (the intuition for this is that since f is well-approximated by the Jacobian close to $x^\star$, choosing an infinitesimal perturbation to u that is not in the range space of the Jacobian will lead to an infeasible linear system). Thus, it is impossible to construct a ball around $u^\star$ that is contained within the solvability set.

Problem Statement: Given $(x^\star, u^\star)$ satisfying (1) such that $\left.\frac{\partial f}{\partial x}\right|_{x^\star, u^\star}$ is non-singular, a norm $\|\cdot\|$ and a number $r > 0$, construct a region $\mathcal{U}_r \subseteq \mathbb{R}^k$ such that $u^\star \in \mathrm{Int}(\mathcal{U}_r)$ and

$$\mathcal{U}_r \subseteq \mathbf{U}_r = \{u : (\exists x : \|x - x^\star\| \leq r, f(x,u) = 0)\}.$$

III. Inner approximations of the solvability region

In this section, we state our main theoretical results characterizing inner approximations of $\mathbf{U}_r$. We then discuss tightness of these inner approximations (how far they are from the boundary of $\mathbf{U}_r$) and how they relate to previous results in the literature.

Lemma 1. *Suppose $(x^\star, u^\star)$ satisfy (1) and that $J_{\star\star}$ is invertible. Define the following quantities:*

$$e(u) = \left\|(J_{\star\star})^{-1}\left(f(x^\star, u) - f(x^\star, u^\star)\right)\right\|,$$
$$h(u) = \max_{\|y\| \leq 1} \left\|(J_{\star\star})^{-1} Q(y,y,u)\right\|,$$
$$g(u) = \left\|(J_{\star\star})^{-1}\left(J_\star(u) - J_{\star\star}\right)\right\|.$$

[1] *Then for any $r > 0$, (1) has a solution in the set $\{x : \|x - x^\star\| \leq r\}$ if*

$$rh(u) + g(u) + \frac{1}{r}e(u) \leq 1. \tag{5}$$

Proof. See Appendix section VI-A. □

a) Interpretation of condition (5): $h(u)$ is a measure of how "nonlinear" the system is, since it bounds the *gain of the quadratic term*, or the maximum norm of $Q(y,y,u)$ given inputs y within the unit ball $\|y\| \leq 1$. Thus, the more nonlinear the system is, the more difficult it is to satisfy (5).

$f(x,u) - f(x^\star, u) \approx J_\star(u)(x - x^\star)$. $g(u)$ bounds the difference in this linear approximation as we move from $u^\star$ to u and hence bounds the change in the linear approximation of the equations around $x^\star$. As the linear approximation changes more, $g(u)$ increases and it becomes harder to satisfy (5).

$f(x^\star, u) - f(x^\star, u^\star) = f(x^\star, u)$ is a measure of how close $x^\star$ is to being a solution to the equations for u - as $f(x^\star, u)$ gets larger, $x^\star$ is further away from being a solution and $e(u)$ controls this deviation. As $x^\star$ gets further away from being a solution, $e(u)$ grows making it harder to satisfy (5).

b) Dependence of (5) on r: The condition (5) is non-monotonic in r, which is un-natural since as the constraints on x get looser (as r increases), $\mathbf{U}_r$ should get larger. This is an artifact of our analysis and is resolved in Theorem 1.

Theorem 1. *Define*

$$H(u) = \begin{cases} 2\sqrt{h(u)e(u)} & \text{if } e(u) \leq r^2 h(u) \\ rh(u) + \frac{e(u)}{r} & \text{otherwise} \end{cases} \tag{6a}$$
$$\mathcal{U}_r = \{u : H(u) + g(u) \leq 1\} \tag{6b}$$

Then for each $u \in \mathcal{U}_r$, there is a solution to (1) with $\|x - x^\star\| \leq r$. Further, $u^\star \in \mathrm{Int}(\mathcal{U}_r)$ and $\mathcal{U}_r$ increases monotonically with r, i.e., $0 < r < r' \implies \mathcal{U}_r \subseteq \mathcal{U}_{r'}$ Further, as $r \to \infty$, we obtain:

$$\mathcal{U}_\infty = \left\{u : 2\sqrt{h(u)e(u)} + g(u) \leq 1\right\} \tag{7}$$

and for each $u \in \mathcal{U}_\infty$ there exists a solution to (1).

