Polychromatic partially spatially incoherent solitons in a noninstantaneous Kerr nonlinear medium

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We analytically and numerically find families of polychromatic partially spatially incoherent solitons in a noninstantaneous Kerr nonlinear medium and analyze their coherence properties. We find that the polychromatic incoherent solitons exist when higher temporal frequency constituents of the light are less spatially coherent than smaller temporal frequency constituents. © 2004 Optical Society of America

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1. INTRODUCTION

Optical spatial solitons made of incoherent light have drawn considerable attention since the experimental observations of solitons made of spatially incoherent light[1] and of white-light solitons.[2] In Ref. 1 the spatially incoherent quasi-monochromatic beam was generated with a laser light that was passed through a rotating diffuser to introduce spatial incoherence. In the subsequent experiment,[2] optical spatial solitons made of temporally and spatially incoherent light were demonstrated by use of an incandescent light bulb as the light source. The key requirement for the observation of these incoherent solitons is the noninstantaneous response of the nonlinear medium. The response time of the medium needs to be much longer than the characteristic time of phase fluctuations of incoherent light.[1,2]

The experimental results in Refs. 1 and 2 were followed by a flurry of theoretical studies of incoherent solitons.[3–21] Several theories for propagation of partially incoherent waves in noninstantaneous nonlinear media were used for the description of incoherent solitons. There are three seemingly different approaches—the coherent density theory,[4] the modal theory,[5] and the mutual coherence function theory[6]—that capture the essential physics involved; these theories were later shown to be equivalent.[7] An intuitive geometrical-optics approach useful for the description of big incoherent solitons has also been suggested.[8] Recently, a statistical physics approach based on a Wigner transform method was developed and used to analyze modulation instability and spatially incoherent solitons.[9] For the study of various types of spatially incoherent solitons in Kerr media,[10–16] the modal theory[5] has been especially fruitful. However, all these theoretical studies[3–21] have considered only spatially incoherent, but quasi-monochromatic (temporally coherent) solitons. Hence these studies are not able to describe the temporally and spatially incoherent solitons observed in Ref. 2.

The dynamics of self-trapping of temporally and spatially incoherent wave packets in noninstantaneous nonlinear media were recently studied numerically.[22] The characteristic features of the self-trapping process and the properties of temporally and spatially incoherent solitons were analyzed.[22] Analytical results regarding such incoherent wave packets were obtained only for the logarithmically saturable (model) nonlinearity.[23]

In this paper we present closed-form solutions representing a family of spatially incoherent solitons with a discrete temporal power spectrum in a Kerr medium. These polychromatic spatially incoherent solitons acquire a quasi-continuous temporal power spectrum when the characteristic width of the wave packet becomes sufficiently large. We use this closed-form solution to numerically identify temporally and spatially incoherent solitons with a continuous temporal power spectrum. The numerical procedure is based on the self-consistency method described in Ref. 5. From our analysis it follows that the polychromatic spatially incoherent solitons exist when higher temporal frequency constituents of the light are less spatially coherent than smaller temporal frequency constituents.

2. EQUATIONS OF MOTION

Let us describe the physical system under consideration. A light source emits spatially and temporally incoherent light continuously in time. Thus the incoherent wave packet is a continuous wave (cw) and not pulsed. For example, in Ref. 2 the light source was a quartz–tungsten–halogen incandescent bulb. A beam made of such temporally and spatially incoherent light enters the noninstantaneous nonlinear medium. The response time of the medium is much slower than the characteristic time of phase fluctuations upon the beam. Thus the medium is unable to follow fast phase fluctuations of incoherent light, but responds to the time-averaged intensity I. The time averages are taken over the response time of the medium. The time-averaged intensity I, and hence the induced index of refraction $\delta n(I)$, are in temporal steady state: $\partial \delta n(I)/\partial t = 0$. Thus equations of motion that describe the system need to describe the evolution of spatial and temporal coherence properties of light along the propagation (z) axis.

For the theoretical description of the problem, we use the linear second-order coherence theory in the space–frequency domain[24] by adding a proper nonlinear term.[25] By assuming linear polarization of the light, we can describe the instantaneous electric field by a complex scalar $\tilde{E}(x, z, t)$. [Because we intend to analyze only $(1+1)D$ solitons in Kerr nonlinearity, only one spatial coordinate x is considered. Namely, the propagation of $(2+1)D$ spatially incoherent beams in a Kerr medium leads to collapse.[26] Hence it is likely that polychromatic spatially incoherent solitons in a Kerr medium would be unstable as well.] The second-order coherence properties of the light are described by the mutual coherence function in the space–time domain[24]:

$$\Gamma(x_1, z_1, x_2, z_2; \tau) = \langle \tilde{E}^*(x_2, z_2, t_2)\tilde{E}(x_1, z_1, t_1)\rangle, \tag{1}$$

where $\langle ... \rangle$ denotes the time average and $\tau = t_1 - t_2$. In the space–frequency domain, the coherence properties are described by the mutual spectral density $\Gamma_\omega(x_1, z_1, x_2, z_2)$ that is defined as the Fourier transform of $\Gamma(x_1, z_1, x_2, z_2; \tau)$[24]:

$$\Gamma(x_1, z_1, x_2, z_2; \tau) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\omega \Gamma_\omega(x_1, z_1, x_2, z_2)\exp(-i\omega\tau). \tag{2}$$

