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ADVANCED GENERALS PROJECT

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A simple potential whose number of minima grows exponentially in the number of variables

1 Introduction

In order to be able to better study systems which have a large N broken permutation symmetry that appear in many physical situations, we tried to come up with a simple class of potentials whose number of distinct minima grows exponentially in the number of variables. As explained in this paper, we have succeeded in this goal. In the second phase of the project, we plan to explore possible applications of our class of potentials.

In order to simplify the analysis, we include only quadratic, and quartic terms in the expansions of the potentials throughout this paper. As a consequence, our potentials always satisfy $V(x_1, ..., x_N) = V(-x_1, ..., -x_N)$, where x_i is the i^{th} variable, and there are N variables total.

Furthermore, we call two minima which occur at positions, $(x_1, ..., x_N)$, and $(y_1, ..., y_N)$ to be "distinct" if their depths are distinct: $V(x_1, ..., x_N) \neq V(y_1, ..., y_N)$. Alternatively, one could also for example compare the eigenvalues of the Hessian matrix evaluated at the minima. These eigenvalues correspond to the masses "felt" in mutually perpendicular directions, and thereby have an immediate physical interpretation. This is easily done for our models; however, we do not present it in this article.

As explained before, our goal is to engineer as simple potential as possible such that the number of distinct minima grows exponentially in the number of variables. In Section 2, we present a particularly simple attempt, and demonstrate that it does not work. In Section 3, we construct a bit more complicated class of potentials, we analyze it, and show that it satisfies our requirement. In Section 4, we evaluate expectation values of a few operators in a potential of our class to demonstrate some of its possible uses. We conclude in Section 5.

2 A "vector" potential

The first logical attempt is to organize the variables in a vector, and say that the potential is a simple function of this vector; that is, if we have N variables, and we denote them with ϕ_i, for $i = 1..N$, we can write a variable vector $\vec{\phi}$, and the potential $V(\vec{\phi})$ will not single out any component of $\vec{\phi}$ as being special in any way. In this section we show that such a potential can have at most $N/2+1$ distinct minima when N is even, and at most $(N+1)/2$ distinct minima when N is odd. Nevertheless, studying this class of potentials is instructive because we use the same approach to study more complicated potentials later.

Under the assumptions above, the most general allowed potential can be written as[1]:

$$V(\vec{\phi}) = \mu \sum_i \phi_i^2 + \alpha \sum_{i,j} \phi_i \phi_j + \beta_1 \sum_i \phi_i^4 + \beta_2 \sum_{i,j} \phi_i^3 \phi_j$$
$$+\beta_3 \sum_{i,j} \phi_i^2 \phi_j^2 + \beta_4 \sum_{i,j,k} \phi_i^2 \phi_j \phi_k + \beta_5 \sum_{i,j,k,l} \phi_i \phi_j \phi_k \phi_l. \quad (1)$$

Note that every potential of this class always has at least two symmetries; $\vec{\phi} \leftrightarrow -\vec{\phi}$ leaves it unchanged, and $\phi_a \leftrightarrow \phi_b$, for any a and b leaves it unchanged. We refer to the latter kind of symmetry as the "exchange symmetry."

We propose the following way of analyzing what is the biggest number of distinct minima any potential from (1) can have. First, we form a vine bottle potential[2] out of the allowed terms. Second, we add a few more allowed terms to form what we call a Coca-cola bottle potential; this potential has local minima at $\vec{\phi}_{min} = (\pm 1, \pm 1, ..., \pm 1)$, all of which look the same, in the precise sense that they are all mutually related by symmetries. All the minima are separated by the saddles on the axes between them. Explicit form of a Coca-cola bottle potential is:

$$V(\phi_1, \phi_2, ..., \phi_N) = -2(aN + b) \sum_i \phi_i^2 + a \left(\sum_i \phi_i^2 \right)^2 + b \sum_i \phi_i^4, \quad (2)$$

where $a, b > 0$ are arbitrary, and the coefficient of the first term has to be fixed in order to have the minima at the positions: $\vec{\phi}_{min} = (\pm 1, \pm 1, ..., \pm 1)$. The first two terms just form a rotationally symmetrical vine bottle potential. For points equally distant from the origin, the last term assumes higher values on the axes compared to the values it takes when evaluated off the axes; therefore it puts the minima off the axes, while putting the saddles on the axes between them. The rotational symmetry of the vine bottle potential is broken now, and we have a potential with 2^N local minima; this is an exponentially large number in N. However, all of the minima have exactly the same depth, and in that sense, they are not distinct; the existing symmetry is still enormous. The analogy with the Coca-cola plastic bottle bottom is obvious, except that this bottle can represent our potential only when $N = 2$; also, the real-world bottle has 5 rather than 4 minima.

