Integer and Fractional Angular Momentum Borne on Self-Trapped Necklace-Ring Beams

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We present self-trapped necklace-ring beams that carry and conserve angular momentum. Such beams can have a fractional ratio of angular momentum to energy, and they exhibit a series of phenomena typically associated with rotation of rigid bodies and centrifugal force effects.

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Many nonlinear-wave systems can be described by the cubic nonlinear Schrödinger equation (NLSE). Solitons in the (1 + 1)D self-focusing version of this equation are stable, displaying interesting physics and applications, yet solitons of (2 + 1)D NLSE are highly unstable [1]. Recent papers [2,3] have proposed self-trapped (2 + 1)D beams that propagate in a stable fashion in a self-focusing Kerr medium: the necklace-ring beams. Necklace beams are shaped like rings whose thickness w is much smaller than their radius R and whose intensity is azimuthally periodically modulated (Fig. 1). Such beams exhibit stable self-trapped propagation for many (>50) physical diffraction lengths [2,3], even though circular (2 + 1)D solitons are inherently unstable in self-focusing Kerr media [1]. In necklace beams, it is the interaction between the spots that stabilizes the structure as a whole. As shown in [3], an isolated individual spot is highly unstable. Furthermore, removing a single spot from the necklace renders the entire necklace unstable. Necklace beams can be thought of as a superposition of two rings carrying equal but opposite topological charge. For such a superposition to exhibit stable propagation, the thickness of the ring must be significantly smaller than its radius, and the thickness must be larger than the azimuthal period. Moreover, the ring has to propagate as one entity, or else the spots walk off each other. In contrast to necklace beams, a single charge-carrying (ring) beam is highly unstable in any self-focusing medium: In a Kerr material it disintegrates, while in saturable nonlinear media it breaks into a number of solitons that can interact with one another [4] or fly off like free particles [5]. This behavior occurs also in quadratic media [6]. Yet necklace beams stay intact and display stable propagation, in Kerr as well as in saturable self-focusing media, if their parameters are chosen properly [2,3]. The stability of self-trapped necklaces (with properly chosen parameters) is unique in soliton science: A superposition of bound solutions (the rings with equal but opposite charge that make up the necklace) is stable, but its individual constituents are *unstable* [7]. Experiments with self-trapped necklaces have already been reported [8]. Here, we present self-trapped necklace beams that carry angular momentum. It is a rare case of self-trapped scalar bright beams that carry angular momentum [9]. In contrast to all known solitons, the angular momentum borne on such necklaces can be a noninteger multiple of the energy. We demonstrate angular momentum and centrifugal force effects.

Solitons that carry angular momentum have been studied in many systems: vortex solitons [10], 3D spiraling of solitons [11], composite solitons [12], quadratic solitons [13,14], and ring solitons in Kerr [15] and cubic-quintic [9] media. Experiments involving transfer of angular momentum carried by light to other forms of angular momentum have been performed [16]. The angular momentum of optical beams is typically associated with azimuthal phase modulation of $\exp(iM\theta)$ [17], which provides angular velocity to every part of the beam with respect to the beam center. The field of solitons is continuous wherever the amplitude is nonzero, that is, everywhere except for the origin. This is because a field discontinuity where the amplitude is nonzero renders the soliton highly unstable, even in a self-defocusing medium. For this reason of stability, for all vortex solitons [having $\exp(iM\theta)$], M is an integer [14]. For the same reason, for all other forms of single solitons carrying angular momentum, Mu is an integer. In the (2 + 1)D cubic self-focusing NLSE,

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2}\left\{\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right\} + |\psi|^2\psi = 0, \quad (1)$$