Proof. See Appendix section VI-B. □

c) Nature of constraint (6b): For many applications, it is convenient to have a *convex inner approximation* of the solvability set since one can do optimization and projection onto a convex set efficiently [19]. Since e, g, h are all norms (or maxima over a set of norms), the constraint (5) is clearly a convex constraint on u [19]. However, the constraint (6b) is

[1] Note that $f(x^\star, u^\star) = 0$ by definition, however we write it explicitly in the definition of $e(u)$ so as to highlight the fact that $e(u)$ is a measure of deviation of u from $u^\star$.

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not necessarily convex. We study the convexity of the set $\mathcal{U}_r$ in the following propositions.

Proposition 1. *If Q, L are independent of u, $\mathcal{U}_r$ is a convex set.*

Proof. If Q, L are independent of u, then so are g and h. In fact, $g = 0$ in this case since J is independent of u as well and we denote the constant value of $h(u)$ by $h^\star$. In this case,

$$H(u) = \begin{cases} 2\sqrt{e(u)\, h^\star} & \text{if } e(u) \leq r^2 h^\star \\ rh^\star + \frac{1}{r} e(u) & \text{otherwise} \end{cases}$$

and the condition for solvability is $H(u) \leq 1$. Thus, the condition for solvability boils down to

$$2\sqrt{e(u)\, h^\star} \leq 1, e(u) \leq r^2 h^\star \iff e(u) \leq \min\left(\frac{1}{4h^\star}, r^2 h^\star\right)$$

OR

$$rh^\star + \frac{1}{r} e(u) \leq 1 \iff e(u) \leq r(1 - rh^\star).$$

Thus, in this case, $\mathcal{U}_r$ can be rewritten as

$$\left\{u : e(u) \leq \max\left(r(1 - rh^\star), \min\left(\frac{1}{4h^\star}, r^2 h^\star\right)\right)\right\}$$

which is a convex set. □

In the general case, it is impossible to express the condition (6b) as a convex set. However, one can express it as a union of convex sets:

Proposition 2. *The set $\mathcal{U}_r$ can be written exactly as an infinite union of convex sets and approximated from the inside as a finite union of convex sets.*

Proof. Define $\mathcal{S}_r(\delta) = \left\{u : r\delta h(u) + g(u) + \frac{e(u)}{r\delta} \leq 1\right\}$. which is a convex set for any value of $\delta \in (0, 1]$. Taking the union of these sets, define $\mathcal{S}_r = \cup_{\delta \in (0,1]} \mathcal{S}_r(\delta)$. It can be shown that $\mathcal{U}_r = \mathcal{S}_r$ (see proof of Theorem 1). Such an infinite union is not a computational tractable object. However, one can discretize the interval $(0, 1]$ using a step-size $t \in (0, 1)$ to obtain the finite set $S_t = \{t, 2t, \ldots, \lfloor \frac{1}{t} \rfloor t\}$. Then, we can use $\mathcal{U}_r^t = \cup_{\delta \in S_t} \mathcal{S}_r(\delta) \subseteq \mathcal{S}_r$ as an inner approximation of $\mathcal{S}_r$ expressed as the union of a finite number of convex sets. As t takes smaller values, this approximation will become more accurate but more complex (union of a larger number of convex sets). □