We are interested in the correlation statistics of the field between points at the same transverse cross section of the beam, i.e., the points for which $z_1 = z_2$, and the spatial coordinates x_1 and x_2 differ. The correlation statistics in this plane are described by the mutual spectral density $B_\omega(x_1, x_2, z) = \Gamma_\omega(x_1, z, x_2, z)$. Under the paraxial approximation, the evolution of $B_\omega(x_1, x_2, z)$ is governed by an integrodifferential equation[25]:

$$\frac{\partial B_\omega}{\partial z} - \frac{i}{2k_\omega}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)B_\omega = \frac{ik_\omega}{n_0}\{\delta n[I(x_1, z)] - \delta n[I(x_2, z)]\}B_\omega(x_1, x_2, z), \tag{3}$$

where $I(x, z) = 1/2\pi \int_0^\infty \mathrm{d}\omega B_\omega(x, x, z)$ denotes the time-averaged intensity. The refractive index is $n^2(I) = n_0^2 + 2n_0\delta n(I)$, where n_0 and $\delta n(I)$ denote the linear and nonlinear contribution, respectively; $k_\omega = \omega n_0/c$.

In deriving the propagation equation [Eq. (3)] we assumed that the medium is dispersionless, i.e., that the linear part of the refractive index n_0 is independent of frequency. Because the coupling term $\delta n(I)$ is independent of time $\partial \delta n(I)/\partial t = 0$, and because Eq. (3) is in the frequency domain, we can include dispersion by substituting $n_0 \to n_0(\omega)$. In this paper, we neglect the effect of dispersion, which enables us to perform analytical calculations. Because the light is not pulsed, but continuously emitted in time (cw), and because $\partial \delta n(I)/\partial t = 0$, dispersion is negligible if $n_0(\omega)$ does not vary significantly over the frequency range.

The mutual spectral density contains information on the spatial coherence properties and the intensity profile at frequency ω. The intensity profile at frequency ω is $I_\omega(x, z) = B_\omega(x, x, z)$, whereas the spatial coherence properties are described in terms of the complex coherence factor at frequency ω[24]:

$$\mu_\omega(x_1, x_2, z) = \frac{B_\omega(x_1, x_2, z)}{[B_\omega(x_1, x_1, z)B_\omega(x_2, x_2, z)]^{1/2}}. \tag{4}$$

The quantity $\mu_\omega(x_1, x_2, z)$ is also referred to as the spectral degree of coherence at frequency ω, or complex degree of spatial coherence at frequency ω.[24]

For the study of Kerr nonlinearity, it is convenient to rewrite the equation of motion [Eq. (3)] in a different form. We use the coherent-mode representation of mutual spectral density[24] and write $B_\omega(x_1, x_2, z)$ as a superposition of modes[5,24,27]:

$$B_\omega(x_1, x_2, z) = \sum_{m=1}^{N_\omega} \lambda_m{}^\omega u_m{}^\omega(x_1, z)u_m{}^{\omega *}(x_2, z). \tag{5}$$

Here $u_m{}^\omega(x, z)$ denote the spatial profiles of the guided modes of the self-induced waveguide at frequency ω. The modes are mutually incoherent, that is, the excitation of one mode is uncorrelated, on the average, with the excitation of some other mode. In Eq. (5), $\lambda_m{}^\omega$ denotes the time-averaged mode-occupancy coefficients.[5] Generally, for every frequency ω, there are $N_\omega \geqslant 0$ different modes, i.e., given the frequency ω, the index m runs from 1 to N_ω. Because $B_\omega(x_1, x_2, z)$ must obey the evolution equation [Eq. (3)] by insertion of Eq. (5) into Eq. (3), it follows that each mode $u_m{}^\omega(x, z)$ must obey the following evolution equation:

$$\frac{\partial^2 u_m{}^\omega(x, z)}{\partial x^2} + 2ik_\omega\frac{\partial u_m{}^\omega}{\partial z} + \frac{2\,\delta n(I)k_\omega{}^2}{n_0}u_m{}^\omega = 0. \tag{6}$$

The time-averaged intensity in terms of modes is $I(x, z) = (2\pi)^{-1}\int_0^\infty \mathrm{d}\omega \Sigma_m \lambda_m{}^\omega |u_m{}^\omega(x, z)|^2$.