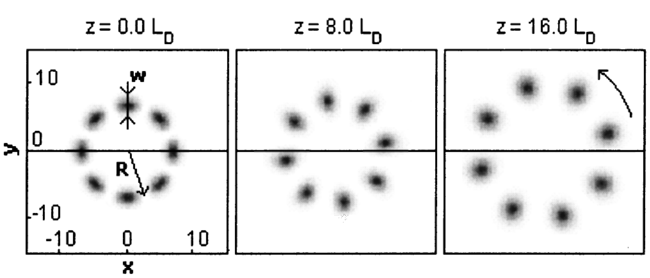

FIG. 1. A rotating necklace with integer L/E. We launch a necklace close to the self-trapped shape. It "breathes" for a short distance, until it reaches the equilibrium shape. This necklace slowly rotates as it propagates. Every necklace slowly expands as it propagates; nevertheless, as seen here, this does not stop the rotation. The input shape is $\sim\psi(r,\theta,z=0) = \mathrm{sech}(r-6.83)\cos(4\theta)\exp(i\theta)$. Dark means high intensity.

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the energy is $E = \iint |\psi|^2\, dx\, dy$ and the angular momentum is $L\hat{\mathbf{z}} = \frac{i}{2} \iint \mathbf{r} \times \{\psi\nabla\psi^* - \psi^*\nabla\psi\}\, dx\, dy$. Generically, any beam that can be written as $\psi(r,\theta) = f(r)\exp(iM\theta)$ has $L/E = M$. Since definitions of neither L or E depend on the nonlinearity, this holds for all nonlinearities. Thus, all optical solitons found so far carry an integer L/E. The angular momentum carried by such a beam, when averaged over the number of photons, is exactly $M\hbar$ per photon [17]. If the light is circularly polarized, the total angular momentum is modified by the spin contribution of $\pm\hbar$ per photon.

The fact that L/E is an integer can be intuitively understood by comparison with quantum mechanics (QM). All (paraxial) optical solitons are described by a Schrödinger-type equation. In QM, the solutions of this equation have a quantized angular momentum which is an integer multiple of $\hbar$, and the total probability $\iint |\psi|^2\, dx\, dy$ is normalized to 1. In classical optics $\iint |\psi|^2\, dx\, dy$ is the total power, which is proportional to the average number of photons. The quantization of L in QM resembles the fact that L/E is an integer for all *solitons* found thus far of a *classical* (2 + 1)D normalized NLSE. But L/E is an integer for solitons not due to quantization reasons, but because a noninteger L/E typically leads to a field discontinuity where the intensity is nonzero; such a discontinuity is thought to be unstable in self-focusing/defocusing media. Here we find self-trapped structures carrying noninteger L/E yet stably propagating for many diffraction lengths: necklace-ring quasisolitons that carry noninteger per-photon angular momentum.

Consider a necklace beam whose input shape is approximately $\psi(r,\theta,z=0) = f(r)\cos(\Omega\theta)$ [18], and add to it angular momentum by multiplying it by $\exp(iM\theta)$ (Ω, M integers). As long as M is reasonably smaller than Ω, we find (numerically) that this necklace is stable for more than 50 diffraction lengths L_D [19]. After 50 L_D we reach our computational limits, but it is plausible that the necklaces are stable for much larger distances. Such shapes have $L/E = M$. Since the symmetry between the spots must be preserved in a stable propagation, the angular momentum is manifested by the rotation of the entire necklace as it propagates (Fig. 1). We find numerically that quantization of L/E means that, for a necklace whose parameters are all fixed (except for its M), only certain angular velocities ω are allowed; these ω's are given by $\approx M/R^2$. Two necklaces that differ only in their radii (have the same M) differ in their ω's by a squared ratio of their radii. The fact that the allowed L/E's are quantized shows a connection between solitons and bound states in QM. Both of these systems are described by very similar wave equations and display several similar properties. The fact that some wave quantities that relate solitons and particles are necessarily quantized in optics was not appreciated so far.

