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Transient stability guarantees for ad hoc dc microgrids

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***Abstract*— Ad hoc electrical networks are formed by connecting power sources and loads without planning the interconnection structure (topology) in advance. They are designed to be installed and operated by individual communities—without central oversight—and as a result are well-suited to addressing the lack of electricity access in rural and developing areas. However, ad hoc networks are not widely used, and a major technical challenge impeding their development (and deployment) is the difficulty of certifying network stability without a priori knowledge of the topology. We develop conditions on individual power sources and loads such that a microgrid comprised of many units will be stable. We use Brayton-Moser potential theory to develop design constraints on individual microgrid components that certify transient stability—guaranteeing that the system will return to a suitable equilibrium after load switching events. Our central result is that stability can be ensured by installing a parallel capacitor at each constant power load, and we derive an expression for the required capacitance.**

I. Introduction

Microgrids are smaller-scale than conventional power systems and can be designed to naturally incorporate distributed renewable resources. These benefits make microgrids attractive for addressing the lack of electricity in remote and rural areas, which continues to affect more than one billion people [1]. The need for universal electricity access and evolving demands on existing bulk power infrastructure have driven extensive development of microgrids in recent years, but the capital-intensive planning process and the need for centralized control continue to impede adoption.

Ad hoc microgrids—microgrids that can be set up without predetermining the network structure—reduce the financial barrier to energy access. Instead, they pose a technical challenge: network stability must be certified before the network topology is known, and the network may be modified after installation depending on the community's needs. We focus on a previously-presented low voltage dc architecture designed specifically for rural electrification [2].

Like all power systems, microgrids are not globally stable, which presents unique control challenges. Further, there are three unusual features of our analysis that require a significant departure from traditional power systems methods: (1) the ad hoc setting, (2) the use of tightly-regulated power electronics at all sources and loads, and (3) transient stability guarantees. Using power electronics to interface sources and loads to the network offer new opportunities for decentralized and autonomous control of power supply and demand [3], but they complicate system stability because they draw constant power from the network to regulate their outputs. The negative incremental resistance ($\partial v/\partial i$) of these loads has a destabilizing effect on power systems [4]. The impact of constant power loads on the stability of conventional microgrids has attracted interest in the controls community [5], [6], [7], [8], [9] and the power electronics community [10], [11]. Previous analyses are based on simplified models of resistive lines that remain stable under mild constraints and arbitrarily high control gains. However, in practical settings the line inductance is a source of instability and cannot be neglected.

To our knowledge, all previous studies have focused on known and fixed topologies, impractically simplified models, and/or linearized models. Our analysis encompasses unknown and changing topologies, realistic subsystem models, and significantly extends our previous conference paper [3] by providing transient stability guarantees for our nonlinear architecture. Robustness to large perturbations is important, especially in low-voltage, low-power networks where each household may be a significant fraction of the total system load. Mathematically, certifying transient stability requires characterizing the extrema *and* attraction basins of our previously presented potential function for networks of unknown topologies. Our main contribution is a set of design-friendly constraints on individual network units (sources, loads, and lines), summarized by Eqs. 23, 31 and 33.

II. Models and Notation

In this section we present models for the interconnecting lines, power electronic loads, and voltage source converters which are analytically tractable and can be adapted to describe many networks. These are based on a previously presented ad hoc microgrid [2].

The electrical network is described as a weighted, directed graph $(\mathcal{V}, \mathcal{E})$ with a total of $|\mathcal{V}|$ nodes (buses) and $|\mathcal{E}|$ edges (lines). Each edge $\alpha \in \mathcal{E}$ represents a tuple $\alpha = (i, j)$ with $i, j \in \mathcal{V}$. A power source or load is attached to each node and we denote the subset of vertex indices corresponding to loads as $\mathcal{L} \subset \mathcal{V}$ with and the subset of source indices as $\mathcal{S} \subset \mathcal{V}$. The state of the system is described by the voltage and current vectors $v \in \mathbb{R}^{|\mathcal{V}|}$ and $i \in \mathbb{R}^{|\mathcal{E}|}$. The topology of the graph is defined by the (transposed) incidence matrix $\nabla \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{V}|}$. Applying ∇ to the voltage vector results in a vector of potential differences across each line, and applying $\nabla^\top$ to the current vector yields the total current flowing out of each node.