Equation (6) is derived to be equivalent to Eq. (3) in the following sense.[7] Given some initial conditions $u_m{}^\omega(x, z = 0)$, from Eq. (5) we know the initial mutual spectral density $B_\omega(x_1, x_2, z = 0)$. By using the evolution equation for the modes [Eq. (6)], we can find the mode profiles $u_m{}^\omega(x, z)$ after some propagation distance z; from these mode profiles, by invoking Eq. (5), we can reconstruct the mutual spectral density $B_\omega(x_1, x_2, z)$. We could have obtained the same value $B_\omega(x_1, x_2, z)$ by using Eq. (3) directly and evolving mutual spectral density from the same initial condition $B_\omega(x_1, x_2, z = 0)$ up to the propagation distance z. In other words, the two approaches, the modal approach given by Eq. (6) and the mutual spectral density approach given by Eq. (3), are equivalent.[7] In fact, Eq. (6) is an extension of a fruitful multimode approach utilized for the description of spatially incoherent, but temporally coherent, solitons.[5,7] This extension allows for a broad frequency power spectrum, and we use it to describe polychromatic spatially incoherent solitons in a Kerr medium.

3. PARTIALLY SPATIALLY INCOHERENT SOLITON WITH A DISCRETE TEMPORAL SPECTRAL DENSITY IN A KERR MEDIUM

In this section we present the closed-form solitary-wave solution of Eq. (6) that represents a family of $(1 + 1)$D polychromatic partially spatially incoherent solitons in a Kerr medium. The nonlinear response of the medium is $2n_0\delta n(I) = n_2 I$, i.e., $n^2(I) \simeq n_0{}^2 + n_2 I$. The temporal spectral density of this soliton(s) is discrete, that is, this soliton is constructed from a number of partially spatially incoherent quasi-monochromatic waves that are propagating at different carrier frequencies: $\omega_1, \omega_2, \ldots, \omega_{N_f}$, where N_f denotes the number of frequency constituents within the beam.

For the purposes of this section, it is convenient to rewrite Eq. (6) so as to describe only evolution of light in a Kerr medium $2n_0\delta n(I) = n_2 I$ and with a discrete temporal power spectrum:

$$\frac{\partial^2 u_m{}^j(x, z)}{\partial x^2} + 2ik_j\frac{\partial u_m{}^j}{\partial z} + \frac{n_2 k_j{}^2}{n_0{}^2} I(x, z) u_m{}^j = 0, \quad (7)$$

where

$$I(x, z) = \sum_{j=1}^{N_f}\sum_{m=1}^{N_j} \lambda_m{}^j |u_m{}^j(x, z)|^2. \quad (8)$$

In Eq. (7), the index j denotes the frequencies $\omega_j \to j$, $k_j = n_0\omega_j/c$. The index j runs from 1 to N_f. The index m runs from 1 to N_j, where $N_j \equiv N_{\omega_j}$ denotes the number of different mutually incoherent modes at a particular frequency ω_j. If $N_f = 1$, i.e., if the beam is constructed from just one frequency, Eq. (7) represents the Manakov system of N_j ($j = 1$) equations.[28] Hence, for $N_f > 1$, Eq. (7) can be regarded as a generalization of the Manakov set of equations, which includes the possibility of evolving light with a number of different frequencies.

We seek soliton solutions in the form $u_m{}^j(x, z) = \nu_m{}^j(x)\exp(i\kappa_m{}^j z)$, where $\nu_m{}^j(x)$ represents the spatial profile of the mth mode at frequency ω_j, and $\kappa_m{}^j$ is the propagation constant of this mode. When $u_m{}^j(x, z) = \nu_m{}^j(x)\exp(i\kappa_m{}^j z)$ is inserted in to Eq. (7), we obtain a Schroedinger-type equation for the modal functions $\nu_m{}^j(x)$:

$$\frac{d^2\nu_m{}^j(x)}{dx^2} - 2k_j\kappa_m{}^j\nu_m{}^j + \frac{n_2 k_j{}^2}{n_0{}^2} I(x)\nu_m{}^j = 0, \quad (9)$$

where

$$I(x) = \sum_{j=1}^{N_f}\sum_{m=1}^{N_j} \lambda_m{}^j |\nu_m{}^j(x)|^2. \quad (10)$$

The potential in this Schroedinger equation is determined by the intensity profile $I(x)$, which is, in turn, determined by the modes $\nu_m{}^j(x)$ themselves. In other words, for the soliton to exist, the self-consistency loop must be closed.