Self-trapped necklace beams slowly expand as they propagate [3,4]. The expansion is a consequence of the net radial force exerted on each spot in the necklace. However, even though the necklace slowly expands, the dynamics is very different (and much slower) than diffractive dynamics: It is uniform and it preserves the shape of the necklace. Once angular momentum is added to a necklace beam, it expands faster. The expansion is still highly dominated by the internal dynamics of the necklace. Nevertheless, for two necklaces that differ only in their M's, the one with larger M expands noticeably faster. This implies that what we observe is actually a centrifugal force in a solitonic system. Furthermore, as a necklace beam expands, its L and E are conserved, implying that ω (the angular velocity) cannot be conserved. This is similar to a skater on ice: If she extends her hands while rotating, her ω decreases. We observe this with necklace beams (Fig. 2). Analytically, the angular phase has to be conserved; otherwise, because ω is quantized, the phase would discontinuously jump from, say, $\exp(i\theta)$ to $\exp(i2\theta)$, which is not physical in the continuous evolution describing the necklace propagation. Thus, as the necklace expands, ωR^2 is conserved. Our numerics confirm this prediction. One can develop a moment of inertia formulation for this system. The moment of inertia for necklaces is $I \approx ER^2$. Since $L = I\omega$ and E and L are conserved, ω has to go down with R^2. This is the first prediction of a "skater on ice" effect, which is so obvious in Newtonian mechanics but is unobserved yet in solitonic systems: the slowing down of angular velocity due to conservation of energy and angular momentum.

Given that, in analogy to QM, only integer L/E values are allowed, we recall that there are objects that carry angular momentum (in the form of spin) in multiples of $\hbar/2$ also. However, such spin is an internal degree of freedom and cannot be reproduced as a manifestation of a spatial property of a wave function. Nevertheless, even in QM, the expectation value of angular momentum can be a noninteger multiple of $\hbar$. We build on this idea to construct stable self-trapped beams that carry noninteger L/E. The necklaces described above have $\psi(r,\theta,z=0) = f(r)\{\exp[i(\Omega + M)\theta] + \exp[-i(\Omega - M)\theta]\}/2$. To create a necklace

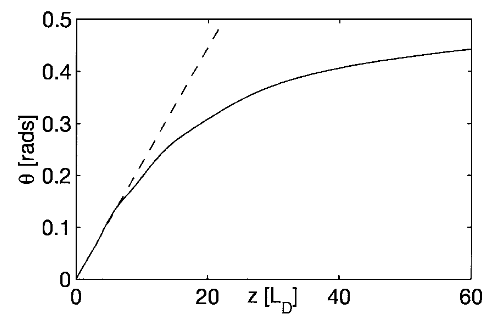

FIG. 2. The rotation angle of the expanding necklace of Fig. 1, as a function of the propagation distance. This particular necklace expands significantly as it propagates. The solid line represents the true instantaneous angle of rotation (measured numerically), whereas the dashed line represents what the instantaneous angle of rotation would have been if the angular velocity were a conserved quantity.

carrying noninteger L/E, we launch $\psi(r, \theta, z = 0) = f(r)\{\exp[i(\Omega + M)\theta] + \exp[-i(\Omega)\theta]\}/2$, as in Fig. 3. Such a necklace has $L/E = M/2$. For an odd M, this necklace has a noninteger L/E. Its intensity is given by $f^2(r)\{1 + \cos[(M + 2\Omega)\theta]\}/2$. In contrast to necklaces that have an even number of spots [3,4], a necklace that carries noninteger L/E has an odd number of spots. Furthermore, in a necklace with an integer (or zero) L/E, adjacent spots are mutually π out of phase (this is why such necklaces expand). This is not the case here since there is an odd number of spots. In order to preserve symmetry between the spots, the angular momentum is manifested in rotation of the necklace, and $\omega = M/(2R^2)$; thus ω is twice slower than for the corresponding necklaces of the previous paragraphs, keeping M and other parameters fixed. Our numerics confirm this prediction, and these necklaces are as stable as the usual necklaces: for many tens of L_D's. We investigate the stability of these necklaces using the same methods as in [2,3].