A. Tightness of the certificate

In this section, we analyze the tightness of Theorem 1 for the special case where Q, L do not depend on u:

$$Q(x, x) + Lx + K(u) = 0. \tag{8}$$

Theorem 2. *Let $(x^\star, u^\star)$ satisfy (8). Define e, g, h as in Theorem 1 and let $h^\star = \max_{\|y\| \leq 1} \left\|(J_{\star\star})^{-1} Q(y, y)\right\|$. For any $r \in (0, \frac{1}{h^\star})$, the system of equations (8) has a solution in the set $\mathcal{B}_r = \{x : \|x - x^\star\| \leq r\}$ if u belongs to the set*

$$\mathcal{U}_r = \left\{u : e(u) \leq \max\left(r(1 - rh^\star), \min\left(\frac{1}{4h^\star}, r^2 h^\star\right)\right)\right\}. \tag{9}$$

Conversely, if u lies outside the set

$$\hat{\mathcal{U}}_r = \{u : e(u) \leq r(1 + rh^\star)\} \tag{10}$$

then (8) does not have a solution in $\mathcal{B}_r$. Thus, $\mathcal{U}_r \subseteq \mathbf{U}_r \subseteq \hat{\mathcal{U}}_r$.

Proof. See Appendix section VI-C □

This result shows that the inner approximation $\mathcal{U}_r$ is tight upto a factor, in the sense that for any u outside $\hat{\mathcal{U}}_r$ (which is simply $\mathcal{U}_r$ scaled by a factor of at most $\frac{1+rh^\star}{1-rh^\star}$) cannot have a solution in $\mathcal{B}_r$. Thus if $r << \frac{1}{h^\star}$ (i.e, the allowed deviation in the solution is small relative to the inverse nonlinearity of the system), the certificate is nearly tight.

B. Computational issues

In this section, we discuss the computation of e, h, g. $e(u)$ is easy to compute since it simply involves taking the norm of an affine function of u. $g(u)$ involves computing induced matrix norm. $h(u)$ involves maximizing the norm of a nonlinear and possibly nonconvex function of x and can be difficult to compute in general. However, in order to apply the certificate (7), a bound on $h(u)$ suffices and this can be computed using convex relaxation techniques. A detailed discussion is beyond the scope of this paper.

IV. Application to Power Systems

In this section, we study the application of results to power systems in detail.

A. Theoretical relationship to previous results

We consider the special case of the AC power flow equations (3) and compare our result to the best preivously-known result on inner approximations [13]. We start with equation (4) from [13]: $V = w + Z\left(\text{diag}\left(\overline{V}\right)\right)^{-1}\overline{s}$ with $s = p + \sqrt{-1}q$(this is simply a restatement of the power flow equations (3) introduced earlier for the case where all the nodes in the network are PQ buses). We rewrite this system as a quadratic system in $\gamma = \frac{w}{V}$ (we assume implicitly that we are not interested in solutions with 0 voltage since this is not practical in a real power system):

$$f(\gamma, s) = [\![\gamma]\!] ([\![w]\!])^{-1} Z\left(\overline{[\![w]\!]}\right)^{-1} [\![\overline{s}]\!]\, \overline{\gamma} + \gamma - \mathbf{1} = 0. \tag{11}$$

where $\mathbf{1}$ is a vector with all entries equal to 1 and $[\![w]\!]$ is a diagonal matrix with entries w. This is in the form studied in the complex extension of Theorem 1 (Theorem 4 in Appendix section VI-D). We simply state our final result on the solvability region computed by applying Theorem 4 to (11) with the nominal solution $\gamma^\star = \mathbf{1}, s^\star = 0$. Define $\zeta(s) = ([\![w]\!])^{-1} Z\left(\overline{[\![w]\!]}\right)^{-1} [\![\overline{s}]\!]$. Then, (11) has a solution if

$$2\|\zeta(s)\, \mathbf{1}\|_\infty + 2\sqrt{\|\zeta(s)\, \mathbf{1}\|_\infty \|\zeta(s)\|_\infty} \leq 1. \tag{12}$$

A complete derivation of this result and detailed applications of this result to power systems can be found in [20]. If we use