Finally, we add to the Coca-cola bottle potential any or all of the terms that appear in (1) as tiny perturbations; this moves positions of the minima, and also their depths

[1] All sums throughout the paper run from 1 to N. Unless explicitly stated, summation over the repeated indices is always assumed; $\phi_i \phi_i \equiv \sum_i \phi_i^2$.

[2] This potential is also sometimes called a Mexican hat potential; $V(\vec{\phi}) = -a\rho^2 + b\rho^4$, where $a, b > 0$, and ρ is the distance from the origin; in this case $\rho^2 \equiv \vec{\phi} \cdot \vec{\phi}$.

by tiny amounts[3]. We intend to experiment with different tiny perturbations to see into how many distinct minima can we break the existing symmetry.

First, we add the following term:

$$\lambda_1 \left(\sum_i \phi_i \right)^2 \equiv \lambda_1 \sum_{i,j} \phi_i \phi_j. \tag{3}$$

The amount by which this term moves the depth of each minima, to first order in λ_1 depends only on the total number of $+1$'s the particular minimum had initially. The minimum that was at $\vec{\phi}_{min1} \equiv (-1, +1, +1, ..., +1)$ before the perturbation now has different depth than the one that was originally at $\vec{\phi}_{min2} \equiv (-1, -1, +1, +1, ..., +1)$[4]. However, the minimum at $(-+++...++)$ still has the same depth as the one at $(++-+++...++)$; to first order in λ_1, all minima that have the same total number of $+$'s still have the same depth. Furthermore, because of the $\vec{\phi} \leftrightarrow -\vec{\phi}$ symmetry, the minima that have exactly M $+$'s have the same depth as the minima that have exactly M $-$'s. All in all, the perturbation (3) broke the symmetry of our 2^N minima, which all had the same depth initially, into $(N/2)+1$ distinct classes of minima when N is even, and into $(N+1)/2$ distinct classes when N is odd.

Unfortunately, we can not do any better than this the way we are doing it. No matter how many different tiny terms we add, even if we do the analysis exactly, rather than only to the first order, we can never increase the number of distinct classes of minima. The reason for this is the exchange symmetry of the original potential (1); since $V(\phi_1, \phi_2, ..., \phi_N) = V(\phi_2, \phi_1, ..., \phi_N)$, the minima at $(-++...++)$ always has exactly the same depth as the one at $(+-++...++)$; all the minima in the quadrants with the same number of $+$'s thus have exactly the same depths[5].

In conclusion, the exchange symmetry forces all the minima with the same number of $+'s$ to have the same depth, while the $\vec{\phi} \leftrightarrow -\vec{\phi}$ symmetry forces the ones with exactly M $+$'s to have the same depth as the ones with exactly M $-$'s. Consequently, one can have at most $(N/2)+1$ distinct classes of minima when N is even, and at most $(N+1)/2$ distinct classes when N is odd. Therefore, using the way we chose to analyze the vector class of potentials, they can not have the number of distinct classes of minima grow exponentially in N.

3 A "matrix" potential

In this Section, we construct a slightly more complicated class of potentials than the one of the previous section. First, in Subsection 3.1, we describe how a potential in this class looks like. Second, in Subsection 3.2 we explain how we plan to analyze the potentials. Next, in Subsection 3.3, we show explicitly that one can always construct at least one potential of our class, whose number of distinct minima grows exponentially

[3] For example we add a small term like $\lambda \sum_{i,j} \phi_i^3 \phi_j$, where λ is small enough so that all original minima stay being minima, and none of the original minima moves out of its original quadrant due to this small perturbation. How small λ has to be to satisfy this depends on N, on how many terms we want to add, etc.

[4] From now on, we denote the minimum that was originally at $(-1, +1, +1, ..., +1)$ simply by $(-++...+)$; similarly, the minimum that started at $(-1, -1, +1, +1, ..., +1)$ is denoted by $(--++...+)$, etc.