Another surprising feature is that, although the probability to find photons is not azimuthally symmetric (hence, the nonuniform azimuthal intensity), the local expectation value of L/E is azimuthally symmetric [20]. That is, we calculate analytically the ratio $L(\theta)/E(\theta)$ at $z = 0$ and find this ratio to be independent of θ, both in the case of the necklace with an even number of pearls (when it equals M) and in the case of a necklace with an odd number of pearls (when it is $M/2$). Because these necklaces rotate as rigid bodies we expect the $L(\theta)/E(\theta)$ not to change significantly during propagation. Thus, in a necklace with an odd number of spots, each photon contributes exactly $M\hbar/2$ the expectation value of the total angular momentum. One might think that a noninteger per-photon angular momentum is because different regions of the beam have different ratios of $L(\theta)/E(\theta)$, but this is not the case; since the shape of each spot is fixed as the necklace propagates, each part of the beam has the same angular velocity with respect to the center of the necklace.

Next, we construct a necklace carrying an *arbitrary* real per-photon angular momentum. Consider a necklace with $\psi(r, \theta, z = 0) = f(r)\{a\exp(iM\theta) + b\exp(iN\theta) + c\exp(-iP\theta) + d\exp(-iQ\theta)\}$; it has $L/E = (a^2M + b^2N - c^2P - d^2Q)/(a^2 + b^2 + c^2 + d^2)$, which can take *any real value.* Not all such necklaces are stable, but one can construct necklaces that are stable for many L_D's. Setting $N = P$, M, and Q to have similar values as N and P, b to be similar to c, and $a, d \ll b, c$, the necklace looks like a "usual" necklace, but with its envelope slightly azimuthally modulated. Small azimuthal perturbations do not destabilize necklaces: we find many such necklaces that are stable for more than $20L_D$, which is plenty for experimental observations. In Fig. 4, we show a necklace that has $d = 0$, $N = P = 8$, $M = 15$, $a = 1$, $b = 7$, and $c = 8$. Therefore, $L/E = -35/38$ for this necklace. As shown in Fig. 5, this necklace indeed has a shape similar to a usual necklace, but with a small azimuthal perturbation. It is interesting to note that in these necklaces the angular momentum is not manifested just in rotation of the necklace, but also in circulation of the modulation of the azimuthal envelope (Fig. 4): Neighboring spots exchange energy and perform a circulation of energy around the necklace, and this is the primary means of transporting the angular momentum upon the propagating beam. The reason for this distinctly different behavior of this necklace from the necklaces with integer or $M/2$ L/E values is symmetry. For a necklace with integer or $M/2$ L/E, the symmetry between the spots is conserved. Thus, if a symmetric necklace is to stay stable, the only way the angular momentum can be manifested is the rotation of the necklace as a whole (Figs. 1 and 3). In contrast, for a necklace described in this paragraph, the symmetry between spots is broken. Thus, spots are allowed to exchange energy and thereby carry angular momentum without a significant rotation of the necklace. Indeed, the frame of the necklace appears stationary (Fig. 4), yet the spots circulate the energy in a preferential direction corresponding to the sign and value of L/E.

The necklace of Figs. 4 and 5 has $L(\theta)/E(\theta)$ which depends on θ. Since in this necklace the spots are not

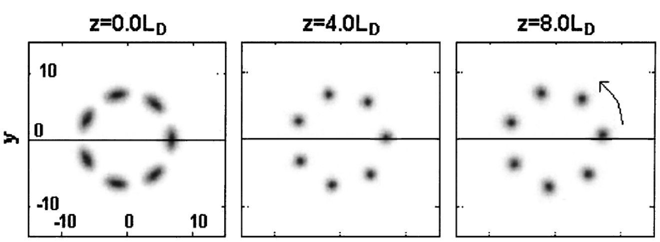

FIG. 3. A necklace with $L/E = 1/2$. This necklace slowly rotates as it propagates. The input shape is approximately $\psi(r, \theta, z = 0) = \text{sech}(r - 6.83)\{\exp(i4\theta) + \exp(-i3\theta)\}/2$.