Each power line is associated with a graph edge $\alpha = (i, j) \in \mathcal{E}$ and is characterized by an inductance $L_\alpha = L_{ij}$

and a resistance $R_\alpha = R_{ij}$. Each line has time constant $\tau_\alpha = L_\alpha / R_\alpha$. Each line current i_α is described by:

$$L_\alpha \dot{i}_\alpha = -R_\alpha i_\alpha + \sum_{k \in \mathcal{V}} \nabla_{\alpha k} v_k, \quad \alpha \in \mathcal{E}\,. \tag{1}$$

Load k is represented by the parallel connection of a capacitance C_k and a constant power load drawing power p_k. In general, constant power loads represent perfectly-regulated power converters, and hence are conservative and general models which can be used to describe many power electronic devices. The capacitor across the input of the power converter is a standard feature of these converters, and is critical for system stability [8]. Each load voltage is described by:

$$C_k \dot{v}_k = -\frac{p_k}{v_k} - \sum_{\alpha \in \mathcal{E}} \nabla_{\alpha k} i_\alpha, \quad k \in \mathcal{L}\,. \tag{2}$$

In this work we assume the source controller, which regulates the converter output voltage, has dynamics much faster than the network. Accordingly, we model them as perfect voltage sources: $v_k = V_0, \quad k \in \mathcal{S}$. The extension to controllable converters with proportional (droop) and integral voltage control is relatively straightforward [3].

Characterizing system stability requires a suitable family of Lyapunov (potential) functions. Extrema of a particular potential correspond to equilibria of the system, and stability can be certified by demonstrating certain properties of the potential. Unfortunately, the presence of constant power loads and lack of global stability in power systems preclude the use of potential functions based on system energy (Hamiltonian potentials). However, the seminal results of Brayton and Moser are applicable to our setting [12], [13]. For nonlinear electrical circuits, they demonstrate how to set up a potential $\mathcal{P}$ and a related matrix $\mathcal{Q}$ that provide a separate (and often more tractable) approach: conditions on $\mathcal{P}$ can be traded for conditions on $\mathcal{Q}$ by using the relation $\mathcal{Q}\dot{x} = -\partial_x \mathcal{P}$.

For our architecture, a representation with the proper structure is given by:

$$\mathcal{G}(v) = \sum_{(i,j) \in \mathcal{E}} \frac{(v_i - v_j)^2}{2R_{ij}} + \sum_{k \in \mathcal{L}} p_k \log v_k \tag{3}$$

$$\begin{aligned}\mathcal{P}(x) = \mathcal{G}(v) &+ \frac{1}{2} \sum_{\alpha \in \mathcal{E}} (\tau_{\max} - \tau_\alpha) L_\alpha \dot{i}_\alpha^2 \\ &+ \frac{\tau_{\max}}{2} \sum_{k \in \mathcal{L}} C_k \dot{v}_k^2\end{aligned} \tag{4}$$

$$\mathcal{Q} = \begin{bmatrix} \mathrm{diag}\left(\tau_{\max} R_\alpha - L_\alpha\right) & -\tau_{\max} \nabla_{\mathcal{E}\mathcal{L}} \\ \tau_{\max} \nabla_{\mathcal{E}\mathcal{L}}^T & \mathrm{diag}\left(C_k - \tau_{\max} \frac{p_k}{v_k^2}\right) \end{bmatrix} \tag{5}$$

x is the state vector $[i_{\mathcal{E}}^T . v_{\mathcal{L}}^T]^T$ and $\dot{i}_\alpha$, $\dot{v}_k$ are given in Eqs. (1) and (2). $\tau_{\max}$ is an upper bound on the line time constants in the network—for convenience, we define it to be strictly larger than the largest time constant: $\tau_{\max} > \max_\alpha \tau_\alpha$. $\nabla_{\mathcal{E}\mathcal{L}}$ refers to the submatrix of ∇ corresponding to the load nodes. Finally, in addition to being notationally convenient, $\mathcal{G}$ is well-studied and is referred to in the literature as the resistive co-content [13]. It is also worth noting, that all equilibria of the system correspond to extrema of $\mathcal{P}$, and vice-versal, every extremum of $\mathcal{P}$ to an equilibrium of the system. Moreover, the extrema of the potential $\mathcal{G}(v)$ correspond to the solutions of the equilibrium power flow equations.

