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Polymer statistics in a random flow with mean shear

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We consider the dynamics of a polymer with finite extensibility placed in a chaotic flow with large mean shear, to explain how the polymer statistics changes with Weissenberg number, Wi, the product of the polymer relaxation time and the Lyapunov exponent of the flow, $\bar{\lambda}$. The probability distribution function (PDF) of the polymer orientation is peaked around a shear-preferred direction, having algebraic tails. The PDF of the tumbling time (separating two subsequent flips), τ, has a maximum estimated as $\bar{\lambda}^{-1}$. This PDF shows an exponential tail for large τ and a small-τ tail determined by the simultaneous statistics of the velocity PDF. Four regimes of Wi are identified for the extension statistics: one below the coil–stretched transition and three above the coil–stretched transition. Emphasis is given to explaining these regimes in terms of the polymer dynamics.

1. Introduction

A number of experimental observations resolving the dynamics of individual polymers (DNA molecules) in a permanent shear flow have been reported by Smith, Babcock & Chu (1999), see also Hur *et al.* (2001). These results and the subsequent theoretical/numerical study of Hur, Shaqfeh & Larson (2000) have focused on the analysis of the power spectral density and simultaneous probability distribution function (PDF) of the polymer extension in the permanent shear, with fluctuations driven by thermal noise. Statistics of polymers and carbon nanotubes in shear flows have been also investigated by Lee, Solomon & Muller (1997) and Hobbie *et al.* (2003). In another experimental development by Groisman & Steinberg (2000, 2001, 2004), a chaotic flow state called by the authors “elastic turbulence” was observed in dilute polymer solutions. This flow consists of regular (shear-like) and chaotic components, the latter being weaker. Resolving an individual polymer in this chaotic steady flow was achieved by Gerashchenko, Chevallard & Steinberg (2004). The coil-stretch transition, predicted by Lumley (1969, 1973) (see also Balkovsky, Fouxon & Lebedev 2000 and Chertkov 2000), was observed in direct single-polymer measurements by Gerashchenko *et al.* (2004).

Here, we discuss the statistics of polymers in a chaotic flow with a relatively large mean shear, which corresponds to the elastic turbulence experiments by Groisman & Steinberg (2000, 2001, 2004). We assume that the effect of velocity fluctuations is stronger than that related to thermal noise, and that polymers are essentially elongated so that the polymer orientation is well defined. Most of the orientational fluctuations occur close to a special direction preferred by the shear. Sometimes the typical fluctuations are interrupted by flips, in which the polymer orientation is reversed. We describe the statistics of the angular orientation and tumbling time, and of the polymer extension. We establish the main features of the PDFs.

We begin by introducing the basic dumb-bell-like equation governing the dynamics of the polymer end-to-end vector in a non-homogeneous flow. If the effect of thermal fluctuations is negligible, the angular part of the polymer dynamics decouples from its extensional counterpart and can be examined separately. Angles ϕ and θ (for in-plane and off-plane orientations, respectively) are counted from the direction prescribed by the shear. We show that the PDF of ϕ is peaked at some small value ϕ_t determined by the velocity fluctuations. The widths (for both angles) of the main part of the angular PDF are also estimated by ϕ_t. We demonstrate that the tails of the joint PDF are algebraic. Then we examine the PDF of the tumbling time, τ, defined as the time between two subsequent flips of the polymer. The PDF of τ, $P(\tau)$, is peaked at a time estimated as $\bar{\lambda}^{-1}$ where $\bar{\lambda}$ is the Lyapunov exponent of the flow. The long-time, $\tau \gg \bar{\lambda}^{-1}$, tail of the PDF is exponential, $\ln[P(\tau)] \sim -\bar{\lambda}\tau$. The statistics of small tumbling times is related to the simultaneous PDF of the velocity gradients. To derive these results we explore the close relation between the angular dynamics and the Lagrangian dynamics of the flow. Then we formulate the basic stochastic equation governing the dynamics of the polymer extension and analyse the structure of the extension PDF which shows a strong dependence on the Weissenberg number, Wi, defined as the product of the polymer relaxation time and $\bar{\lambda}$. We consider four cases corresponding to qualitatively different PDF behaviours. We explain how the typical extension R depends on Wi and examine the tails of the extension PDF, for R less than and larger than its typical value. The structure of the tails is complicated, consisting in some cases of a number of asymptotic sub-intervals. We explain a dynamical origin of all the sub-intervals. To illustrate our generic analytical results, we present graphs, corresponding to the four different regimes, obtained by direct numerical simulation made within a model of short-correlated velocity statistics and of the so-called FENE-P modelling. We conclude by discussing the applicability conditions for our approach and the validity of the assumptions made.

