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Minimal Potentials with Very Many Minima

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We demonstrate, by construction, that simple renormalizable matrix potentials with S_N, as opposed to $O(N)$, symmetry can exhibit an exponentially large number of inequivalent deep local minima.

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There are many situations where behavior of great complexity arises, or is thought to arise, from simple underlying equations. Extensively studied cases include chaos, turbulence, and spin glasses. Chaos and turbulence involve long-term dynamics and extended spatial structures, while spin glasses involve an element of randomness. Here we will analyze a much simpler case (the simplest known to us) involving a static, deterministic, and very symmetrical system, wherein simple equations exhibit quite a complicated space of solutions. In particular, we present a simple class of potentials in n-component order parameters, whose number of local minima unrelated by symmetry grows exponentially in n. Our model is closely related to ones commonly used in studying large N limits of quantum field theory [1], differing only in that the assumption of some continuous symmetry among the fields [e.g., $O(N)$] is replaced by a discrete permutation symmetry (basically S_N). Of course, it is just such permutation symmetries which arise in studies of quenched disorder in long range models of spin glasses, so there is a close connection to that branch of spin glass theory [2,3]. In some circumstances the flexibility afforded by imposing less symmetry might allow better extrapolations than the traditional one, in the sense that $1/N$ corrections might be made smaller, and more complex behaviors captured.

To put the later results in perspective, and to highlight the minimal requirements for complexity in our framework, let us first consider an example that does not work. Suppose that we have N order parameters ϕ_i, for $i = 1, \ldots, N$. For definiteness, we focus our attention on no more than quartic potentials; such potentials are also renormalizable. The most general renormalizable potential symmetric under the S_N permuting these parameters, and under a change in all of their signs simultaneously, is

$$V(\vec{\phi}) = \mu \sum_i \phi_i^2 + \alpha \sum_{i,j} \phi_i \phi_j + \beta_1 \sum_i \phi_i^4 + \beta_2 \sum_{i,j} \phi_i^3 \phi_j + \beta_3 \sum_{i,j} \phi_i^2 \phi_j^2 + \beta_4 \sum_{i,j,k} \phi_i^2 \phi_j \phi_k + \beta_5 \sum_{i,j,k,l} \phi_i \phi_j \phi_k \phi_l . \quad (1)$$

Varying with respect to ϕ_a, we find that for an extremum ϕ_a must obey a cubic equation. The equation can be written in a form that is the same for every value of a. That is, for a particular fixed minimum, the equation can be written as a cubic equation in ϕ_a, so that the coefficients of the various terms (when evaluated as constant numbers for the particular minimum in question) are the same for all values of a. This cubic equation has at most three real roots. Of these, at most two are local minima. Therefore, for any local minimum, the different components of the order parameter will take at most two distinct values. Thus for large values of N many of the components will be equal. Let us suppose there are n_1 components with value r_1, and n_2 components with value r_2, where $n_1 + n_2 = N$ and $n_2 \leq n_1$. Then (for given n_1, n_2) the conditions for an extremum will be two polynomial equations of degree 3 in the variables r_1 and r_2. In general, these will have at most nine solutions. Taking into account that there are at most two solutions when $n_2 = 0$, for generic values of the parameters μ, α, and $\beta_1, \ldots, \beta_5$ in the potential we readily bound the number of distinct minima

depths by $(9N + 4)/2$ for N even, and $(9N - 5)/2$ for N odd. We expect that with more care this number could be further reduced. Nongeneric values presumably correspond either to fine tuning of the parameters, which is not physically realistic, or to enhanced symmetry, which renders mathematically distinct solutions physically equivalent. In any case, one does not find here a straightforward possibility for the exponential growth in the number of physically distinct minima that we will encounter shortly.

Instead of vector order parameters, let us now consider matrices. For simplicity we require that our matrices be symmetric, so M_{ij} and M_{ji} are the same variable. Each index runs from 1 to N; thus, there are $N(N + 1)/2$ independent order parameters in the matrix $\mathbf{M}$. We define the ath "row-column" of a matrix to be the union of the ath row and the ath column of the matrix; it is the set of all M_{ia}'s and all M_{ai}'s for all i's.

We assume the potential is symmetric under $\mathbf{M} \to -\mathbf{M}$ and under permutation of the values of the labels; none of the row-columns is to be singled out in any way. For example, one can take a matrix and, every time one sees index 3 in the matrix, replace it with index 7, and vice versa. Thus entries M_{37} and M_{73} stay the same, entries M_{33} and M_{77} get interchanged, and, for all other i's, M_{i3} swaps with M_{i7}, and M_{3i} swaps with M_{7i}. We refer to this symmetry as the "exchange symmetry" of the potential.