[5] Since any two quadrants which have the same total number of $+$'s are always related by an exchange symmetry, one can see that considering other parameters of the potential evaluated at the two minima does not help; the two minima in question look physically the same no matter how one chooses to look at them.

in the number of variables. Finally, in Subsection 3.4, we discuss how strict constraints are imposed on our class of potentials if one requires them to have the number of distinct minima classes grow exponentially in the number of variables.

3.1 What is it?

The next logical step after the vector potential is to consider a matrix potential. We impose that the matrix is symmetric, so that M_{ij} and M_{ji} are actually the same variable. We define the a^{th} "row-column" of a matrix to be the union of the a^{th} row, and the a^{th} column of the matrix; it is the set of all M_{ia}'s and all M_{ai}'s for all i's. In this class of potentials, none of the row-columns are singled out in any way by the potential. Therefore, the allowed quadratic terms in a potential are:

$$M_{ii}M_{ii}, M_{ii}M_{ij}, M_{ii}M_{jj}, M_{ij}M_{ij}, M_{ii}M_{jk}, M_{ij}M_{ik}, \text{ and } M_{ij}M_{kl}, \tag{4}$$

where the sumation over all indices in sight, even if they are not repeated, is assumed as a convention from now on; each index runs from 1 to N, implying that there are $N(N+1)/2$ independent variables in the matrix $\mathbf{M}$. Since there are quite a few allowed quartic terms, we will not write them all out. We just want to emphasize that the terms as confusing as $M_{ij}M_{jk}M_{kl}M_{li}$, and $M_{ii}M_{ij}M_{jk}M_{kl}$ are fair game now.

We will not include terms of any order other than quadratic, and quartic in the expansion of the potentials. Consequently, the potentials still have the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry. Furthermore, any potential of this class is invariant under the exchange of any two row-columns. For example, one can take a matrix, and every time one sees index 3 in the matrix, he/she replaces it with index 7, and vice-versa. This way entry M_{37}, and entry M_{73} stay the same, while the entries M_{33} and M_{77} swap places; for all other i's, M_{i3} swaps with M_{i7}, and M_{3i} swaps with M_{7i}. Nevertheless, this whole operation leaves every potential of this class unchanged. We refer to this symmetry as the "row-column exchange symmetry," or simply as the "exchange symmetry." These are the only two symmetries that apply for all potentials in our class. Consequently, we refer to them as "the symmetries" of this class of potentials.

Due to the two existing symmetries, we expect that many of the quadrants will be related by a symmetry of the potential, and their minima will therefore never be distinct. However, we hope that we can succeed to have the number of distinct minima classes to grow at least as fast as $a^{N(N+1)/2}$ for at least some $a > 1$, and for large N.

3.2 How do we analyze it?

We plan to analyze this class of potentials in the same way we analyzed the "vector potentials." First, we form a Coca-cola bottle potential of the allowed terms. In the matrix form, taking into account that M_{ij} and M_{ji} are the same variable, the potential (2) takes the form:

$$V(\mathbf{M}) = -\left[\frac{aN(N+1)}{2} + b\right]\left[M_{ii}^2 + M_{ij}^2\right] + \frac{a}{4}\left[M_{ij}^2M_{kl}^2 + 2M_{ii}^2M_{jk}^2 + M_{ii}^2M_{jj}^2\right] + \frac{b}{2}\left[M_{ij}^4 + M_{ii}^4\right], \tag{5}$$

where $a, b > 0$ are arbitrary. We adjusted the first term so that initially all the minima are at:

$$\begin{bmatrix} \pm 1 & \pm 1 & . & . & . & \pm 1 \\ \pm 1 & \pm 1 & & & & \pm 1 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & . \\ \pm 1 & \pm 1 & . & . & . & \pm 1 \end{bmatrix}, \tag{6}$$

and they all look the same, in the precise sense that they are all mutually related by symmetries.

Now, we experiment with adding more small allowed terms to try to separate the minima into an exponentially large number of classes of minima of distinct depth. The positions of the minima, and their depths will move as we vary the amounts of the various small terms we are adding. We want to be careful that the terms we are adding are small enough so as not to make any minimum into a not-minimum, and also not to move any minimum outside of its original quadrant.