2. Model

We consider a single polymer molecule advected by a chaotic/turbulent flow (i.e. the polymer moves along a Lagrangian trajectory of the flow) and stretched by velocity inhomogeneity. The polymer stretching is characterized by the molecule's end-to-end separation vector, $\boldsymbol{R}$, satisfying the following dumb-bell-like equation (see e.g. Hinch 1977; Bird *et al.* 1987):

$$\partial_t R_i = R_j \nabla_j v_i - \gamma(R) R_i + \zeta_i. \tag{2.1}$$

Here γ is the polymer relaxation rate and ζ_i is the Langevin force. The velocity gradient $\nabla_j v_i$ is taken at the molecule position. The velocity difference between the polymer end points is approximated in (2.1) by the first term of its Taylor expansion in the end-to-end vector. This is justified if the polymer size is less than the velocity correlation length. The relaxation rate γ in (2.1) is a function of the extension R which varies from zero up to a maximum value $R_{\max}$ corresponding to a fully stretched polymer. We assume that the relaxation is Hookean for $R \ll R_{\max}$, i.e. $\gamma(R)$ is approximated well by a constant $\gamma(0)$ there, while it diverges (the polymer becomes stiff) for $R \to R_{\max}$.

We focus on the situation in which the effect of velocity fluctuations is stronger than that of thermal fluctuations, so that the Langevin force $\boldsymbol{\zeta}$ in (2.1) can be neglected. We

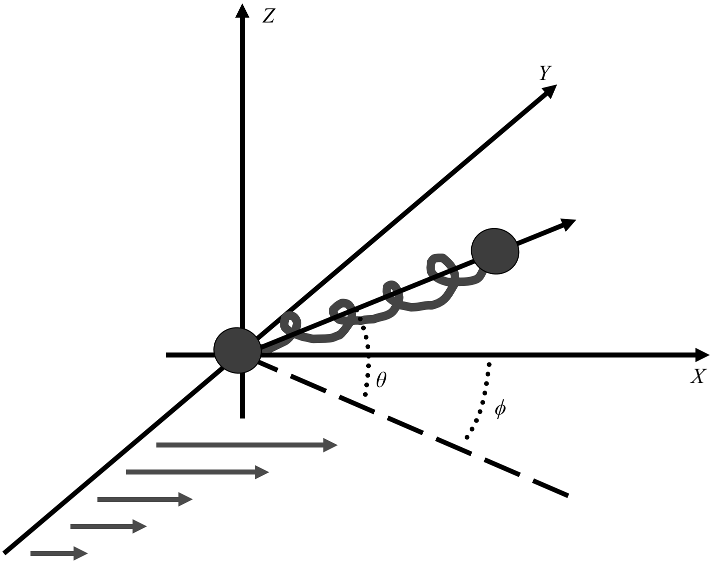

FIGURE 1. Polymer orientation geometry.

consider the case in which the steady shear flow is accompanied by weaker random velocity fluctuations, the setting realized in the elastic turbulence experiments by Groisman & Steinberg (2000, 2001, 2004). We choose the coordinate frame associated with the shear flow, as shown in figure 1, where the mean flow is characterized by the shear velocity $(sy, 0, 0)$ and s is positive. Then the polymer end-to-end vector $\boldsymbol{R}$ is conveniently parameterized by the spherical angles ϕ and θ: $R_x = R\cos\theta\cos\phi$, $R_y = R\cos\theta\sin\phi$, $R_z = R\sin\theta$. In terms of these variables, (2.1) (with the Langevin term omitted) transforms into the following set of equations:

$$\partial_t\phi = -s\sin^2\phi + \xi_\phi, \tag{2.2}$$

$$\partial_t\theta = -s\sin\phi\cos\phi\sin\theta\cos\theta + \xi_\theta, \tag{2.3}$$

$$\partial_t \ln R = -\gamma(R) + s\cos^2\theta\cos\phi\sin\phi + \xi_\parallel, \tag{2.4}$$

where ξ_ϕ, ξ_θ and $\xi_\parallel$ are random variables related to the fluctuating component of the velocity gradient. Note, that the angular (orientational) dynamics described by (2.2), (2.3) decouples from the dynamics of the polymer extension, R, and that, at $\gamma = 0$, (2.4) describes a divergence of neighbouring Lagrangian trajectories.