Given these constraints, the allowed quadratic terms in a potential are

$$M_{ii}M_{ii},\ M_{ii}M_{ij},\ M_{ii}M_{jj},\ M_{ij}M_{ij},$$
$$M_{ii}M_{jk}, M_{ij}M_{ik}, \quad \text{and} \quad M_{ij}M_{kl}\,. \tag{2}$$

Here, and hereafter, summation over all indices, even if they are not repeated, is always assumed. There are many allowed quartic terms. Terms as highly structured as $M_{ij}M_{jk}M_{kl}M_{li}$ and $M_{ii}M_{ij}M_{jk}M_{kl}$ are fair game now.

We now demonstrate, by explicit construction, exponential proliferation of inequivalent local minima in this case. Our strategy is to use a subset of the allowed terms to construct a very simple potential with many isolated local minima. These are equivalent under a symmetry of the simplified potential, but not under the smaller symmetry of our full class of allowed potentials. We then lift the degeneracy (and physical equivalence) of these minima in a controlled way be perturbing with additional allowed terms, in such a way that they remain local minima.

To begin, we form what we call a plastic-soda-bottle potential out of the allowed terms (in contrast to a wine-bottle potential, a plastic-soda-bottle potential in two variables has four symmetrically arranged dips):

$$V(\mathbf{M}) = a\left[\sum_{i,j}(1 - M_{ij}^2) + \sum_i (1 - M_{ii}^2)\right]^2 + b\left[\sum_{i,j}(1 - M_{ij}^2)^2 + \sum_i (1 - M_{ii}^2)^2\right], \tag{3}$$

where $a, b > 0$ are arbitrary, and no summation over the indices is assumed. All the local minima lie at:

$$\begin{bmatrix} \pm1 & \pm1 & . & . & . & \pm1 \\ \pm1 & \pm1 & & & & \pm1 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ \pm1 & \pm1 & . & . & . & \pm1 \end{bmatrix}. \tag{4}$$

They are all related by the accidental symmetry of the plastic-soda-bottle potential, which allows both independent changes in the signs of individual components and interchange of any two components (not just row-columns).

Now we experiment numerically by adding in more of the allowed terms. The positions of the minima, and their depths, change as we vary the amounts of the various small terms we are adding. We are careful that the terms we are adding are small enough so as not to destabilize any minimum or change the sign of any of the order parameters at the position of any of the minima. Let us add the terms of the form $M_{ij}M_{jk}M_{kl}M_{li}$ and $M_{ii}M_{ij}M_{jk}M_{kl}$ with small coefficients, with $N = 2$, 3, 4, 5, 6, and track the depth of each minimum numerically. We then count the number of distinct numerical values for the potential at the perturbed minima. Local minima with distinct energies must be physically inequivalent, i.e., unrelated by an underlying symmetry. The results are exhibited in Fig. 1; it appears that the number of distinct minima classes grows exponentially in the number of the order parameters, in response to only these two particular terms for the perturbations. Now we discuss how this proliferation can be understood.

Let us show that the number of minima which are not related by the exchange symmetry, or by the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry, grows exponentially in the number of the order parameters. We focus our attention on one very particular subset of all minima, and prove that the number of minima in this subset that is not related by any of the allowed

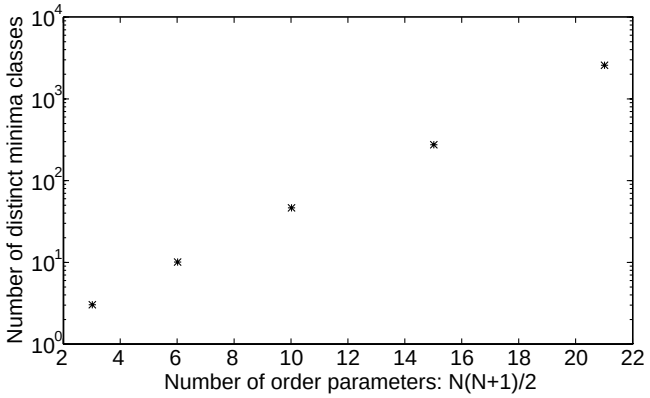

FIG. 1. On a semilog plot, the number of distinct minima classes versus the number of order parameters appears as a straight line. This is evidence that the number of distinct minima grows exponentially with the number of order parameters.