III. Stability of a Two Bus System

In this section we consider the two bus system shown in Figure 1, to provide a simple introduction to the techniques we will use in the next section to analyze ad hoc networks with no topology constraints. The dynamic equations of the system are given by:

$$C\dot{v} = -\frac{p}{v} + i \tag{6}$$

$$L\dot{i} = -Ri + V_0 - v\,. \tag{7}$$

Applying the definitions of $\mathcal{G}$ and $\mathcal{P}$ from Sec. II yields:

$$\mathcal{G}(v) = \frac{(V_0 - v)^2}{2R} + p \log v \tag{8}$$

$$\mathcal{P}(x) = \mathcal{G} + \frac{1}{2}(\tau_{\max} - \tau) L\dot{i}^2 + \frac{\tau_{\max}}{2} C\dot{v}^2\,. \tag{9}$$

The relationship between the voltage v and the load p at equilibrium (the "nose curve") is shown in Figure 2. The largest load that can be supported is the apex of the nose curve at $p = P_0 = V_0^2/4R$, which corresponds to a load bus voltage of $V_0/2$. For all $p < P_0$, there are two solutions:

$$V_{\mathrm{high}}(p) = \tfrac{V_0}{2}\left(1 + \sqrt{1 - \tfrac{p}{P_0}}\right) \tag{10}$$

$$V_{\mathrm{low}}(p) = \tfrac{V_0}{2}\left(1 - \sqrt{1 - \tfrac{p}{P_0}}\right) \tag{11}$$

V_{high} is a stable equilibrium point (a minimum of $\mathcal{G}$) and V_{low} is an unstable equilibrium point (a maximum of $\mathcal{G}$). When the power exceeds P_0, $\mathcal{G}$ does not have any extrema and the system has no equilibria. For the network to achieve a minimum voltage of $V_0 > V_{\min} > V_0/2$, the largest load that can be supported is $P_{\max} = V_{\min}(V_0 - V_{\min})/R$.

To analyze transient stability, we first define a "switching event" to be any time t such that the load power changes instantaneously:

$$p(t^-) \neq p(t^+)\,. \tag{12}$$

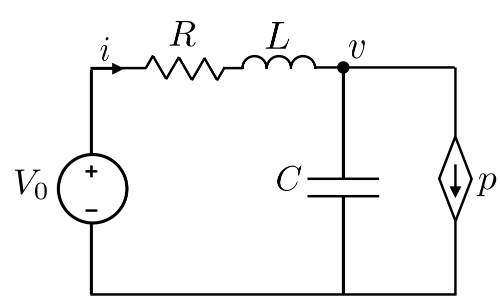

Fig. 1. Description of two bus network

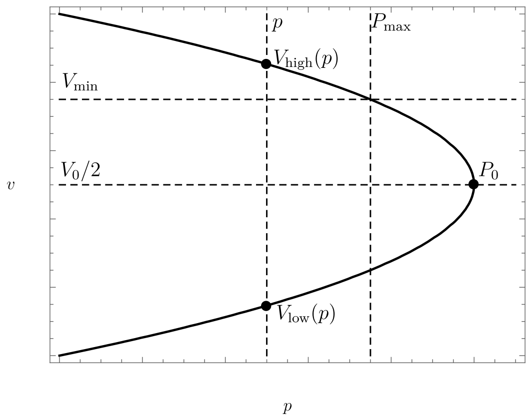

Fig. 2. Nose curve demonstrating the relationship between power and load bus voltage

Hereafter, $^+$ and $^-$ refer to quantities evaluated at t^+ and t^-. All state variables are continuous: $i^- = i^+$ and $v^- = v^+$. We assume that the system is at equilibrium before the switching event: $v^- = V_{\text{high}}^-$.

According to Lyapunov theorem, convergence to a stable equilibrium point is guaranteed whenever 1) the "energy" is strictly decreasing

$$\dot{\mathcal{P}} = -R(\tau_{\max} - \tau)\dot{i}^2 - \left(C - \frac{\tau_{\max} p}{v^2}\right)\dot{v}^2 < 0 \tag{13}$$

and 2) $\mathcal{P}(v) > \mathcal{P}(V_{\text{high}})$ for $v \neq V_{\text{high}}$. To satisfy these requirements, a lower bound on the load bus voltage during transients, V_{tr}, is introduced such that $v \geq V_{\text{tr}}$ and

$$C_k > \frac{\tau p_k^{\max}}{V_{\text{tr}}^2} \tag{14}$$

such that matrix Q, given in equation 5, is positive definite in the restricted voltage domain. To satisfy the second requirement and ensure the stable equilibrium point is in this domain the value of V_{tr} is constrained, $V_{\min} \geq V_{\text{tr}} \geq V_{\text{low}}$. For simplicity, we impose a stricter lower bound and assume $V_{\text{tr}} \geq V_0/2 \geq V_{\text{low}}$. Therefore, for convergence to a stable equilibrium:

$$\mathcal{P}^+(V_{\text{high}}^-) < \mathcal{G}^+(V_{\text{tr}}) \tag{15}$$

given that $\mathcal{P}^+(V_{\text{tr}})$ is bounded from below by $\mathcal{G}^+(V_{\text{tr}})$. The upper bound on $\mathcal{P}^+$ can be found by characterizing the capacitor current as

$$C\dot{v}(t^+) = \frac{p^- - p^+}{V_{\text{high}}^-}, \tag{16}$$

which then yields

$$\mathcal{P}^+ = \mathcal{G}^+(V_{\text{high}}^-) + \frac{\tau}{2C}\left(\frac{p^- - p^+}{V_{\text{high}}^-}\right)^2. \tag{17}$$

This representation of $\mathcal{P}$ makes explicit that larger values of C decrease the overall system "energy." A resulting bound on C is $C > C_{\text{tr}}(p^-, p^+)$ where C_{tr} is given by:

$$C_{\text{tr}}(p^-, p^+) = \frac{\tau}{2(\mathcal{G}^+(V_{\text{tr}}) - \mathcal{G}^+(V_{\text{high}}^-))}\left(\frac{p^- - p^+}{V_{\text{high}}^-}\right)^2$$

This expression can be reduced to a minimum capacitance bound, analogous to Eq. (14), by characterizing the "worst-case" switching event $p^- \to p^+$. The sufficient condition on capacitance are given by

$$\begin{aligned} C > \max_{p^-,\, p^+} \; & C_{\text{tr}}\left(p^-, p^+\right) \\ \text{subject to} \quad & p^- \leq P_{\max} \\ & p^+ \leq P_{\max} \end{aligned} \tag{18}$$

This optimization problem can easily be solved computationally to provide a lower bound on the parallel load capacitance.

IV. Generalization to Networks

In this section we generalize our analysis to arbitrary networks. We begin with a few simple assumptions: we assume our network has one strongly-connected component with at least one source. The topology of the graph is not restricted, but we assume that the aggregate resistance of all lines is bounded from above by $\sum_{(i,j)\in\mathcal{E}} R_{ij} \leq R_{\max}$. This bound is used to define the natural unit of power $P_0 = V_0^2/4R_{\max}$ corresponding to the maximum loadability of the two-bus topology. We use p_Σ to denote the instantaneous sum of all the loads, i.e. $p_\Sigma = \sum_{k\in\mathcal{L}} p_k$ and assume an upper bound $P_{\max}$ on the overall loadability: $p_\Sigma \leq P_{\max} < P_0$. We assume that the minimum voltage, $V_{\min}$, is selected in advance as a design choice. Additionally, the power consumption of each load is bounded from above: $p_k \leq p_k^{\max}$. We also introduce x^{sep} and v^{sep} to refer to the high voltage equilibrium point, existence and uniqueness of which is guaranteed by the following Lemma [14]:

Lemma 1. *Whenever $p_\Sigma < P_0$, there exists exactly one solution to the power flow equations $\partial_v \mathcal{G} = 0$ with all load buses satisfying*

$$v_k > V_{\text{high}}(p_\Sigma). \tag{19}$$

At the same time, all other equilibria have at least one load bus $\kappa \in \mathcal{L}$ such that

$$v_\kappa < V_{\text{low}}(p_\Sigma). \tag{20}$$

Proof. See [14], Supplementary Theorem 1 for the formal proof. □

Inequality (19) demonstrates that exactly one feasible equilibrium point is guaranteed to exist for any $p_\Sigma \leq P_{\max}$ if and only if $R_{\max}$, $P_{\max}$, $V_{\min}$, and V_0 satisfy $V_{\text{high}}(P_{\max}) \geq V_{\min}$ with V_{high} given by Eq. (10). This condition is equivalent to

$$P_{\max} \leq \frac{V_{\min}(V_0 - V_{\min})}{R_{\max}}. \tag{21}$$

Both inequalities (19) and (20) become tight for the two-bus system—in this sense, the two bus topology (one source separated from one load p_Σ by a line of resistance $R_{\max}$) is the "worst-case" topology for equilibrium point feasibility. This observation also implies that the condition (21) is both necessary and sufficient for existence of feasible equilibrium in an ad hoc setting.