3. Angular statistics

The statistics of the velocity fluctuations is assumed to be homogeneous in time. In a statistically stationary velocity field, the angular statistics is stationary as well, characterized by the joint PDF, $\mathscr{P}(\phi, \theta)$, which is a periodic function of the angles with period π for both ϕ and θ. Thus, it is sufficient to consider $\mathscr{P}(\phi, \theta)$ within the bounded domain (torus) $-\pi/2 < \phi, \theta < \pi/2$. According to (2.2), (2.3), $\mathscr{P}(\phi, \theta)$ is symmetric with respect to θ but it is not symmetric with respect to ϕ. Therefore, the average value of ϕ, $\phi_t = \langle\phi\rangle$, is non-zero. In our setting, ϕ_t is positive. The value of ϕ_t can be estimated by balancing the deterministic and stochastic terms on the right-hand side of (2.2). The weakness of the random term in comparison with s implies $\phi_t \ll 1$. Quantity ϕ_t also estimates typical fluctuations of ϕ about its mean value. If the random terms in (2.2), (2.3) are assumed to be comparable, it immediately follows that the typical value of θ fluctuations is estimated by ϕ_t as well.

Note that the equation, $\partial_t r_i = r_j \nabla_j v_i$, describing the dynamics of separation $\boldsymbol{r}$ between two neighbouring fluid particles (moving along nearby Langangian trajectories), leads to the same angular dynamics as determined by (2.2), (2.3). For $r = |\boldsymbol{r}|$ one derives $\partial_t \ln r = s \cos^2\theta \cos\phi \sin\phi + \xi_\parallel$, where $\xi_\parallel$ represents the direct (as opposed to indirect through fluctuations in ϕ) effect of velocity fluctuations. It follows from (2.2) that for typical fluctuations (when $\phi, \theta \ll 1$) ξ_ϕ competes with $s\phi^2$. Assuming that $\xi_\parallel \sim \xi_\phi$ one finds that $\xi_\parallel$ is negligible in comparison with $s\phi$, because the relevant values of ϕ are small, $\phi \ll 1$. Therefore, for small ϕ, θ, one arrives at $\partial_t \ln r = s\phi$. This equation establishes the relation between the angular dynamics of the polymer and the dynamics of Lagrangian separation. For the Lyapunov exponent, defined as the mean logarithmic rate of divergence of Lagrangian trajectories, $\bar{\lambda} \equiv \langle \partial_t \ln r \rangle = s\phi_t$.

It is natural to expect that the Lagrangian velocity correlation time is $\bar{\lambda}^{-1}$, that is also a characteristic time of the ξ_ϕ and ξ_θ variations. Then, comparing the left-hand sides of (2.2), (2.3) with the first terms on their right-hand sides (for $\phi, \theta \ll 1$), one concludes that the angular correlation time can be estimated by the same quantity $\bar{\lambda}^{-1} = (s\phi_t)^{-1}$. Next, equating the terms on the right-hand sides of (2.2), (2.3), one derives $\xi_\phi \sim \xi_\theta \sim s\phi_t^2 \ll s$. The last inequality reflects the assumed weakness of the velocity gradient fluctuations compared to the shear rate, s.

4. Tails of the angular PDFs

Let us consider the domain $|\phi|, |\theta| \gg \phi_t$, where the random terms in (2.2), (2.3), ξ_ϕ and ξ_θ, are negligible. The angular dynamics is purely deterministic in this domain leading to the following dependence of the angles on time t:

$$\cot\phi = s(t - t_0), \qquad \tan\theta = c\sin\phi, \tag{4.1}$$

where t_0 and c are constants. The vector $\boldsymbol{R}$ reverses its direction as t increases. Therefore, (4.1) describes a single flip of the polymer. Due to the assumed homogeneity in time of the velocity statistics, t_0 is homogeneously distributed. Recalculating the measure dt_0 in the PDF of the angles in accordance with (4.1), one derives

$$\mathscr{P}(\phi, \theta) = U(\tan\theta/\sin\phi)\sin^{-3}\phi\,\cos^{-2}\theta. \tag{4.2}$$

The function U reflects possible variations in the parameter c (its statistics), which is determined from the initial conditions for the deterministic evolution. These conditions have to be found from matching (4.1) to those defined for the stochastic domain $|\phi|, |\theta| < \phi_t$. One concludes that the function U is sensitive to the angular dynamics in the stochastic domain and, respectively, to details of velocity fluctuations, i.e. the function is non-universal. Note, that (4.2) is identical to the one found by Hinch & Leal (1972) in the context of a solid rod tumbling in shear flow caused by thermal fluctuations.