Let us present a closed-form soliton solution of Eq. (7). This soliton draws on the closed-form solutions for the partially spatially incoherent, but temporally coherent, solitons found by Carvalho *et al.* in Ref. 14. Here we identify a family of polychromatic spatially incoherent solitons with the intensity profile

$$I(x) = I_0 \operatorname{sech}^2(x/x_0). \quad (11)$$

Thus every soliton within the family has an intensity profile that is secant hyperbolic squared. The frequencies that constitute this polychromatic soliton are specified by the following relation (see Ref. 14):

$$\omega_j = \frac{c}{x_0}\left[\frac{j(j+1)}{n_2 I_0}\right]^{1/2}, \qquad j = 1, 2, \ldots, N_f. \quad (12)$$

When the expressions for the intensity profile and the frequencies ($k_j = n_0\omega_j/c$) from Eqs. (11) and (12), respectively, are inserted into Eq. (9), it follows that the propagation constants are[14]

$$\kappa_m{}^j = \frac{m^2}{2}\frac{1}{k_j x_0{}^2} = \frac{m^2}{2}\left[\frac{n_2 I_0}{j(j+1)}\right]^{1/2}\frac{1}{n_0 x_0}, \qquad m = 1, 2, \ldots, j, \quad (13)$$

whereas the mode profiles $\nu_m{}^j(x)$ must obey[14]

$$\frac{d^2\nu_m{}^j}{d(x/x_0)^2} + [j(j+1)\operatorname{sech}^2(x/x_0) - m^2]\nu_m{}^j = 0. \quad (14)$$

From Eq. (13) it follows that the number of modes that are used to describe the mutual spectral density at a frequency ω_j is $N_j = j$.[14] Furthermore, from Eq. (14) it follows that the profiles of the modes $\nu_m{}^j(x)$ are given by the Legendre polynomials [see Eqs. (4) and (6) in Ref. 14]:

$$\nu_m{}^j(x) \propto P_j{}^m\left(\tanh\frac{x}{x_0}\right). \quad (15)$$

Most importantly, from Ref. 14 it follows that the modal weights $\lambda_m{}^j$ corresponding to each frequency ω_j can be adjusted so that

$$\sum_{m=1}^{N_j} \lambda_m{}^j |\nu_m{}^j(x)|^2 = I_j \operatorname{sech}^2(x/x_0), \quad (16)$$

where I_j is determined by the power contained within the ω_j frequency constituent of the beam. Equation (16) means that the modal weights $\lambda_m{}^j$ at every frequency ω_j can be adjusted so that the intensity profile of every frequency constituent of the beam is $\operatorname{sech}^2(x/x_0)$. This is a consequence of a particular choice of total intensity profile [Eq. (11)] and the choice of frequency constituents ω_j [Eq.

(12)]. To obtain the total intensity profile from the modes, we must perform a sum of Eq. (16) over all frequencies ω_j [see Eq. (8)]:

$$I(x) = \sum_{j=1}^{N_f} \sum_{m=1}^{N_j} \lambda_m{}^j |\nu_m{}^j(x)|^2 = \sum_{j=1}^{N_f} I_j \operatorname{sech}^2(x/x_0). \tag{17}$$

Because we started the calculation by assuming that the intensity profile is $I(x) = I_0 \operatorname{sech}^2(x/x_0)$ [see Eq. (11)], the only remaining condition that needs to be satisfied for the self-consistency loop to be completed, that is, for Eq. (10) to hold, is

$$\sum_{j=1}^{N_f} I_j = I_0. \tag{18}$$

Thus in Eqs. (11)–(18) we have constructed an analytical solution representing a family of polychromatic partially spatially incoherent solitons in a noninstantaneous nonlinear Kerr medium. Because every frequency constituent has an intensity profile that is $\propto \operatorname{sech}^2(x/x_0)$, the number of frequency constituents within the soliton depends on our choice. Namely, we can choose N_f to be arbitrarily large and then choose the peak of intensity profile at frequency ω_j, that is, $I_j \geqslant 0$ almost at will, because only the condition of Eq. (18) needs to be satisfied. For example, we can construct a soliton with two frequencies, say ω_2 and ω_6, by putting all the power within these two constituents, which means that $I_2 + I_6 = I_0$; in this case all other values are $I_j = 0$ for $j \neq 2$ and $j \neq 6$. It should be noted that the construction of these spatially incoherent solitons with discrete temporal power spectrum by a combination of quasi-monochromatic solitons found by Carvalho *et al.*[14] corresponds to the linear superposition principle for partially spatially incoherent solitons described in Ref. 21.

For the sake of the clarity of the exposition, let us explicitly write down the polychromatic spatially incoherent soliton constructed from four frequencies, ω_2, ω_3, ω_4, and ω_5, given by Eq. (12). Denoting $S = \operatorname{sech}(x/x_0)$ and $T = \tanh(x/x_0)$, the normalized profiles

$$\tilde{\nu}_m{}^j(x) = \left(\frac{\lambda_m{}^j}{I_j}\right)^{1/2} \nu_m{}^j(x) \tag{19}$$

are as follows[14]:

$$\text{For } j = 2, \qquad \tilde{\nu}_2{}^2(x) = S^2, \qquad \tilde{\nu}_1{}^2(x) = ST;$$

