Optimal paths on the road network as directed polymers

Alex Solon, Guy Bunin, Sherry Chu, Mehran Kardar

[arXiv:1706.00489]
Directed polymer in a random medium (DPRM)

Statistics of a stretched chain in a disordered environment

- Fracture line of a torn paper sheet
- Domain wall in a 2d Ising model with random bonds
- Vortex lines in superconductors
Directed polymer in a random medium (DPRM)

Statistics of a stretched chain in a disordered environment

- Fracture line of a torn paper sheet
- Domain wall in a 2d Ising model with random bonds
- Vortex lines in superconductors

Mapping to:
- the fluctuations of a growing interface (KPZ equation)
- the noisy Burger equation
DPRM scaling

\[ E[h(x)] = \int_0^d \left[ \frac{\gamma}{2} \left( \frac{dh}{dx} \right)^2 + V(x, h) \right] dx \]

Random potential

\[ P[h] \propto \exp \left( -\frac{1}{k_B T} E[h] \right) \]
DPRM scaling

\[ E[h(x)] = \int_0^d \left[ \frac{\gamma}{2} \left( \frac{dh}{dx} \right)^2 + V(x, h) \right] dx \]

Random potential

\[ P[h] \propto \exp \left( -\frac{1}{k_B T} E[h] \right) \]

- \[ \langle (E - \bar{E})^2 \rangle \sim d^{2\beta} \]
- \[ \langle (h - \bar{h})^2 \rangle \sim x^{2\zeta} \]

(1+1)d: \( \beta = 1/3, \zeta = 2/3 \)

Short-range disorder
DPRM scaling

\[ E[h(x)] = \int_0^d \left[ \frac{\gamma}{2} \left( \frac{dh}{dx} \right)^2 + V(x, h) \right] dx \]

\[ P[h] \propto \exp \left( -\frac{1}{k_B T} E[h] \right) \]

\[ \langle (E - \bar{E})^2 \rangle \sim d^{2\beta} \]

\[ \langle (h - \bar{h})^2 \rangle \sim x^{2\zeta} \]

(1+1)d: \( \beta = 1/3, \zeta = 2/3 \)

Short-range disorder

Universal fluctuations

\[ \sigma P(E) \]
Optimal path on the road network

DPRM at $T = 0 \implies$ Configuration minimizing energy
Optimal path on the road network

DPRM at $T = 0 \iff$ Configuration minimizing energy
Optimal path on the road network

DPRM at $T = 0 \implies$ Configuration minimizing energy
Optimal path on the road network

DPRM at $T = 0 \implies$ Configuration minimizing energy

- Fastest route by car
- Shortest route in distance

Travel time or distance $\sim$ Energy of a DPRM
Questions

- Do optimal paths follow universal scaling laws?

$$V(x) = ?$$
Questions

Do optimal paths follow universal scaling laws?

What does $P(V)$ look like? → Look at short paths

Can we forget about the details of the network on large scale? → Increase distance
Fractal structure

DPRM model
[Halpin-Healy and Zhang, Physics Reports 95]

Shortest path from Munich
\[ d = 300 \text{km} \]
Short paths

Three data sets
Open Street Map
Open Source Routing Machine
Short paths

Three data sets
Open Street Map
Open Source Routing Machine

Paths between points at distance $d = 1\text{km}$
Short paths

Three data sets
Open Street Map
Open Source Routing Machine

Paths between points at distance $d = 1$km

Shortest path

Fastest path

Universal scaling at short distance!
Larger distances

\[ \frac{P(L|d)}{P_m} \]

\[ (L - L_m)/d^{\beta} \]

\[ \beta = 0.66 \]

\[ d = 3 \text{ km} \]
\[ d = 10 \text{ km} \]
\[ d = 30 \text{ km} \]
\[ d = 100 \text{ km} \]

Tracy-Widom

\[ \approx \] same distributions

Different exponents

\[ \beta \in [0.58, 0.75] \]
Larger distances

\[ \frac{P(L|d)}{P_m} \]

\[ \beta = 0.66 \]

\[ (L - L_m)/d^\beta \]

DPRM model with power-law noise

\[ \frac{P(E|d)}{P_m} \]

\[ \beta = 1/3 \]

\[ (E - E_m)/d^\beta \]
Larger distances

\[ \frac{P(L|d)}{P_m} \quad \frac{P(E|d)}{P_m} \]

\( \frac{(L - L_m)}{d^\beta} \approx \text{same distributions} \)

Different exponents

\( \beta \in [0.58, 0.75] \)

\( \beta = 0.66 \)

\( \beta = 1/3 \)
Transverse deviation

\[ \Delta h(x) - a(x) = \Delta h \sim x^{\zeta} \]

fixed \( d = 1000\text{km} \)

\( \zeta > \frac{2}{3} \)

Europe, \( \zeta = 0.72 \)

US, \( \zeta = 0.75 \)

Asia, \( \zeta = 0.69 \)
Transverse deviation

\[ \Delta h(x) - a \text{ (km)} \]

Europe, \( \zeta = 0.72 \)

US, \( \zeta = 0.75 \)

Asia, \( \zeta = 0.69 \)

\[ \Delta h \sim x^{\zeta} \]

fixed \( d = 1000 \text{km} \)

\( \zeta > 2/3 \)

Can we explain the different scaling exponents?
LR correlations in the noise change the exponents

$$\langle \eta(x, y)\eta(x', y') \rangle = |y - y'|^{2\rho-1}\delta(x - x')$$

[Chu and Kardar, PRE 2016]
Look at correlations of road density

Discretized maps

\[ C(r) = \langle \rho(x + r)\rho(x) \rangle - \langle \rho \rangle^2 \]
Look at correlations of road density

Discretized maps

\[ C(r) = \langle \rho(x + r)\rho(x) \rangle - \langle \rho \rangle^2 \]

Slow decay
Conclusion

- Universal power-law distribution at short scale
- Scaling laws
  Convergence to $\approx$ universal distribution on larger scale
- Non-universal exponents due to LR correlations