Equation (4.1) shows that in the deterministic regime the angle ϕ decreases uniformly with time (that is except for the jump from $-\pi/2$ to $+\pi/2$ at $t = t_0$). Therefore, the stationary PDF for the angular degrees of freedom, $\mathscr{P}(\phi, \theta)$, corresponds to a non-zero probability flux from positive to negative ϕ related to a preferred (clockwise) direction of the polymer's rotations in the (X, Y)-plane. Formally, the probability flux goes out through $\phi = -\pi/2$ and the same flux comes back (enters) through $\phi = \pi/2$ ($\pi/2$ and $-\pi/2$ are identical by our construction) thus keeping the total probability equal to unity.

The PDF of ϕ, P_ϕ, can be obtained from the joint PDF: $P_\phi = \int \mathrm{d}\theta\, \mathscr{P}(\phi, \theta)$. Integrating (4.2) over θ one obtains the following expression valid for $|\phi| \gg \phi_t$:

$$P_\phi \equiv \int \mathrm{d}\theta\, \mathscr{P}(\phi, \theta) = C\phi_t \sin^{-2}\phi, \tag{4.3}$$

where the constant C is of order unity. Let us reiterate that, thinking dynamically, (4.3) originates from the deterministic flips bringing ϕ from its most probable domain $\sim \phi_t$ to the observation angle. Equation (4.3) describes the probability flux: as determined by (2.2), $P_\phi \partial_t \phi$ is constant in the deterministic region.

Consider the PDF of θ, $P_\theta = \int \mathrm{d}\phi\, \mathscr{P}(\phi, \theta)$. The naive result for the PDF tail following from (4.2) is $P_\theta \propto \theta^{-2}$, provided $1 \gg |\theta| \gg \phi_t$. However, one should be careful, since (4.2) does not cover a particular angular domain, characterized by $|\phi| < \phi_t$ and $|\theta| \gg \phi_t$, which should be analysed separately. In this domain, one can neglect ξ_θ in (2.3). Assuming also $|\theta| \ll 1$, one arrives at $\partial_t \ln(\theta) = -s\phi$, where $s\phi$ can be treated as a random variable independent of θ. It follows that the tail of the θ PDF is related to the long (compared to the correlation time $\bar{\lambda}^{-1}$) period when ϕ fluctuates around some negative value, $\sim -\phi_t$. (These fluctuations can be interrupted by flips.) Then θ at the end of a period T is estimated according to $\ln(\theta/\phi_t) \sim Ts\phi_t$. The probability W to observe such a long atypical period is estimated by $\ln W \sim -\bar{\lambda}T$. These estimates, recalculated in the PDF of θ, $P_\theta = \mathrm{d}W/\mathrm{d}\theta$, give an algebraic tail $P_\theta \propto |\theta|^{-a}$, where the exponent a is a positive number of order unity sensitive to the statistics of the ϕ fluctuations. (Therefore, a is not universal.) Note that this algebraic tail is analogous by its origin to the algebraic tail of the polymer extension PDF discussed by Balkovsky *et al.* (2000, 2001) and by Chertkov (2000).

Thus one finds two different contributions to the PDF tail: one is related to the deterministic motion, described by (4.1), while the other is associated with the stochastic evolution in the domain $|\phi| < \phi_t, |\theta| \gg \phi_t$. For $1 \gg |\theta| \gg \phi_t$, both contributions are algebraic, $\propto |\theta|^{-2}$ and $\propto |\theta|^{-a}$, respectively. The deterministic contribution, $\propto |\theta|^{-2}$, dominates if $a > 2$, while the stochastic contribution, $\propto |\theta|^{-a}$, dominates otherwise.

5. Tumbling time statistics

As follows from (4.1), the deterministic process, which defines the polymer turn (because ϕ changes essentially only during the deterministic part of the dynamics), is faster than the stochastic wandering taking place at small angles, $|\phi|, |\theta| \sim \phi_t$. Therefore it is convenient to define the tumbling time, τ, as the time separating two subsequent crossings in ϕ of the special angle $\pm\pi/2$, in the middle of the deterministic domain. Since the major contribution to τ originates from the stochastic wandering in the ϕ_t vicinity of $\phi = 0$, the position of the τ-PDF maximum and its width are both estimated by the correlation time $(s\phi_t)^{-1}$, because this is the only relevant characteristic time of the stochastic evolution.