Theorem 1. *The function $\mathcal{G}(v)$ is strictly convex whenever all load voltages satisfy $v_k > V_0/2$ and $p_\Sigma < P_0$.*

Proof. The quadratic form associated with the Hessian can be represented as

$$w^T \partial_{vv}\mathcal{G}(v) w = \sum_{(i,j)\in\mathcal{E}} \frac{(w_i - w_j)^2}{R_{ij}} - \sum_{k\in\mathcal{L}} \frac{p_k}{v_k^2} w_k^2 \tag{22}$$

Where we formally define $w_k = 0$ whenever $k \in \mathcal{S}$. Consider a set of paths $\Pi_k \subset \mathcal{E}$ with $k \in \mathcal{L}$ connecting each bus k to one of the sources. Then $w_k = \sum_{(i,j)\in\Pi_k}(w_i - w_j)$. Define $R_{\Pi_k} = \sum_{(i,j)\in\Pi_k} R_{ij}$. The term w_k^2 in (22) can be then bounded with the help of Jensen's inequality as

$$\begin{aligned} w_k^2 &= \left(\sum_{(i,j)\in\Pi_k} (w_i - w_j)\right)^2 \\ &= R_{\Pi_k}^2 \left(\sum_{(i,j)\in\Pi_k} \frac{R_{ij}}{R_{\Pi_k}} \frac{w_i - w_j}{R_{ij}}\right)^2 \\ &\le R_{\Pi_k}^2 \sum_{(i,j)\in\Pi_k} \frac{R_{ij}}{R_{\Pi_k}} \left(\frac{w_i - w_j}{R_{ij}}\right)^2 \\ &\le R_{\max} \sum_{(i,j)\in\Pi_k} \frac{(w_i - w_j)^2}{R_{ij}} \end{aligned}$$

Hence, for the Hessian quadratic form we have

$$\begin{aligned} w^T \partial_{vv}\mathcal{G}(v) w &> \sum_{(i,j)\in\mathcal{E}} \frac{(w_i - w_j)^2}{R_{ij}} \left(1 - \sum_{k:(i,j)\in\Pi_k} \frac{p_k R_{\max}}{v_k^2}\right) \\ &> \sum_{(i,j)\in\mathcal{E}} \frac{(w_i - w_j)^2}{R_{ij}} \left(1 - \frac{4 p_\Sigma R_{\max}}{V_0^2}\right) > 0 \end{aligned}$$

□

Corollary 1. *The voltage profile v^{sep} minimizes the function $\mathcal{G}$ in the domain $v > V_0/2$. Furthermore in the same domain, and for arbitrary currents, we obtain $\mathcal{P}(x) \ge \mathcal{G}^{\text{sep}} = \mathcal{G}(v^{\text{sep}})$ and $\mathcal{P}(x) = \mathcal{G}^{\text{sep}}$ for $v = v^{\text{sep}}$ and $i = \operatorname{diag}(R_k^{-1})\nabla v^{\text{sep}}$.*

Next, we identify the conditions for the decay of the Lyapunov function $\mathcal{P}$ in the transiently acceptable domain of $\mathcal{T} = \{x : v_k > V_{\text{tr}} > V_0/2\}$.

Lemma 2. *Whenever the capacitances on all the load buses satisfy*

$$C_k > \frac{\tau p_k^{\max}}{V_{\text{tr}}^2}, \tag{23}$$

one has $d\mathcal{P}(x(t))/dt \le 0$ for $x \in \mathcal{T}$.

Proof. This result follows directly from the relation $\dot{\mathcal{P}} = -\dot{x}^T \mathcal{Q} \dot{x}$ and positive definiteness of the matrix $\mathcal{Q}$ as defined in equation (5). □

These two results imply that any sublevel set of $\mathcal{P}$ inside $\mathcal{T}$ is invariant and any trajectory starting inside such a sublevel set converges to the equilibrium point x^{sep}. These sublevel sets are compact as the function $\mathcal{P}$ is bounded from below by a convex $\mathcal{G}$. To estimate the largest sublevel set of $\mathcal{P}$ that is contained in the transient domain $\mathcal{T}$ we prove the following theorem.