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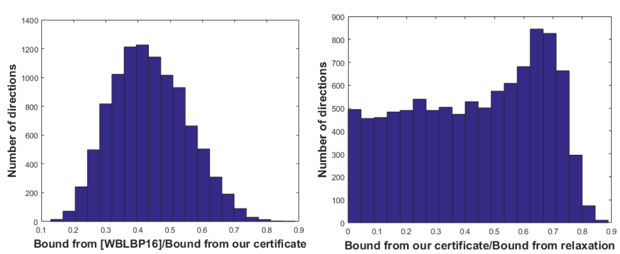
the bound $\|\zeta(s)\mathbf{1}\|_\infty \leq \|\zeta(s)\|_\infty$, this condition is implied by the condition

$$\|\zeta(s)\|_\infty \leq \frac{1}{4} \tag{13}$$

which is the region defined by corollary 1 in [13]. Thus our analysis produces a stronger result than that from [13] for $s^\star = 0, \gamma^\star = \mathbf{1}$. Another advantage of our framework in this case is that we can obtain a region around any nominal solution $(\gamma^\star, s^\star)$ with nonsingular Jacobian, while [13] requires a stronger assumption.

B. Numerical results

In this section, we perform numerical studies applying the techniques from this paper to the AC power flow equations. We choose a random configuration of injections $\hat{s}$ [2] and find the largest scaling factor t such that $t\hat{s}$ satisfies (12) (i.e., is in our inner approximation of the solvability set) and (13) (i.e., is in the inner approximation constructed in [13]) - we denote these $t_{\hat{s}}, t'_{\hat{s}}$ respectively. We also compute the smallest value of t such that there are provably no solutions to the power flow equations for injections $t\hat{s}$ - we denote this value $t^\star_{\hat{s}}$. Thus, we have $t'_{\hat{s}} \leq t_{\hat{s}} \leq t^\star_{\hat{s}}$. $t'_{\hat{s}}/t_{\hat{s}}$ can be used to compare our results



(a) Comparison of certificate (12) and [13] (histogram of $t'_{\hat{s}}/t_{\hat{s}}$)

(b) Comparison of certificate (12) and relaxation (histogram of $t_{\hat{s}}/t^\star_{\hat{s}}$)

with those from [13] (figure 1a) - if this ratio is much smaller than 1, it means that we improve upon [13] significantly. $t_{\hat{s}}/t^\star_{\hat{s}}$ can be used to compare our results with (an outer approximation of) the true solvability set(figure 1b). If this ratio is close to 1, we are nearly tight (close to the boundary of the solvability set). The plots show that along most directions, we improve upon [13] by a factor of at least 2 and that there are directions along which where are nearly touching the boundary of the solvability set. There are, however, directions along which our region is away from the boundary of the solvability set. By choosing the appropriate norm, the approximation can be made tight in specific directions.

V. Conclusions

We have developed a general framework for computing inner approximations of the solvability regions of affinely parameterized quadratic systems of equations. Choosing the right norm is critical for constructing large inner approximations of the solvability region - investigating how to make this choice is a direction of future work. We will also explore applications of the regions to checking robust feasibility (with respect to uncertain parameters) and robust optimization (over controllable parameters). Generalizing the results of this paper to affinely parameterized nonlinear equations (beyond quadratic) and more general constraint sets on x (beyond norm balls) is also an important and interesting direction for further work.

[2] These are chosen uniformly random from the set of all unit vectors so as to capture uncertainty both in loads and in distributed renewable generation like rooftop solar.

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VI. Appendix

Theorem 3 (Brouwer's fixed-point Theorem)**.** *Let $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous mapping and $S \subset \mathbb{R}^n$ be a compact convex set. Then, if $F(S) \subseteq S$, F has a fixed-point in S.*

A. Proof of Lemma 1

Proof. Let $\theta = f(x^\star, u)$, $dx = x - x^\star$, $\Delta J(u) = J_\star(u) - J_{\star\star}$ and $M = (J_{\star\star})^{-1}$. Adding and subtracting θ from the LHS of the equation $f(x,u) = 0$, we obtain

$$Q(x,x,u) - Q(x^\star, x^\star, u) + L(dx, u) + \theta = 0.$$

Expanding $Q(x,x,u) = Q(x^\star + dx, x^\star + dx, u)$ using the tri-linearity of Q, we obtain

$$\begin{aligned} &2Q(x^\star, dx, u) + Q(dx, dx, u) + L(dx, u) + \theta = 0 \\ &= Q(dx, dx, u) + J_\star(u)\, dx + \theta \\ &= Q(dx, dx, u) + \Delta J(u)\, dx + \theta + J_{\star\star} dx. \end{aligned}$$