Considering the PDF tail for $\tau \gg \bar{\lambda}^{-1}$, one observes that if a flip does not occur for a long time, then this delay can be interpreted in terms of the large number, $\bar{\lambda}\tau$, of independent unsuccessful attempts to pass (clockwise in ϕ) the stochastic domain $|\phi| < \phi_t$. The probability of the delayed flip is given by the product of the probabilities of these $\bar{\lambda}\tau$ events, resulting in $\ln P_\tau \sim -\bar{\lambda}\tau$, for $\tau \gg \bar{\lambda}^{-1}$.

The left, $\tau \ll \bar{\lambda}^{-1}$, tail of the tumbling time PDF is non-universal because it is sensitive to details of the velocity field statistics. Indeed, it is determined by the special configurations of the velocity field that force ϕ to pass through the stochastic region atypically fast. (Those configurations are vortices with clockwise rotation of the fluid in the (X, Y)-plane leading to negative values of ξ_ϕ that are larger than $s\phi_t^2$

in absolute value.) Analysing the anomalously fast revolutions of the polymer, one finds that ξ_ϕ from the right-hand side of (2.2) may be considered as time-independent. (Here again, the natural assumption is that the correlation time of the velocity field fluctuations is of the same order as the inverse Laypunov exponent in the flow, $\sim \bar{\lambda}^{-1}$.) Then the direct solution of (2.2) gives $\tau = \pi/\sqrt{|\xi_\phi| s}$, where we have assumed that the major contribution to τ comes from the domain of small ϕ, $\phi \ll 1$. This estimate holds if $s \gg |\xi_\phi| \gg s\phi_t^2$. For $1/s \ll \tau \ll \bar{\lambda}^{-1}$ one arrives at the following expression for the PDF of τ: $P_\tau = 2\pi^2\tau^{-3}s^{-1}P_\xi(-\pi^2/[\tau^2 s])$, where P_ξ is the single-time PDF of ξ_ϕ.

6. Polymer extension statistics

For most of the R dynamics, the basic dynamical equation (2.4) can be simplified. First, the main contribution to the R dynamics stems from the region of small angles where $\sin\phi$ can be replaced by ϕ and $\cos\theta$ by unity. Second, the term $\xi_\parallel$ is potentially important only in the stochastic region where ξ_ϕ competes with $s\phi^2$. Assuming $\xi_\phi \sim \xi_\parallel$, we conclude that $\xi_\parallel$ is negligible in comparison with $s\phi$ there. Therefore

$$\partial_t \ln R = -\gamma(R) + s\phi. \tag{6.1}$$

Note that (6.1) is inapplicable when R is close to its minimum value during a flip (since the angles ϕ, θ are of order unity there). Note also that (6.1) is correct for $R \gg R_T$ (R_T is the typical size of the polymer in the absence of the flow) where the Langevin force can be neglected.

The statistics of R is determined by the interplay of the two terms on the right-hand side of (6.1). Since the average value of $s\phi$ is equal to the Lyapunov exponent, $\bar{\lambda}$, the dimensionless parameter characterizing the statistics of the polymer extension is the Weissenberg number, $Wi = \bar{\lambda}/\gamma(0)$, which grows with the strength of the shear, or/and, of the velocity fluctuations. At $Wi = 1$, when the two terms on the right-hand side of (6.1) balance, the system undergoes the so-called coil-stretched transition, see Lumley (1969, 1973), Balkovsky *et al.* (2000, 2001), and Chertkov (2000) for details. We find, however, that in the specific case of strong shear considered here additional qualitative changes in the PDF of R occur at $Wi > 1$ so that the overall picture is richer than in the case of isotropic velocity statistics. Below we describe the extension PDF as a function of Wi.

To illustrate our generic analytical results we plot in figure 2 four graphs of the extension PDF obtained by numerical simulations based on a modification of (6.1), $\partial_t \ln R = -\gamma + s\sin\phi\cos\phi$, which allows the correct reproduction of the flips, and (2.2) with the stochastic term ξ_ϕ chosen to be δ-correlated in time. The simulations were done with $\gamma(R) = \gamma(0)/(1 - R^2/R_{\max}^2)$, corresponding to the so-called FENE-P model of the polymer elasticity, see e.g. Bird *et al.* (1987).

6.1. $Wi < 1$, $\alpha > 0$

We begin by discussing the case $Wi < 1$, for which the polymer is only weakly stretched. Then, typically, the molecule stays in the 'coil' state characterized by the thermal size, R_T, which emerges as the result of a balance between the Langevin–driven extension and contraction (relaxation) related to polymer elasticity. Recall that due to a large number of monomers in the polymer molecule, the thermal-noise-induced length, R_T, is much smaller than the maximal polymer extension, $R_{\max}$.