Theorem 2. *The function $\mathcal{P}$ evaluated at the boundary $\partial\mathcal{T}$ is bounded from below such that $\mathcal{P}(x) \ge \mathcal{G}_{\text{tr}}^+$ where*

$$\mathcal{G}_{\text{tr}}^+ = \frac{(V_{\text{tr}} - V_0)^2}{2R_{\max}} + p_\Sigma \log V_{\text{tr}} \tag{24}$$

Proof. Given $\mathcal{P} \ge \mathcal{G}$, it is sufficient to establish the bound on $\mathcal{G}$. Given that x belongs to the boundary $\partial\mathcal{T}$, there exists a load bus κ with $v_\kappa = V_{\text{tr}}$. Consider a path $\Pi \subset \mathcal{E}$ connecting the bus κ to some source in the system and define $R_\Pi = \sum_{(i,j)\in\Pi} R_{ij}$ Applying the same Jensen's inequality approach as in Theorem 1, we show that the potential $\mathcal{G}$ satisfies

$$\begin{aligned} \mathcal{G}(v) &= \sum_{(i,j)\in\mathcal{E}} \frac{(v_i - v_j)^2}{2R_{ij}} + \sum_{i\in\mathcal{L}} p_i \log v_i \\ &\ge \sum_{(i,j)\in\Pi} \frac{(v_i - v_j)^2}{2R_{ij}} + \sum_{i} p_i \log V_{\text{tr}} \\ &\ge \frac{(v_\kappa - V_0)^2}{2R_\Pi} + p_\Sigma \log V_{\text{tr}} \\ &\ge \frac{(V_{\text{tr}} - V_0)^2}{2R_{\max}} + p_\Sigma \log V_{\text{tr}} \end{aligned}$$

□

To certify transient stability of the system after the switching event, we estimate the corresponding value of the Lyapunov function $\mathcal{P}$. Specifically we consider an event when only one of the loads $\kappa \in \mathcal{L}$ experiences switching, changing its power from p_κ^- to p_κ^+. In this case, the following result bounds the value of the effective kinetic energy immediately after the switching event:

Lemma 3. *For a single load switching event in a network initially at equilibrium, where only one load $\kappa \in \mathcal{L}$ changes, the bound on the "kinetic energy" is given by*

$$\frac{\tau_{\max}}{2} \sum_{k\in\mathcal{L}} C_k \dot{v}_k^2 \le \frac{\tau_{\max}}{2C_\kappa} \left(\frac{p_\kappa^- - p_\kappa^+}{V_{\text{high}}^-}\right)^2 \tag{25}$$

Proof. Due to the continuity of the state variables, and relation (1), one has $L\dot{i}(t^-) = L\dot{i}(t^+) = 0$, implying that only the term that includes $\dot{v}$ contributes to the "kinetic energy." Given that $p_k \ge 0$ for all $k \in \mathcal{L}$ and $\dot{v}_k = 0$ for $k \ne \kappa$

$$\begin{aligned} \frac{\tau_{\max}}{2} \sum_{k\in\mathcal{L}} C_k \dot{v}_k^2 &= \frac{\tau_{\max}}{2} \frac{1}{C_\kappa} \left(-\frac{p_\kappa^+}{v_\kappa^-} - \sum_{\alpha\in\mathcal{E}} \nabla_{\alpha\kappa} i_\alpha\right)^2 \\ &\le \frac{\tau_{\max}}{2C_\kappa} \left(\frac{p_\kappa^- - p_\kappa^+}{v_\kappa^-}\right)^2 \\ &\le \frac{\tau_{\max}}{2C_\kappa} \left(\frac{p_\kappa^- - p_\kappa^+}{V_{\text{high}}^-}\right)^2 \end{aligned}$$

□

Next, we prove the following two lemmas to establish the bounds on the "potential energy" after the switching event.

Lemma 4. *At any equilibrium point, the potential $\mathcal{G}$ can be represented as*

$$\mathcal{G}(v) = \sum_{i\in\mathcal{L}} \left[\frac{p_i}{v_i}\frac{(V_0 - v_i)}{2} + p_i \log(v_i)\right] \tag{26}$$

Proof. Assume that i_k with $k \in \mathcal{V}$ are the nodal currents leaving the sources or the loads. The global current conservation law implies that

$$\sum_{k\in\mathcal{S}} i_k = -\sum_{k\in\mathcal{L}} i_k \tag{27}$$

On the other hand, whenever the voltage on all the source buses is given by V_0, and there is no capacitor charging/discharging current at equilibrium, it follows from the energy conservation that the energy produced by the sources is equal to energy consumed by the loads plus the energy dissipated in the lines, or more formally

$$\sum_{k\in\mathcal{S}} i_k V_0 = -\sum_{k\in\mathcal{L}} i_k v_k + \sum_{(i,j)\in\mathcal{E}} \frac{(v_i - v_j)^2}{R_{ij}} \tag{28}$$