symmetries grows exponentially with N^2. This subset consists of all minima that can be written in the form

$$\begin{bmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{A^T} & \mathbf{C} \end{bmatrix}, \tag{5}$$

where, when N is even, all matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ have $N/2$ rows and $N/2$ columns; when N is odd, $\mathbf{B}$ has $(N+1)/2$ rows and $(N+1)/2$ columns, while $\mathbf{C}$ has $(N-1)/2$ rows and $(N-1)/2$ columns; consequently, $\mathbf{A}$ has $(N+1)/2$ rows and $(N-1)/2$ columns. Furthermore, we require that the matrix $\mathbf{B}$ has only positive values on the diagonal [from now on, denoted by $(+)$], while the matrix $\mathbf{C}$ has only negative values on the diagonal [from now on, denoted by $(-)$], while any other entry of $\mathbf{B}$ and $\mathbf{C}$ is "free" to be either $(+)$ or $(-)$ (as long as $M_{ij} = M_{ji}$). Note that the number of such free entries grows with N^2 for large N. Finally, all entries of the matrix $\mathbf{A}$ are fixed; if N is even, all elements on or above the diagonal are $(+)$'s, while all elements below the diagonal are $(-)$'s; if N is odd, entry A_{kl} is $(+)$ if $k \leq l$, and $(-)$ otherwise.

The reason we focus our attention on this particular subset is that none of its elements are related by the symmetries of our class of potentials, as we now discuss. The proof proceeds in two steps. First, ignoring the existence of the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry, we prove that exchange symmetry alone cannot change one member of a subset into another. Second, we prove that the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry does not cause any further problems.

We propose a "painting scheme" to keep track of where each entry of the matrix moves during the exchange process. Paint each row-column with a different color. Each M_{ij} for $i \neq j$ is covered with two layers of distinct paints; M_{ii} is covered with two layers of the same paint. Use "light colors" if $(+)$ is on the diagonal entry of the row-column you are painting, and use "dark colors" if $(-)$ is on the diagonal entry. Each entry M_{ij} for $i \geq j$ is now labeled uniquely by its two colors; of course, M_{ij} has the same colors as M_{ji}, but they are the same variable anyway.

Say the second row-column is yellow and the fifth row-column is green. Exchanging indices 2 and 5 makes the fifth row-column yellow and the second row-column green. Using the coloring scheme, it is easy to keep track of where each particular entry moved during the exchange. Say the eleventh row-column was blue initially, and we want to know where the entry $M_{2,11}$ ended up after the exchange, we look for the square of the matrix that is covered precisely by the yellow, and the blue, paint and conclude that the entry in question is now at the position $M_{5,11}$.

Note that every entry of the $\mathbf{B}$ matrix initially contains only light colors, while the matrix $\mathbf{C}$ contains only dark colors. In contrast, every entry of matrix $\mathbf{A}$ is painted with precisely one light and one dark color.

Now, we start with a matrix $\mathbf{M}_1$ and permute it into a matrix $\mathbf{M}_2$ so that both of these matrices are elements of our preferred subset. Note that all rows of $\mathbf{A}_2$ and $\mathbf{B}_2$ are painted with light colors, while all columns of $\mathbf{A}_2$ and $\mathbf{C}_2$ are painted with darker colors; this is so because $\mathbf{B}_2$ has only $+$'s on the diagonal, while $\mathbf{C}_2$ has only $-$'s on the diagonal. Therefore, the set of all entries of $\mathbf{A}_1$ is exactly the same as the set of all entries of $\mathbf{A}_2$; only these entries are such as to have exactly one light and one dark color. Suppose that the light colors we have are yellow, orange, red and pink, and suppose $N = 8$. Furthermore, suppose that $\mathbf{A}_1$ has the first row yellow, the second row orange, etc. Since two entries that were the same row-column before the exchanges stay in the same row-column after the exchanges, the only way to get exactly 4 $(+)$'s in the first row of $\mathbf{A}_2$ is to have the first row of $\mathbf{A}_2$ yellow. This implies that the first row-column of $\mathbf{M}_2$ is yellow. Furthermore, the only way to have exactly 3 $(+)$'s in the second row of $\mathbf{A}_2$ is to have the second row of $\mathbf{A}_2$ orange, implying that the second row-column of $\mathbf{M}_2$ is orange, etc. This way we determine the position of all light colors, and thereby determine uniquely everything about the matrix $\mathbf{B}_2$. In a similar manner, we determine everything about the matrix $\mathbf{C}_2$. Therefore $\mathbf{A}_1 = \mathbf{A}_2$, $\mathbf{B}_1 = \mathbf{B}_2$, and $\mathbf{C}_1 = \mathbf{C}_2$, so that $\mathbf{M}_1 = \mathbf{M}_2$. Therefore, there is no symmetry that relates any two elements of this particular subset.