$$\text{for } j = 3, \qquad \tilde{\nu}_3{}^3(x) = \sqrt{\frac{15}{16}}S^3,$$

$$\tilde{\nu}_2{}^3(x) = \sqrt{\frac{5}{2}}S^2T,$$

$$\tilde{\nu}_1{}^3(x) = \frac{1}{4}S(4 - 5S^2).$$

$$\text{For } j = 4, \qquad \tilde{\nu}_4{}^4(x) = \sqrt{\frac{7}{8}}S^4,$$

$$\tilde{\nu}_3{}^4(x) = \frac{3}{4}\sqrt{7}S^3T,$$

$$\tilde{\nu}_2{}^4(x) = \frac{\sqrt{2}}{4}S^2(6 - 7S^2),$$

$$\tilde{\nu}_1{}^4(x) = \frac{1}{4}ST(4 - 7S^2).$$

$$\text{For } j = 5, \qquad \tilde{\nu}_5{}^5(x) = \frac{\sqrt{210}}{16}S^5,$$

$$\tilde{\nu}_4{}^5(x) = \frac{1}{2}\sqrt{21}S^4T,$$

$$\tilde{\nu}_3{}^5(x) = \frac{\sqrt{42}}{16}S^3(8 - 9S^2),$$

$$\tilde{\nu}_2{}^5(x) = \frac{\sqrt{7}}{2}S^2T(2 - 3S^2),$$

$$\tilde{\nu}_1{}^5(x) = \frac{1}{8}S(8 - 28S^2 + 21S^4).$$

These modes induce the potential $I(x)$ that traps them:

$$\begin{aligned} I(x) &= \sum_{j=2}^{5} \sum_{m=1}^{j} \lambda_m{}^j |\nu_m{}^j(x)|^2 \\ &= \sum_{j=2}^{5} \sum_{m=1}^{j} I_j |\tilde{\nu}_m{}^j(x)|^2 \\ &= (I_2 + I_3 + I_4 + I_5)\operatorname{sech}^2(x/x_0) \\ &= I_0 \operatorname{sech}^2(x/x_0), \end{aligned} \tag{20}$$

i.e., $I_2 + I_3 + I_4 + I_5 = I_0$, which is a particular case of Eq. (18).

The spatial coherence properties at frequency ω are described in terms of the complex coherence factor μ_ω at frequency ω [see Eq. (4)]. The complex coherence factors $\mu_\omega(x, 0)$ are plotted in Fig. 1 at frequencies ω_2, ω_3, ω_4, and ω_5. Evidently, the characteristic width of the complex coherence factor is smaller at larger frequencies. This is attributed to the fact that the number of modes at frequency ω_j is $N_j = j$, that is, N_j increases with the increase of frequency ω_j, and generally the multimode spatially incoherent soliton is more spatially incoherent if the number of excited modes is larger.[3] This means that, for this soliton to exist, the spatial correlation distance must decrease with the increase of frequency.

Let us show that, in the limit of big incoherent solitons,[8] the discrete temporal power spectrum of the polychromatic soliton becomes quasi-continuous. Suppose that we use light with the temporal power spectrum restricted to the interval $[\Omega_{\min}, \Omega_{\max}]$. From Eq. (12) it follows that only a certain number of frequencies ω_j,

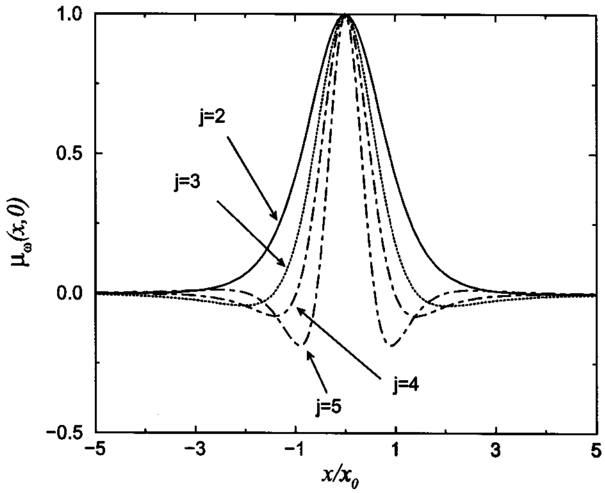

Fig. 1. Complex coherence factors $\mu_\omega(x, 0)$ of the spatially incoherent soliton with the discrete temporal power spectrum. The function μ_ω is plotted for frequencies ω_j, where $j = 2, 3, 4, 5$ [see Eq. (12)]. The characteristic width of $\mu_\omega(x, 0)$ is evidently smaller at larger frequencies ($\omega_2 < \omega_3 < \omega_4 < \omega_5$).