To start, we experiment with $N = 2, 3, 4, 5, 6$; the terms we add as small perturbations are $M_{ij}M_{jk}M_{kl}M_{li}$ and $M_{ii}M_{ij}M_{jk}M_{kl}$. We find numerically the exact depth of each minima. Because of the limitations imposed by our computer, we were not able to check the relation for larger N's.

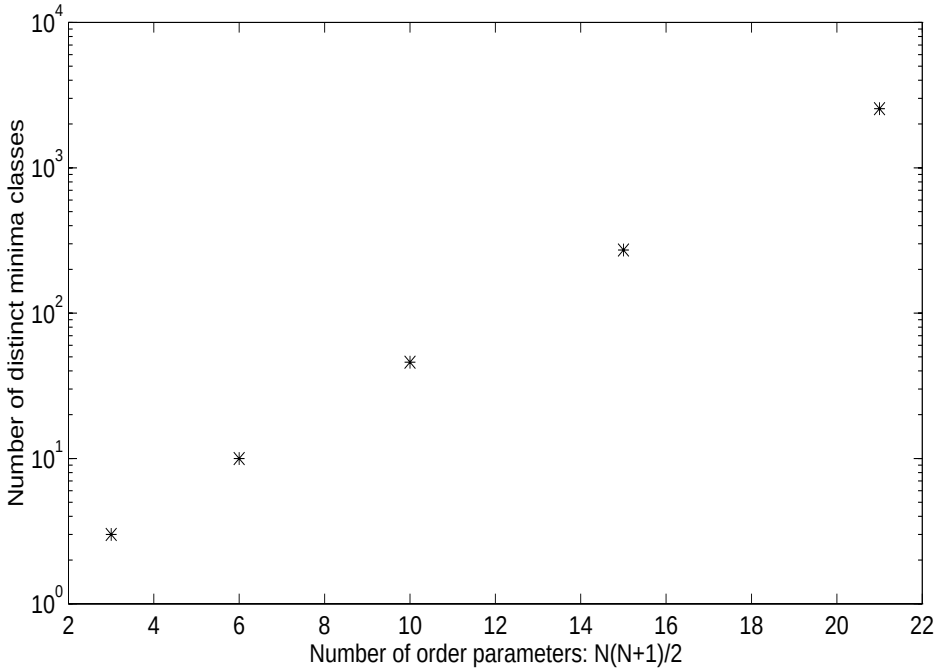

Figure 1: On a semi log plot, the number of distinct minima classes versus the number of order parameters appears as a straight line. This is a pretty convincing proof that the number of distinct minima grows exponentially with the number of variables.

As one can see in Figure 1, it seems that the number of distinct minima classes does grow exponentially in the number of variables, using only the two particular terms for the perturbations.

3.3 Proof that the number of distinct minima classes can grow exponentially fast in the number of variables

Although it seems from Figure 1 that the number of distinct minima classes grows exponentially with the number of variables for the small values of N and for the potential we tried, we would like to show quite generally that this can be true for any N. Consequently, in this section we show explicitly that for every N, one can always construct a potential of our class whose number of distinct minima classes is an exponential in the number of variables. The proof itself is probably not useful in constructing any realistic potential. In fact, the potential we construct is rather artificial; its only purpose is to show that there exists at least one potential with the desired characteristics. However, as we discuss in Subsection 3.4, it is clear from this proof that almost any potential one builds using most of the allowed terms has an exponentially large number of distinct minima classes.

The proof proceeds in two steps. First, we show that the number of quadrants which are not related by the exchange symmetry, or by the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry grows exponentially in the number of variables. Next, we show that starting from a Coca-cola bottle potential, through adding additional small terms to the potential, one can always break the degeneracy of the minima, so that any two minima which are unrelated by an exchange symmetry, or by the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry, always have distinct depth. Since there is an exponentially large number of classes of such minima, this completes the proof.

3.3.1 Proof that number of quadrants unrelated by the symmetries grows exponentially fast in the number of the order parameters

We focus our attention on one very particular subset of all quadrants, and prove that the number of quadrants in this subset, which are not related by any of the allowed symmetries, grows faster than $a^{N(N+1)/2}$ for at least some $a > 1$, and for large N; this provides a lower bound on the total number of quadrants unrelated by a symmetry, thereby completing this part of the proof.