Combining the definition (3) with the relations (27) and (28) one arrives at (26). □

Lemma 5. *Following an arbitrary or single-load switching event, the potential energy of the system immediately after the switching occurred is bounded as $\mathcal{G}^+ \leq \mathcal{G}^+_{\rm ini}$ where*

$$\mathcal{G}^+_{\rm ini} = \frac{p^-_\Sigma}{2}\frac{V_0 - V^-_{\rm high}}{V^-_{\rm high}} + p^+_\Sigma \log V_0 \tag{29}$$

Proof. Noting that Lemma 1 defines the lower bound of voltage level $v_k \geq V_{\rm high}(p^-_\Sigma)$, while the upper bound is $v_k \leq V_0$, we have

$$\begin{aligned}\mathcal{G}^+(v(t^-)) &= \sum_{i\in\mathcal{L}} \left[\frac{p^-_i}{v^-_i}\frac{(V_0 - v^-_i)}{2} + p^+_i \log(v^-_i)\right] \\ &\leq \frac{V_0 - V^-_{\rm high}}{2V^-_{\rm high}} \sum_{i\in\mathcal{L}} p^-_i + \log(V_0)\sum_{i\in\mathcal{L}} p^+_i \\ &\leq \frac{p^-_\Sigma}{2}\frac{V_0 - V^-_{\rm high}}{V^-_{\rm high}} + p^+_\Sigma \log(V_0)\end{aligned} \tag{30}$$

□

These two lemmas allow us to derive the following central result of this paper.

Theorem 3. *The system starting at stable equilibrium and experiencing an arbitrary single-load switching event returns back to a stable equilibrium point whenever the capacitors installed on every load satisfy:*

$$C_\kappa > \max_{p^-_\Sigma,\, p^+_\Sigma} C_{\kappa,\rm tr}\left(p^-_\Sigma, p^+_\Sigma\right) \tag{31}$$

$$\text{subject to} \quad \begin{aligned} &p^-_\Sigma \leq P_{\max} \\ &p^+_\Sigma \leq P_{\max} \\ &|p^+_\Sigma - p^-_\Sigma| \leq p^{\max}_\kappa \end{aligned}$$

where

$$C_{\kappa,\rm tr}\left(p^-_\Sigma, p^+_\Sigma\right) = \frac{\tau_{\max}}{2\left(\mathcal{G}^+_{\rm tr} - \mathcal{G}^+_{\rm ini}\right)}\left(\frac{p^-_\Sigma - p^+_\Sigma}{V^-_{\rm high}}\right)^2 \tag{32}$$

Proof. This follows directly from the combination of Lemmas 3, 5 and Theorem 2. □

Remark 1. *For each $V_{\rm tr}$, the sufficient bound for C defined by* (31) *exists only for small enough values of power, $p^{\max}_\kappa \leq P^{\rm crit}_\kappa$. Above those levels, stability cannot be certified as the initial energy $\mathcal{G}^+_{\rm ini}$ may exceed $\mathcal{G}^+_{\rm tr}$ for some admissible values of p^-_Σ, p^+_Σ.*

Remark 2. *Numerical simulation demonstrates that the worst case switching scenario (the scenario that maximizes $C_{\kappa,\rm tr}$) corresponds to $p^-_\Sigma = P_{\max} - p^{\max}_\kappa$ and $p^+_\Sigma = P_{\max}$, that is load κ switching on to its maximum power and bringing the total network loading to $P_{\max}$.*

The nature of the derivation above implies that the condition (31) is sufficient but not necessary. The following Lemma introduces a necessary condition:

Lemma 6. *For a system to maintain stability in an ad hoc setting, it is necessary that each load capacitance satisfies*

$$C_\kappa > \frac{\tau p^{\max}_k}{V^2_{\min}} \tag{33}$$

Proof. As follows from the discussion in section III, violation of this condition results in the loss of asymptotic stability for a two-bus system with load power $p^{\max} = P_{\max}$. □

Tighter necessary conditions (not shown) can be obtained numerically by simulating transients in specific networks, and these conditions are in close agreement with (33).