The particular case $N = 8$ is just illustrative; what we said generalizes immediately to any even N, and with only minor modifications to the case where N is odd.

Now we prove that, during the whole process of transforming $\mathbf{M}_1$ into $\mathbf{M}_2$, one always has to multiply the matrix with -1 an even number of times. When N is odd, we have to end up with less $(-)$'s than $(+)$'s on the diagonal of $\mathbf{M}_2$, which is the same as for the diagonal we started with; however, none of the entries of the diagonal ever moves off the diagonal during the process. When N is even, we have to end up with less $(-)$'s than $(+)$'s in the matrix $\mathbf{A}_2$, and we already proved during Step 1 of this proof that $\mathbf{A}_1$ consists of the same set of elements as $\mathbf{A}_2$. Therefore, the matrix has to be multiplied with -1 an even umber of times during the process, both when N is odd and when N is even. Since the operation of multiplication with -1 treats all of the elements of the matrix indiscriminately, it does not matter at all when during the process we perform these operations; we could perform all of them before doing anything else, but then, we might as well not do them at all, since multiplying the matrix with -1 an even number of times leaves the matrix unchanged.

This concludes our proof that the number of local minima of the special potential that are unrelated by any symmetry of the general potential grows exponentially in the number of the order parameters for large N.

Physical intuition suggests that, unless two minima have a very good reason to have the same depths (e.g., an underlying symmetry of the full potential), generically one would not expect them to have equal depths. Since the potentials of our class support an exponentially large number of minima unrelated by symmetry, we expect that such potentials generally have a number of distinct depths at local minima that are exponential in the number of order

parameters, unless the equations that determine them are insensitive to the symmetry-breaking structure. That is the behavior indicated by our numerical work. It differs markedly from the vector case. The following consideration makes it plausible that the degeneracy among the physically distinct minima, which occurs for our initial plastic-soda-bottle potential, is lifted by perturbation with certain of the allowed potential terms. The derivative with respect to M_{ij} of a term such as $M_{ab}M_{bc}M_{cd}M_{da}$, that is, $M_{ib}M_{bc}M_{cj}$, probes the whole structure of $\mathbf{M}$ in a way this is significantly different for each value of ij. Unlike in the vector case, here the response to the perturbation, in principle, knows enough about (contains enough independent measures of) the order parameter to encode its detailed structure. In the vector case, one would need to go to Nth order terms, of the type $\phi_1\phi_2\cdots\phi_N$, or higher to encounter similar sensitivity.

We now examine the properties of some particular cases of our potentials, thus showing concretely how various minima become physically inequivalent.

To keep things as simple as possible, we add just a tiny perturbation to the initial plastic-soda-bottle potential. Because the perturbations are tiny, we are justified in evaluating the changes in the potential only to first order; we say that the depth of each minimum moves by whatever perturbation we are adding evaluates to at the original position of the minimum in question; these positions are given in (4). To first order, the degeneracy cannot be broken into an exponentially large number of minima classes; for example, a quartic term that involves as many as eight different indices can assume at most $\mathcal{O}(N^8)$ different values when evaluated at the positions given in (4), since it is a sum of N^8 terms, each of which can be either a $+1$ or a -1. Even if we add all of the allowed terms, each multiplied by an arbitrary tiny coefficient, at lowest order we still have at best a power law breaking of the degeneracy.

Nevertheless, the number of distinct minima one can get by analyzing only to first order is quite large, especially if we include many allowed terms to create the perturbation. Furthermore, for small perturbations, the expectation values of different operators typically do not differ significantly if we evaluate the changes in the depths only to first order, as opposed to evaluating them exactly. Moreover, in practice we sort the minima into energy bins of finite width in our plots. If our perturbation breaks the degeneracy to first order into, say, $\mathcal{O}(N^8)$ distinct minima classes and $N = 6$, we have, in principle, up to $\sim 10^6$ distinct minima. Our plots typically involve 200 bins; it does not matter that we evaluate the depth changes to first order only.