Rescaling the above equation by M, we define

$$G(dx) = -M\left(\Delta J(u)\, dx + Q(dx, dx, u) + \theta\right)$$

so that $f(x,u) = 0 \iff G(dx) = dx$. If $\|dx\| \leq r$,

$$\begin{aligned} &\|G(dx)\| \leq \|MQ(dx, dx, u)\| + \|M\Delta J(u)\, dx\| + \|M\theta\| \leq \\ &\max_{\|dx\| \leq r} \|MQ(dx, dx, u)\| + \max_{\|dx\| \leq r} \|M\Delta J(u)\, dx\| + \|M\theta\| \\ &= r^2 h(u) + r g(u) + e(u) \end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from the fact that $\|dx\| \leq r$ and the third equality follow from the definition of g, h, e and the fact that the terms inside the norm are homogeneous (of order 2, 1 and 0) in dx and the definition of the matrix norm $\|M\Delta J(u)\| = \max_{\|dx\| \leq 1} \|M\Delta J(u)\, dx\|$.
If $r^2 h(u) + r g(u) + e(u) \leq r$, we have that G maps the set $\{dx : \|dx\| \leq r\}$ onto itself. Thus, Brouwer's Theorem (Theorem 3) guarantees existence of a solution to $G(dx) = dx$ (or equivalently to $f(x,u) = 0$) with $\|dx\| = \|x - x^\star\| \leq r$. Dividing both sides by r, we obtain $rh(u) + g(u) + \frac{1}{r} e(u) \leq 1$. This is precisely the condition defined in Lemma 1 and completes the proof of Lemma 1. □

B. Proof of Theorem 1

Proof. Since we are required to find a solution with $\|dx\| \leq r$, it also suffices to find a solution with $\|dx\| \leq r\delta$ for some $\delta \in (0,1]$. Define $\mathcal{S}_r(\delta)$ as in Proposition 2. By Lemma 1, for every $\delta \in (0,1]$, every $u \in \mathcal{S}_r(\delta)$ is guaranteed to have a solution with $\|dx\| \leq r\delta$ and hence with $\|dx\| \leq r$. We can take the union of these regions to obtain $\mathcal{S}_r = \cup_{0<\delta\leq 1} \mathcal{S}_r(\delta)$. This infinite union can also be characterized in a finite fashion: $u \in \mathcal{S}_r$ if and only if $\exists \delta \in (0,1] : \delta r h(u) + g(u) + \frac{e(u)}{\delta r} \leq 1$. We can choose the value of δ to minimize the LHS and $u \notin \mathcal{S}_r$ if and only if the inequality is not satisfied for this optimal choice of δ. Since the LHS of the inequality is a strongly convex function over the set of positive δ and bounded below by 0, it has a unique optimal value. The derivative of the LHS wrt δ is $rh(u) - \frac{e(u)}{r\delta^2}$ so that the function is decreasing on the interval

$$\delta \in \left(0, \frac{1}{r}\sqrt{\frac{e(u)}{h(u)}}\right).$$

Hence, if $\frac{1}{r}\sqrt{\frac{e(u)}{h(u)}} \leq 1$, the optimal value of δ is $\frac{1}{r}\sqrt{\frac{e(u)}{h(u)}}$. Otherwise, the optimal value of δ is 1. Thus, $\mathcal{S}_r$ is equal to

$$\left\{ u : \begin{cases} 2\sqrt{h(u)\, g(u)} + g(u) \leq 1 \text{ if } e(u) \leq r^2 h(u) \\ rh(u) + \frac{e(u)}{r} + g(u) \leq 1 \text{ otherwise} \end{cases} \right\}$$