At the scales larger than R_T the thermal noise is irrelevant and the extension dynamics is described by (6.1). We are interested in the statistics of R for large, $R \gg R_T$, a problem already analysed in detail by Balkovsky *et al.* (2000, 2001),

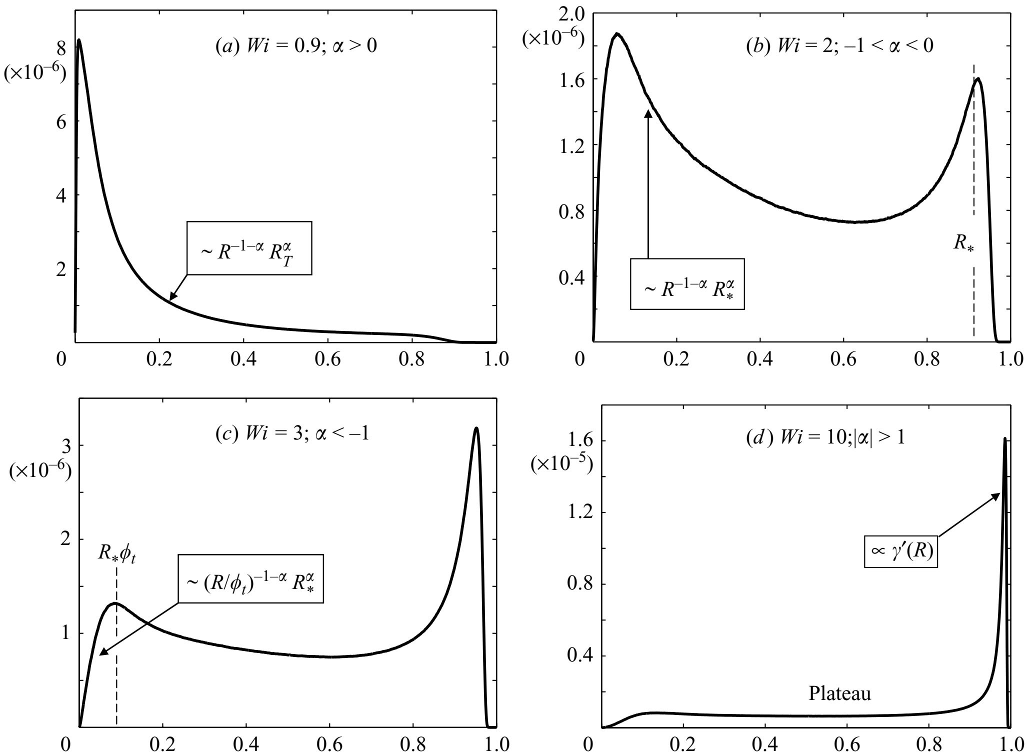

FIGURE 2. PDF of the polymer extension, R, measured in the units of maximal extension, for different values of Wi, obtained from numerical simulations of the stochastic equations explained in the text.

Chertkov (2000), where it was shown that the extension PDF is

$$P(R) \propto R^{-1-\alpha}, \tag{6.2}$$

with $\alpha > 0$. Equation (6.2) holds for $R_{\max} \gg R \gg R_T$ where $\gamma(R)$ weakly deviates from $\gamma(0)$. The situation is reflected in figure 2(a), where the algebraic tail is clearly seen. The positive value of the exponent α in (6.2) guarantees that the normalization integral $\int \mathrm{d}R\, P(R)$ converges in the region $R \gg R_T$. Thus, the normalization coefficient in (6.2) is $\sim R_T^{-\alpha}$. The exponent α decreases as Wi increases, and it crosses zero at the coil–stretch transition, where $Wi = 1$.

The algebraic tail (6.2) is related to a long (compared to the correlation time $\bar{\lambda}^{-1}$) process of polymer extension from the typical value R_T to the current value of the extension, $R \gg R_T$. Note that this long extension does not mean that the right-hand side of (6.1) is always positive during the process of extension since ϕ fluctuates and $s\phi$ is larger than γ only on average. Moreover, the process consists of alternating stochastic and deterministic portions (polymer flips during the later ones). Although R decreases during the first half of the flip, the initial extension is restored (returns to its initial value) during the second half. Overall, the flips do not influence the extension process. The probability W of this atypically long extension process depends exponentially on its duration T, since W is a product of independent probabilities, each representing a sub-processes of duration $\bar{\lambda}^{-1}$. This gives the estimate $\ln W \sim -\bar{\lambda}T$. On the other hand, in accordance with (6.1), $\ln(R/R_T) \sim \bar{\lambda}T$. Combining the two estimates, we arrive at the algebraic tail (6.2) for the extension PDF, $P = \mathrm{d}W/\mathrm{d}R$.