Typical results are displayed in Fig. 2. Plots (A) and (B) from that figure demonstrate that one can get quite a rich structure by using only a few of the allowed terms. Furthermore, the breaking of degeneracy is quite extensive even when we work to first order only. When we include more than one perturbative term, the degeneracy breaking is even bigger, producing quite a rich structure even at

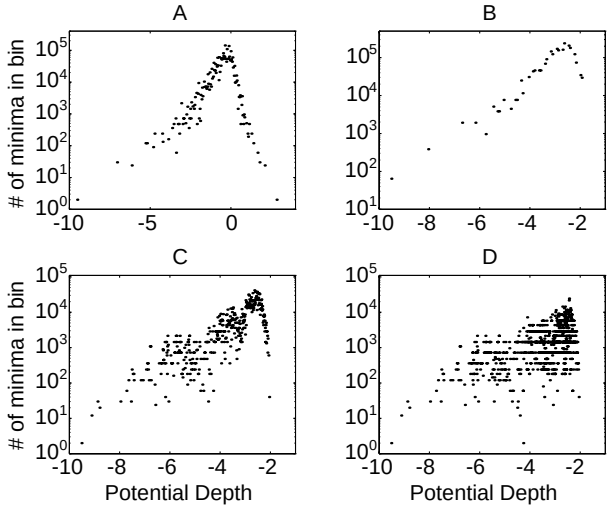

FIG. 2. Plots of bin occupation numbers versus the changes in minima depths, evaluated to first order in the small perturbations. All plots are for $N = 6$. The x axes are in arbitrary units. The width of the bins in plots (A)–(C) is 10^{-2} energy units, and in plot (D) is 2.5×10^{-4} energy units. The term added in plot (A) was $-M_{ii}M_{ij}M_{jk}M_{kl}$, while in plot (B) it was $-M_{ij}M_{jk}M_{kl}M_{li}$. To create plot (C), we add 20 different terms, with random coefficients multiplying them. Plot (D) is exactly the same as plot (C), except with much higher bin resolution.

first order. This is visible in plots (C) and (D) of Fig. 2, where we included 20 of the allowed terms, with random coefficients multiplying them. Plot (D) has a very high resolution of almost 40 000 bins for the whole plot; both plots are for exactly the same potential. Note that in these plots we count the total number of minima, so that minima are counted as distinct even if they are related by a symmetry. Thus, much of the degeneracy is intrinsic, and will not be broken in any order of approximation.

An example of the general type of structure described here arises in the analysis of QCD with many flavors of quarks at high density. For three flavors, the color-flavor locking condensate takes the form [4]

$$\langle q_a^\alpha q_b^\beta \rangle = U_\gamma^\alpha U_\delta^\beta (\kappa_1 \delta_a^\gamma \delta_b^\delta + \kappa_2 \delta_b^\gamma \delta_a^\delta), \qquad (6)$$

where the Greek indices refer to color and the Latin to flavor. For our present purposes we are suppressing various inessential complications (spin, chirality, momentum dependence), and emphasizing the existence of the matrix degree of freedom U, which parametrizes the degenerate vacua associated with the spontaneous symmetry breaking $\mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(3)_{\mathrm{flavor}} \to \mathrm{SU}(3)_{\mathrm{color+flavor}}$.

It seems that for $3k$ flavors the favored condensation is repeated color-flavor locking [5]. Thus we take the *ansatz:*

$$\langle q_a^\alpha q_b^\beta \rangle = \sum_{i=1}^{k} U_\gamma^{(i)\alpha} U_\delta^{(i)\beta} (\kappa_1 \delta_{a-3i+3}^\gamma \delta_{b-3i+3}^\delta + \kappa_2 \delta_{b-3i+3}^\gamma \delta_{a-3i+3}^\delta), \qquad (7)$$

corresponding to the symmetry breaking $SU(3)_{\text{color}} \times SU(3k)_{\text{flavor}} \rightarrow SU(3)_{\text{color+diagonal}} \times S_k$. The residual SU(3) acts on the flavor indices in blocks of 3, and the permutation symmetry S_k implements block interchanges.

Now the question arises as to how the energy depends on the relative alignment of the $U^{(i)}$. Nontrivial relative alignments violate the permutation symmetry. We will not attempt to determine here whether this actually occurs in the ground state, or in other low-lying states, but we do want to point out that to analyze this question one would need to consider potentials resembling those discussed above, featuring permutation rather than rotation symmetry in internal space. This case is intermediate in complexity between the vector and matrix cases discussed above, in that the permutation acts on a single index (as in the vector case), but the objects being permuted are chosen from a complicated manifold, rather than being a simple choice of sign. Symmetry-breaking correlations of the type $\langle U^{(i)} U^{(j)} \rangle \sim M^{(ij)}$ could produce an effective matrix structure in the permutation index.

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