which is equal to the set $\mathcal{U}_r = \{u : H(u) + g(u) \leq 1\}$. Since $\mathcal{U}_r = \mathcal{S}_r = \cup_{0<\delta\leq 1} \mathcal{S}_r(\delta)$, $\mathcal{U}_r$ is obviously monotone in r: $0 < r < r' \implies \mathcal{U}_r \leq \mathcal{U}_{r'}$. Further, we have that $H(u^\star) = 0 = g(u^\star)$, so that $\exists \epsilon > 0$ such that $H(u) + g(u) \leq 1 \quad \forall u : \|u - u^\star\|_\infty \leq \epsilon$ so that $u^\star \in \mathrm{Int}(\mathcal{U}_r)$. □

C. Proof of Theorem 2

Proof. From proposition 1 , the condition for existence of a solution in $\mathcal{B}_r$ from theorem 1 reduces to $e(u) \leq \max\left(r(1 - rh^\star), \min\left(\frac{1}{4h^\star}, r^2 h^\star\right)\right)$. Conversely, suppose that $e(u) > r(1 + rh^\star)$ and that (8) has a solution in $\mathcal{B}_r$. Using the derivation of Lemma 1, we arrive at $-MQ(dx, dx) - M\theta = dx$ (since $\Delta J = 0$ in this case as only the constant term depends on 0 and hence the Jacobian is independent of u). Taking norms and applying the triangle equality, we get $\|dx\| + \|MQ(dx, dx)\| \geq e(u)$. If $\|dx\| \leq r$, then we know that $\|MQ(dx, dx)\| \leq h^\star r^2$. Thus, we get $r(1 + rh^\star) \geq e(u)$ which is a contradiction to our assumption. Thus, if $u \in \hat{\mathcal{U}}_r$, no solutions exist in $\mathcal{B}_r$. □

D. Complex equations

Let $x \in \mathbb{C}^n, u \in \mathbb{C}^k$ and consider the system

$$f(x,u) = Q(x, \overline{x}, u) + L(x,u) + K(u) = 0 \tag{14}$$

where $Q : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{C}^n, L : \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{C}^n, K : \mathbb{C}^n \mapsto \mathbb{C}^n$ are complex-valued trilinear, bilinear and linear functions. Define

$$J(x,u;y) = \left.\frac{\partial f}{\partial x}\right|_{x,u} y + \left.\frac{\partial f}{\partial \overline{x}}\right|_{x,u} \overline{y}, J_\star(u;y) = J(x^\star, u; y)$$

$$J_{\star\star} = \begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{x^\star,u^\star} & \left.\frac{\partial f}{\partial \overline{x}}\right|_{x^\star,u^\star} \\ \overline{\left.\frac{\partial f}{\partial \overline{x}}\right|_{x^\star,u^\star}} & \overline{\left.\frac{\partial f}{\partial x}\right|_{x^\star,u^\star}} \end{pmatrix}, (J_{\star\star})^{-1} = \begin{pmatrix} M_\star & N_\star \\ \overline{N_\star} & \overline{M_\star} \end{pmatrix}$$

Theorem 4. *Let $(x^\star, u^\star)$ satisfy* (14) *and suppose that $J_{\star\star}$ be invertible. Define the following quantities:*

$$\begin{aligned} e(u) &= \left\| M_\star f(x^\star, u) + N_\star \overline{f(x^\star, u)} \right\|, \\ h(u) &= \max_{\|y\| \leq 1} \left\| M_\star Q(y, \overline{y}, u) + N_\star \overline{Q(y, \overline{y}, u)} \right\|, \\ g(u) &= \max_{\|y\| \leq 1} \left\| M_\star J_\star(u;y) + N_\star \overline{J_\star(u;y)} - y \right\|. \end{aligned}$$

Define the sets $\mathcal{U}_r, \mathcal{U}_\infty$ as in Theorem 1. Then, the conclusions of Theorem 1 are true.

Proof. Similar to Theorem 1. □