6.2. $Wi > 1$, $-1 < \alpha < 0$

Above the coil–stretch transition, when $\bar{\lambda}$ exceeds $\gamma(0)$, the polymers become strongly extended. In this stretched state the typical size of the polymer, R_*, is much larger than R_T. Considering the average of (6.1) one finds $\gamma(R_*) = \bar{\lambda}$.

The left tail of the PDF, corresponding to $R_T \ll R \ll R_*$, is governed by the same algebraic law (6.2) (figure 2*b*). However, now $\alpha < 0$, meaning that the normalization integral $\int dR\, P(R)$ has a major contribution for $R \sim R_*$. Therefore, restoring normalization, one derives $P \sim R_*^{\alpha} R^{-1-\alpha}$ for the $R \ll R_*$ tail. The transition from positive to negative α implies an important change in the nature of the dynamical configuration corresponding to the algebraic tail: extension, as a typical process for $\alpha > 0$, is replaced by contraction for negative α, so that initially typical extension, $\sim R_*$, contracts through a long, $T \gg \bar{\lambda}^{-1}$, (multi-tumbling) evolution. (Physical arguments, clarifying the origin of the algebraic tail, are identical to the ones presented above for $Wi < 1$.)

The right tail, corresponding to extreme extensions, $R_{\max} - R \ll R - R_*$, can also be explained in general terms. The domain is characterized by extremely fast relaxation, so that the left-hand side of (6.1) can be neglected: $\gamma(R) = s\phi$. Moreover, because of the fast nature of the polymer relaxation in the extreme case, ϕ simply follows the respective random term in (2.2), i.e. the left hand side of (2.2) can also be neglected resulting in $s\phi^2 = \xi_\phi$. In other words, the extreme configuration is produced through a fast anticlockwise revolution of the polymer to a large (in comparison with the typical value ϕ_t) positive angle, $1 \gg \phi \gg \phi_t$, driven by anomalous fluctuations in ξ_ϕ. Recalculating the PDF of ξ_ϕ to $P(R)$ one arrives at $P(R) = 2s^{-1}\gamma\gamma' P_\xi(\gamma^2/s)$, where P_ξ is the simultaneous PDF of the velocity gradient term ξ_ϕ. Note that the expression for $P(R)$ is not restricted to the case considered in this subsection but applies to the description of extreme extensions in all regimes.

6.3. $Wi > 1$, $\alpha < -1$

Once α crosses -1, R_* becomes maximum of the extension PDF, $P(R)$. This modification in the PDF shape is accompanied by the emergence of a plateau on the left from the maximum (see figure 2*c*), associated with an additional contribution to the PDF related to the deterministic angular dynamics.

Let us explain the origin of the plateau. For angles which are smaller than unity but larger than ϕ_t, R and ϕ satisfy, $\partial_t \ln R = s\phi$ and $\partial_t \phi = -s\phi^2$, as follows from (2.2), (6.1). Integrating these equations one derives $R = A|t - t_0|$ where t_0 and A are constants, the latter being estimated by $A \sim \bar{\lambda} R_*$. Assuming that the time t_0 is homogeneously distributed (due to the assumed homogeneity of the velocity statistics), and recalculating the measure dt_0 to the PDF of R, one arrives at $P(R) = C/R_*$ (C is an R-independent $O(1)$ constant) corresponding to the plateau seen in figure 2(*c*).

The 'deterministic' contribution to $P(R)$, $\sim 1/R_*$, discussed above, does not cancel the 'stochastic' one $\sim R_*^{\alpha} R^{-1-\alpha}$ corresponding to the long contraction which starts at R_*; these contributions co-exist. For $-1 < \alpha < 0$, the stochastic contribution dominates, in full agreement with the discussion of § 6.2. When $\alpha < -1$, the situation is reversed and the deterministic contribution dominates.