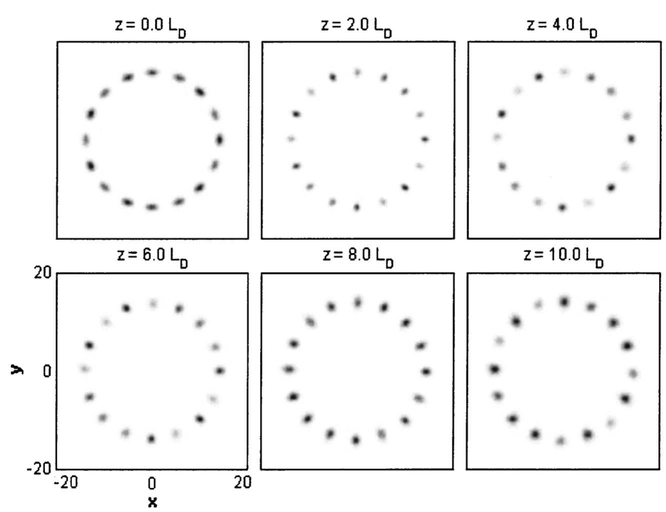

FIG. 4. A necklace with $L/E = -35/38$. This necklace is stable for $8L_D$. Necklaces with better stability are such that the energy exchange between the spots is slow, so it is not visible in a gray-level figure: for example, a necklace with $L/E = 261/1634$ that is stable for at least $50L_D$.

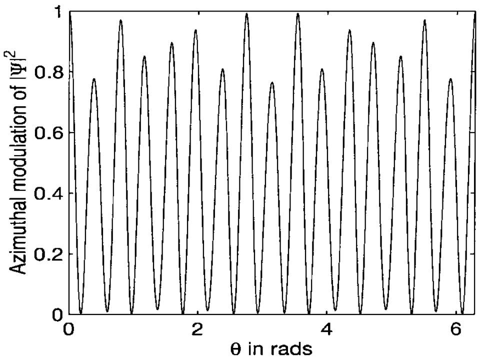

FIG. 5. Azimuthal intensity profile of the necklace of Fig. 4.

"rigid" as with the necklaces from Figs. 1 and 3, different parts of the necklace can have different angular velocities. Thus, the noninteger L/E of these beams does not imply that the angular momentum per photon is a noninteger multiple of $\hbar$. This is in contrast with the $M/2$ case, where the angular momentum per photon is $\hbar M/2$ everywhere.

Note that all necklaces described here are indeed self-trapped. If we start with a shape that is close to the equilibrium necklace shape, as long as this necklace is stable, both L and E are conserved: the L and E carried away by radiation are negligible compared to the initial L and E values. The self-trapped necklace conserves its initial L/E to even much better accuracy, although a tiny fraction of L and E is carried away through radiation.

To the best of our knowledge the necklaces described here are the only self-trapped shapes that have a noninteger per-photon angular momentum (in units of $\hbar$). Necklace beams can be constructed in many nonlinear wave equations. For example, one might think about converting the fractional angular momentum per particle carried by a necklace with $L/E = M/2$ into the angular momentum carried by the spin. This will imply rotation of the polarization state in optics, or spin-orbit interaction in a coherent system, such as a Bose-Einstein condensate. Such a conversion should be even more interesting when the necklaces are made of few photons only [21] (as opposed to a macroscopic number of photons [22]). Another exciting possibility is to investigate atomic necklaces in Bose-Einstein condensates.

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[18] $f(r)$ must go at least as fast as r^2 close to the origin, so no singularities appear when this ψ is substituted into the NLSE; this issue is explained in [3,4].

[19] The diffraction length L_D equals 1 normalized unit. If the nonlinearity is turned "off" in Eq. (1), then a beam of $\psi(x, z = 0) = \exp(-x^2/2)$ expands by $\sqrt{2}$ within $1L_D$.

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