The subset we are focusing on consists of all quadrants which can be written in the following form:

$$\begin{bmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{A^T} & \mathbf{C} \end{bmatrix}, \tag{7}$$

where in the case N is even, all matrices **A**, **B**, and **C** have $N/2$ rows and $N/2$ columns. In contrast, when N is odd, **B** has $(N+1)/2$ rows and $(N+1)/2$ columns, while **C** has $(N-1)/2$ rows and $(N-1)/2$ columns; consequently, **A** has $(N+1)/2$ rows, and $(N-1)/2$ columns when N is odd. Furthermore, the matrix **B** has only $+$'s on the diagonal, while the matrix **C** has only $-$'s on the diagonal; any other entry of **B**, and **C** is "free" to be either a $+$, or a $-$; note that the number of such "free" entries grows faster than b^{N^2} for some $b > 1$ and for large N. In contrast, all the entries of the matrix **A** are fixed; if N is even, all elements on the diagonal, and above the diagonal are $+$'s, while all the elements below the diagonal are $-$'s; if N is odd, entry A_{ij} is $+$ if $i \leq j$, and $-$ otherwise.

For example, if $N = 8$, any element of the subset looks like:

$$\begin{bmatrix} + & ? & ? & ? & + & + & + & + \\ ? & + & ? & ? & - & + & + & + \\ ? & ? & + & ? & - & - & + & + \\ ? & ? & ? & + & - & - & - & + \\ + & - & - & - & - & ? & ? & ? \\ + & + & - & - & ? & - & ? & ? \\ + & + & + & - & ? & ? & - & ? \\ + & + & + & + & ? & ? & ? & - \end{bmatrix}, \tag{8}$$

where $?$ can be either a $+$ or a $-$, as long as it is consistent with the requirement $M_{ij} = M_{ji}$; i.e. the elements below the diagonal are fixed once we pick the elements above the diagonal. Similarly, when $N = 7$, every element of the set looks like:

$$\begin{bmatrix} + & ? & ? & ? & + & + & + \\ ? & + & ? & ? & - & + & + \\ ? & ? & + & ? & - & - & + \\ ? & ? & ? & + & - & - & - \\ + & - & - & - & - & ? & ? \\ + & + & - & - & ? & - & ? \\ + & + & + & - & ? & ? & - \end{bmatrix}, \tag{9}$$

with similar requirements as for the case $N = 8$.

The reason we focus our attention on this particular subset is that the "gauges" for the symmetries of our class of potentials are naturally fixed by the elements on the diagonal, and by the elements of the matrix **A**. The number of elements of the subset grows exponentially with the number of variables. Nevertheless, as we prove below, none of the elements of the subset can be "permuted into each other" through the allowed symmetries of the potential.

Question: Suppose one starts with matrix $\mathbf{M_1}$, which is an element of the subset described above. We associate with this matrix its sub-matrices $\mathbf{A_1}$,$\mathbf{B_1}$, and $\mathbf{C_1}$. Then, one exchanges as many row-columns as one wants, in any order. During the process, one is also allowed to multiply the whole matrix with -1 as many times as he/she likes. However, after all this, he/she ends up with the matrix $\mathbf{M_2}$ which is also an element of the subset, and we associate with it its sub-matrices $\mathbf{A_2}$,$\mathbf{B_2}$, and $\mathbf{C_2}$. Is it necessary that $\mathbf{M_1} \equiv \mathbf{M_2}$?

Answer: Yes! But, note that the affirmative answer to this question implies that we have completed the proof of Subsection 3.3.1.

Proof: Proof proceeds in two steps. First, ignoring the existence of the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry, we prove that the exchange symmetry alone satisfies the proposition. Second, we prove that the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry does not cause any further problems.

Step 1: For now, we ignore the existence of the $\mathbf{M} \leftrightarrow -\mathbf{M}$ symmetry; the only symmetry is the exchange symmetry. The crucial point to realize is that if some two entries of the matrix are in the same row-column initially, then they are still in the same row-column after any number of exchanges, although their relative position within the row-column might change.

We propose a "painting scheme" to keep track where each entry of the matrix moves during the exchange process. This scheme also makes it easier to visualize what is

going on. Paint each row-column with different color. Consequently, each M_{ij} for $i \neq j$ is covered with two layers of distinct paints; M_{ii} is covered with two layers of the same paint. Make sure to use "light colors" if $+$ is on the diagonal entry of the row-column you are painting, and "dark colors" if $-$ is on the diagonal entry. Each particular entry M_{ij} for $i \geq j$ is is now labeled uniquely by its two colors; of course, M_{ij} has the same colors as M_{ji}, which suits us because they are the same variable anyway.