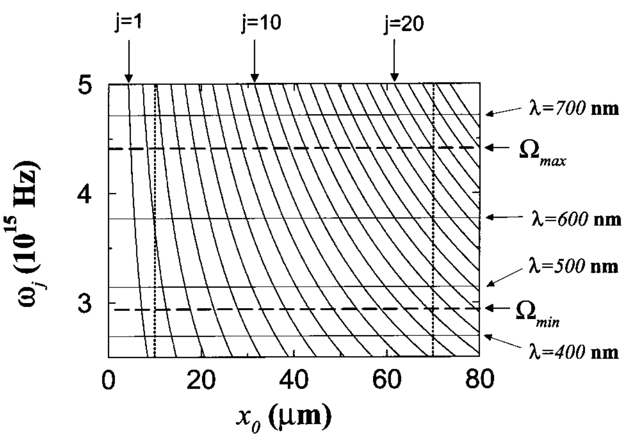

Fig. 2. Frequencies ω_j [see Eq. (12)] versus the characteristic width x_0. Two horizontal dashed lines show the frequencies $\Omega_{\min}$ and $\Omega_{\max}$. Four thin horizontal lines represent the frequency values corresponding to 400, 500, 600, and 700 nm. Two vertical dotted lines are placed at $x_0 = 10$ and 70 μm. We can see that the number of ω_j values within the interval $[\Omega_{\min}, \Omega_{\max}]$ increases with the increase of x_0. The separation between adjacent frequencies decreases with an increase of x_0.

where $j = j_{\min}, j_{\min} + 1,..., j_{\min} + j_{\text{total}}$, are within the interval $[\Omega_{\min}, \Omega_{\max}]$. This is pictorially presented in Fig. 2, which shows ω_j versus x_0. For Fig. 2, $\Omega_{\min} = 2.94 \times 10^{15}$ Hz ($\Omega_{\max} = 4.41 \times 10^{15}$ Hz), which corresponds to a 641-nm (427-nm) vacuum wavelength. The strength of the nonlinearity is $n_2 I_0 = 0.0004$. If $x_0 = 10\,\mu$m, there is just one frequency ω_j (the one for $j = 2$) within the interval $[\Omega_{\min}, \Omega_{\max}]$; the interval $[\Omega_{\min}, \Omega_{\max}]$ is shown as two dashed horizontal lines. However, if $x_0 = 70\,\mu$m, there are seven frequencies ω_j (the ones for $j = 14,15,..., 20$) within the interval $[\Omega_{\min}, \Omega_{\max}]$. Given the parameters n_2 and I_0, both $j_{\min}$ and j_{total} increase with the increase of the characteristic width of the soliton (see Fig. 2). As x_0 increases, the spacing between adjacent frequency constituents is approximately

$$\omega_{j+1} - \omega_j \simeq \frac{c}{x_0 \sqrt{n_2 I_0}}. \tag{21}$$

We can see that the separation between adjacent frequencies that constitute the frequency power spectrum decreases with the increase of the characteristic width x_0 of the soliton. When we sufficiently increase x_0, the width $\omega_{j+1} - \omega_j$ becomes sufficiently small, and the discrete spectrum can be regarded as quasi-continuous. Thus, in the limit of big incoherent solitons, our solution represents the polychromatic incoherent solitons, with the quasi-continuous spectrum of light in the Kerr-like nonlinearity.

4. TEMPORALLY AND SPATIALLY INCOHERENT SOLITON WITH A CONTINUOUS TEMPORAL SPECTRAL DENSITY IN A KERR MEDIUM

In this section we find a temporally and spatially incoherent soliton with a continuous temporal spectral density in a Kerr medium by using numerical methods. The temporal power spectrum of this soliton is broad, and therefore it represents a white-light soliton, such as the one observed in Ref. 2. Equation (6) is solved self-consistently by use of the method described in Ref. 5. The temporal power spectrum of this soliton is spanned in the interval $[\Omega_{\min}, \Omega_{\max}]$. For the purposes of numerical calculations, the frequency-domain interval is discretized on a uniform grid of N_f points. The frequencies from the grid are

$$\omega_j = \left(j - \frac{1}{2}\right)\Delta\omega, \qquad j = 1, 2,..., N_f,$$

where

$$\Delta\omega = (\Omega_{\max} - \Omega_{\min})/N_f. \tag{22}$$

In fact, because we discretize the frequency domain, we are actually solving the discretized equation [Eq. (7)] numerically. However, by gradually reducing the separation between the frequency constituents $\Delta\omega$, we check whether the soliton found numerically converges to the continuous spectral density soliton as $\Delta\omega \to 0$. In that sense, the numerically found soliton solution of Eq. (7) represents the soliton solution of Eq. (6).

Let us present the white-light soliton in a noninstantaneous Kerr medium. The induced nonlinearity at the peak of the soliton is $n_2 I_0 = 0.0004$, and the soliton width is $x_0 = 10\,\mu$m. The temporal power spectrum is rectangular (i.e., the spectral density is uniform) within the interval $[\Omega_{\min}, \Omega_{\max}] = [2.94, 4.41] \times 10^{15}$ Hz. The width of the temporal power spectrum is $(\Omega_{\max} - \Omega_{\min})/\omega_c = 0.4$; ω_c denotes the central frequency $\omega_c = (\Omega_{\max} + \Omega_{\min})/2$. For the numerical calculations, the frequency-domain interval is divided into $N_f = 81$ frequency constituents, i.e., the grid refinement is $\Delta\omega/\omega_c \simeq 0.005$. Because the same features of the white-light soliton are observed as the grid refinement is gradually reduced from $\Delta\omega/\omega_c \sim 0.04$ down to $\Delta\omega/\omega_c \simeq 0.005$, we conclude that the numerically found solution represents the spatially and temporally incoherent soliton, with a truly continuous spectral density.