V. Discussion

A comparison of the lower bounds on C, normalized by $C_0 = \tau_{\max}/R_{\max}$, during a switching event is presented in Figure 3. The larger load demanded requires a larger capacitance to ensure stability. The small signal stability constraint (Eq. 33) is necessary while the others (Eqs. (2) and (31)) are sufficient. The gap between these constraints gives an indication of how conservative the sufficient bounds are.

The capacitance requirements imposed by the three constraints are affected by the choice of $V_{\rm tr}$. Increasing the levels of $V_{\rm tr}$ decreases the requirements imposed by (23) but increases the ones from (31). Furthermore, high levels of $V_{\rm tr}$ result in relatively small critical levels of load power as discussed in 1. For example, given $V_{\rm tr} = 0.66V_0$ as in Figure 3, the maximum load power consumption is $P^{\rm crit}_\kappa \approx 0.47P_0$. A trade-off therefore exists between power demanded and magnitude of the capacitance as well as between choice of $V_{\rm tr}$ and $P^{\rm crit}_k$.

The stability analysis conducted in this work allows for the development of a streamlined design process in which the rules suggested here can be applied offline, before

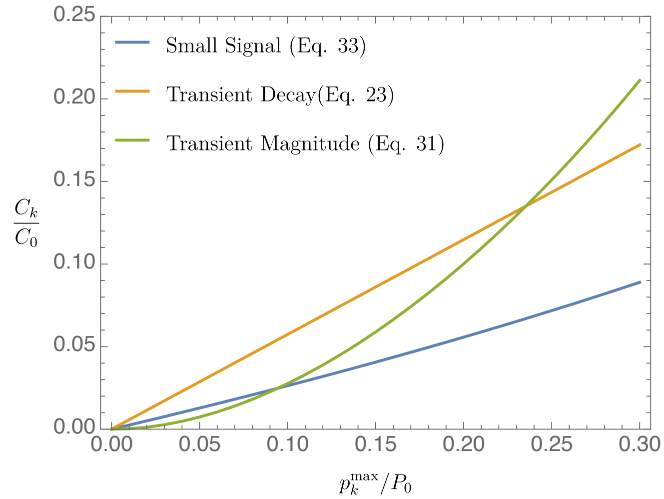

Fig. 3. Comparison of lower bounds on C_k/C_0 for $V_{\mathrm{tr}} = 0.66V_0$.

the network structure is fixed. The design process can be summarized as:

1) Define the acceptable voltage levels based on the converter constraints: nominal voltage V_0, minimum acceptable equilibrium voltage level $V_{\min}$, minimum acceptable voltage during transients V_{tr}.
2) Select system parameters: the upper bound on the maximum system loading $P_{\max}$ and the maximum line resistance $R_{\max}$ (determined by the line material, diameter, and length).
3) For each load k with maximal power $p_k^{\max}$ size the capacitance in accordance to (23) and (31).

Because this process is independent of the network topology, it does not need to be repeated for each community. Instead, it can be performed once and used to develop, for example, electricity access "kits": a set of modular power sources, loads, and lines that could be produced in bulk and easily adapted to the needs of individual communities without oversight.

VI. Conclusions and Path Forward

The development of ad hoc networks has been impeded by the difficulty of guaranteeing the stability of arbitrary interconnections of power electronic subsystems a priori. In this paper we have relied on a number of classical control theory techniques, most notably Brayton-Moser potentials, and recent approaches to the analysis of load flow equations, to develop conditions under which ad hoc dc microgrids are stable. In particular, we have expanded our small signal analysis from [3] to establish constraints on individual load capacitance values (Theorem 3) that make the system robust to large loading changes. These constraints allowed for the development of a simple design procedure to reduce planning and aid in the development of ad hoc DC microgrids.

This work introduces several exciting paths to be further explored. First is the generalization of the results to more detailed load and source models and characterization of the class of admissible power electronic controllers that can be used in the ad hoc setting. Similarly, more research is required to understand how the stability can be enforced in the presence of secondary control loops on sources [15], [3], [16] that aim to achieve economically optimal dispatch. The line inductances limit the acceptable levels of integral gains, and should be accounted for in the design of higher level control layers. Finally, practical restrictions on the network topology (for example, considering only networks with one source) may be able to provide tighter bounds on the required capacitance. In this paper, we have addressed some challenges inherent to ad hoc microgrids and hope that these challenges continue to be resolved, with the ultimate goal of enabling the large-scale deployment of truly decentralized power systems.

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