The plateau extends from $R \sim R_*$ down to $R \sim R_*\phi_t$, where $R_*\phi_t$ is the smallest value of R one can achieve under the condition that when the flip begins the initial extension is R_*. However, if the initial extension is smaller than R_*, a deterministic flip could bring it to values that are smaller than $R_*\phi_t$. Therefore, to understand even smaller values of R, $R < R_*\phi_t$, one should consider flips which begin with an anomalously small initial value R_0, $R_0 < R_*$ (prepared by some preliminary and long stochastic processes, of the type discussed above). The probability density of achieving

R_0 during the preparatory stage is estimated according to (6.2): $\sim R_*^{\alpha} R_0^{-1-\alpha}$. On the other hand, any R_0 that lies between R and R/ϕ_t transforms dynamically as a result of a fast flip into the current value of extension R with the same R-independent probability, which now can be estimated as $\sim 1/R_0$. Therefore, one arrives at an estimate valid for $R < R_* \phi_t$ that explains the probability decrease at the smallest R seen in figure 2(*c*):

$$P \sim \int_R^{R/\phi_t} R_0^{-1} \mathrm{d}R_0 R_0 \left[R_*^{\alpha} R_0^{1+\alpha} \right] \sim R_*^{-|\alpha|} \left(R/\phi_t \right)^{|\alpha|-1}. \tag{6.3}$$

One can observe the bump in figure 2(*c*) separating the plateau and the power tail. To explain the bump, one simply needs to account for the angular nonlinearity in the estimate of the plateau just discussed.

6.4. $Wi \gg 1$, $\alpha \ll -1$

The larger Wi is, the closer R_* approaches $R_{\max}$. Then the condition of fast relaxation, $R\gamma'(R) \gg \gamma$, which has already allowed us to analyse the $-1 < \alpha < 0$ regime, also applies to the region in the vicinity of R_*. The smallness of the ratio $\gamma/(R\gamma')$ suggests that the left-hand side of (6.1) can be replaced by zero, resulting in $\gamma(R) = s\phi$. Expressing the PDF of R through the PDF of ϕ one arrives at $P(R) \sim s^{-1}\gamma'(R)P_\phi(\gamma/s)$, where it is also assumed that $\gamma(R)/s < \phi_t$. In this special domain of R and ϕ, variations of P_ϕ are slow, so that the main dependence on R is due to the factor $\gamma'(R)$. The expression for $P(R)$ also applies to the left (for smaller values of R) of R_* provided $R\gamma'(R)/\gamma$ is large. In this domain, the PDF can be estimated as $P \sim \gamma'(R)/\gamma(R_*)$. At even smaller values of R, the PDF has a plateau, $P \sim \gamma(0)/[R_*\gamma(R_*)]$ (which generalizes our previous result to the $Wi \gg 1$ case). This explains the complex behaviour of the PDF of R shown in figure 2(*d*).

To conclude, the PDF of R demonstrates complex and rich Wi-dependent behaviour related to a number of distinct processes governing polymer dynamics. We observed that the typical value of the extension, associated with the stochastic wandering of the polymer orientation around the shear-preferred direction, increases with Wi. In the region of maximal stretching, near $R_{\max}$, the major contribution to the PDF originates from the fast adjustment of the polymer extension to the current value of the angular degrees of freedom. We identified special contributions to the PDF tails associated with fast (deterministic) flips and long (stochastic) extension/contraction processes.

7. Discussion

We have studied the dynamics and statistics of the polymer molecules placed in a chaotic flow with mean shear. Encouraged by qualitative agreement of our results with the newest experimental data of Gerashchenko *et al.* (2004), we anticipate that the rich variety of theoretical predictions presented in the paper will help guide and test future experimental works in this field. In this context it is important to discuss applicability conditions and limitations of our approach.

Our theory is based on the simple dumb-bell-like equation (2.1). This equation is obviously approximate, taking into account only one variable (end-to-end vector $\boldsymbol{R}$) of generally more complex polymer dynamics. Therefore, it is important to assess the effects of more realistic modelling. It was our assumption that the mean flow can be approximated by a perfect shear flow, whereas in reality flow parameters demonstrate spatial inhomogeneities. Even though these variations were not included in our derivations, our results remain valid if the variations along the Lagrangian trajectories occur on time scales larger than $\bar{\lambda}^{-1}$ and also if the local flow does not deviate strongly from the shear configuration. Then, the PDFs discussed in this paper adjust adiabatically to the current values of the parameters becoming slow functions of spatial position. If the regular part of the flow is elongational, polymer flips become forbidden in the ideally deterministic regime while fluctuations will still generate some tumbling.

Polymer tumbling was first observed in the steady shear flow experiments by Smith *et al.* (1999) and Hur *et al.* (2001) in which orientational fluctuations were driven by thermal noise, while our analysis has focused primarily on the case of tumbling driven by velocity fluctuations. Therefore, even though all of our results are directly applicable to the elastic turbulence setting of Groisman & Steinberg (2000, 2001, 2004), examination of the statistics of the Langevin–driven tumbling and of angular and extension probability distributions is a separate task. These problems will be examined elsewhere.

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