Figure 3 shows the intensity profile $I(x)$ of this soliton. It differs slightly from the $\mathrm{sech}^2(x/x_0)$ intensity profile of the analytically obtained polychromatic soliton. Figure 4 shows the complex coherence factor at the representative frequencies $\Omega_{\min}$, ω_c, and $\Omega_{\max}$. We can see that the width of the complex coherence factor, which corresponds to the spatial correlation distance $l_s(\omega)$,[24] is smaller for higher frequencies. The same feature is observed in the analytic example above (see Fig. 1). The intensity profile at a particular frequency ω,

$$I_\omega(x) = (2\pi)^{-1} \sum_{m=1}^{N_f} \lambda_m{}^\omega |u_m{}^\omega|^2, \tag{23}$$

is displayed in Fig. 5 at three representative frequencies $\Omega_{\min}$, ω_c, and $\Omega_{\max}$. We observe that the intensity profile $I_\omega(x)$ is more narrow, with larger peaks at higher frequencies. This differs from the discrete spectral density soliton studied in Section 3. Namely, for that soliton

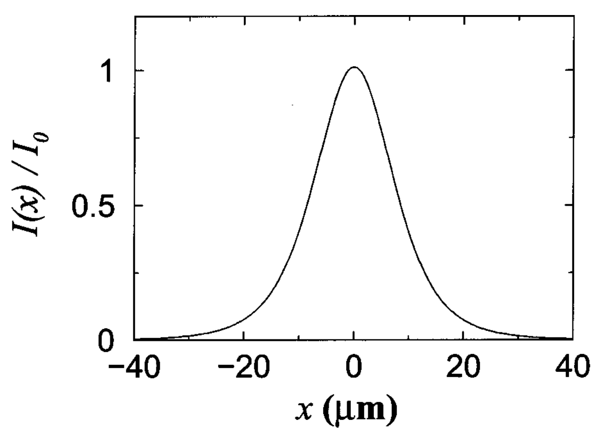

Fig. 3. Normalized intensity profile $I(x)/I_0$ of the incoherent white-light soliton in a noninstantaneous Kerr medium.

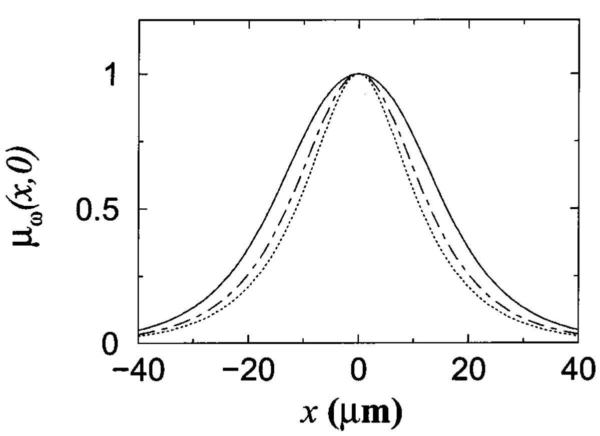

Fig. 4. Complex coherence factors $\mu_\omega(x, 0)$ of the incoherent white-light soliton for three representative frequencies $\Omega_{\min}$ (solid curve), ω_c (dotted–dashed curve), and $\Omega_{\max}$ (dotted curve). The functions $\mu_\omega(x, 0)$ are narrower at larger frequencies.

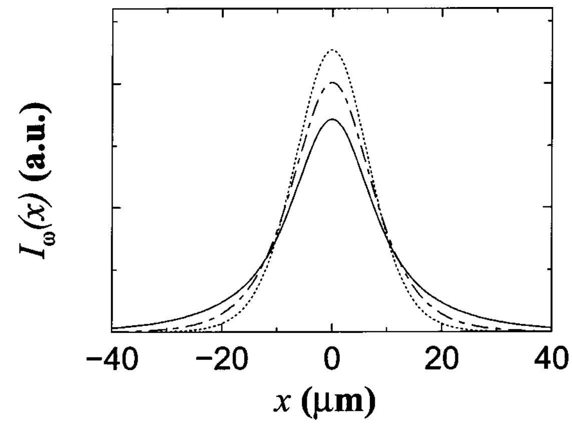

Fig. 5. Intensity profiles $I_\omega(x)$ of the incoherent white-light soliton at three representative frequencies $\Omega_{\min}$ (solid curve), ω_c (dotted–dashed curve), and $\Omega_{\max}$ (dotted curve). The functions $I_\omega(x)$ are more narrow with higher peaks at larger frequencies.

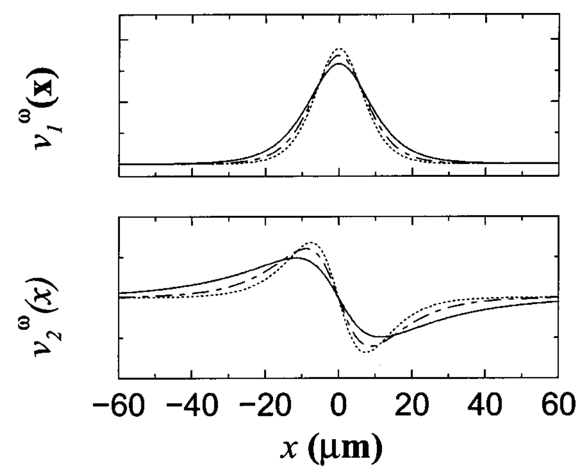

Fig. 6. Two excited modes $\nu_2{}^\omega(x)$ and $\nu_1{}^\omega(x)$ at three frequencies $\Omega_{\min}$ (solid curve), ω_c (dotted–dashed curves), and $\Omega_{\max}$ (dotted curves). The modal structure is similar at every frequency $\omega \in [\Omega_{\min}, \Omega_{\max}]$. The only difference is that the modes at lower-frequency constituents are more stretched than the modes at higher frequencies.

$I_{\omega_j}(x) = I_{\omega_l}(x)$, for any choice of frequencies ω_j and ω_l, which was a consequence of a particular choice of the intensity profile [Eq. (11)], a particular choice of frequency constituents [Eq. (12)], and a particular choice of modal weights [Eq. (16)].

The observation from Fig. 5 regarding the intensity profiles at various frequencies of the continuous spectral density soliton can be explained as follows. Let us assume that the mth mode $\nu_m{}^\omega(x)$ is excited at all frequencies $\omega \in [\Omega_{\min}, \Omega_{\max}]$. This mode satisfies the same equation for each ω, but with different coefficients that depend on frequency [e.g., see Eq. (6)]. Consequently, the profiles of the modes will have the same topological structure (e.g., see Fig. 6). However, modes corresponding to higher frequencies are less stretched than the modes corresponding to lower frequencies as is illustrated in Fig. 6. This is explained as follows. All frequency constituentsof the beam are within the same induced waveguide, and the characteristic width of this waveguide is measured by each frequency constituent in terms of its own characteristic length scale—the wavelength. Consequently, higher-frequency constituents (smaller wavelengths) effectively see a larger induced waveguide and the modes corresponding to higher frequencies will be less stretched, and more concentrated at the center of the soliton, than the modes corresponding to smaller frequencies (see Fig. 6). The same observation is via Eq. (23) reflected into the intensity profiles $I_\omega(x)$ (see Fig. 5). To be precise, we should say that this argument is valid when the spectral density is rectangular, i.e., when the power within different frequency constituents is equal. However, if this is not the case, all conclusions hold for the normalized functions $\hat{I}_\omega(x) = I_\omega(x)/\int \mathrm{d}x I_\omega(x)$ and normalized modes $\hat{\nu}_m{}^\omega(x) = \nu_m{}^\omega(x)/(\int \mathrm{d}x I_\omega(x))^{1/2}$.

For the numerically obtained soliton above, the number of excited modes at each frequency ω is $N_\omega = 2$; these are the two lowest modes $u_1{}^\omega(x, z)$ and $u_2{}^\omega(x, z)$ per each ω. The explanation provided above clearly explains the profiles of the modes $u_m{}^\omega(x, z)$ (see Fig. 6) and functions $I_\omega(x)$ (see Fig. 5). The same behavior of the profiles $I_\omega(x)$, but less pronounced, is observed in white-light solitons in saturable nonlinearity of the type $\delta n(I) = -1/(1 + I)$.[25] Furthermore, the dynamical self-trapping of an incoherent wave packet (white-light quasi-soliton) in Kerr media has also revealed that, during the self-trapping process, the functions $I_\omega(x)$ that are initially the same for each frequency tend to evolve into profiles equivalent to the one presented in Fig. 5.[25]

5. CONCLUSION

In summary, we have presented a family of closed-form soliton solutions representing partially spatially incoherent solitons with a discrete temporal power spectrum in a noninstantaneous Kerr medium. Spectral density of these solitons becomes quasi-continuous as the size of the soliton is sufficiently increased. By using this solution as a starting point, we have numerically identified temporally and spatially incoherent soliton with a broad and continuous temporal power spectrum. From both the analytical family of solitons and the numerically identified soliton, it follows that higher temporal frequency constituents of the light are less spatially coherent than smaller temporal frequency constituents. For future research we foresee the study of interactions of incoherent white-light solitons. Spatially incoherent, but temporally coherent, dark solitons have already been studied experimentally and theoretically.[29,30] However, the problem of dark white-light solitons is yet unexplored. The problem of spectral density at the darkest spot of a dark white-light soliton seems intriguing in view of the recent study of universal pattern of colors near an isolated phase singularity.[31]

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