Say the 2^{nd} row-column is yellow, and the 5^{th} row-column is green. Exchanging indices 2 and 5 makes the 5^{th} row-column yellow, and the 2^{nd} row-column green. Using the coloring scheme, it is easy to keep track where each particular entry moved during the exchange. Say the 11^{th} row-column was blue initially, and we want to know where the entry $M_{2,11}$ ended up after the exchange; we look for the square of the matrix that is covered precisely by the yellow, and the blue paint, and conclude that the entry in question is now at the position $M_{5,11}$.

Note that initially every entry of the $\mathbf{B}$ matrix contains only light colors, while the matrix $\mathbf{C}$ contains only dark colors. In contrast, every entry of matrix $\mathbf{A}$ is painted with precisely one light, and one dark color.

Now, we start with a matrix $\mathbf{M_1}$ and permute it into a matrix $\mathbf{M_2}$ so that both of these matrices are elements of the subset. First, note that all rows of $\mathbf{A_2}$ and $\mathbf{B_2}$ are painted with light colors, while all columns of $\mathbf{A_2}$ and $\mathbf{C_2}$ are painted with darker colors; this is so because $\mathbf{B_2}$ has only $+$'s on the diagonal, while $\mathbf{C_2}$ has only $-$'s on the diagonal. Therefore, the set of all entries of $\mathbf{A_1}$ is exactly the same as the set of all entries of $\mathbf{A_2}$; only these entries are such as to have exactly one light, and one dark color. Suppose that the light colors we have are: yellow, orange, red and pink, and suppose $N = 8$. Furthermore, suppose that $\mathbf{A_1}$ has the 1^{st} row yellow, the 2^{nd} row orange, etc. Since two entries that were in the same row-column before the exchanges stay in the same row-column after the exchanges, the only way to get exactly $4+$'s in the 1^{st} row of $\mathbf{A_2}$ is to have the 1^{st} row of $\mathbf{A_2}$ yellow. This implies that the 1^{st} row-column of $\mathbf{M_2}$ is yellow. Furthermore, the only way to have exactly $3+$'s in the 2^{nd} row of $\mathbf{A_2}$ is to have the 2^{nd} row of $\mathbf{A_2}$ orange, implying that the 2^{nd} row-column of $\mathbf{M_2}$ is orange, etc. This way we determine the position of all light colors, and thereby determine uniquely everything about the matrix $\mathbf{B_2}$. In a similar manner, we determine everything about the matrix $\mathbf{C_2}$.

Therefore, we proved that $\mathbf{A_1} = \mathbf{A_2}$, $\mathbf{B_1} = \mathbf{B_2}$, and $\mathbf{C_1} = \mathbf{C_2}$ thereby implying that $\mathbf{M_1} = \mathbf{M_2}$, as we sought to prove. Of course, the particular case $N = 8$ is just illustrative; everything we said generalizes immediately to any even N. Furthermore, everything we said can also be applied with only minor modifications for the case N is odd. Therefore, we are done proving Step 1.

Step 2: We prove that during the whole process of transforming matrix $\mathbf{M_1}$ into matrix $\mathbf{M_2}$, one always has to multiply the matrix with -1 a total of an even number of times. The way to see this differs a bit in the case when N is even, and when N is odd. When N is odd, we have to end up with less $-$'s than $+$'s on the diagonal of $\mathbf{M_2}$, which is the same as for the diagonal we started with; however, none of the entries of the diagonal ever moves off the diagonal during the process. Similarly, in the case N is even, we have to end up with less $-$'s than $+$'s in the matrix $\mathbf{A_2}$, and we already proved during Step 1 of this proof that $\mathbf{A_1}$ consists of the same set of elements as $\mathbf{A_2}$. Therefore, the matrix has to be multiplied with -1 an even number of times during the process, both when N is odd, and when N is even.

Since the operation of multiplication with -1 treats all the elements of the matrix

indiscriminately, it does not matter at all when during the process we perform these operations; in particular, we could instead perform all of them before doing anything else; but then, we might as well not do them at all, since multiplying the matrix with -1 an even number of times leaves the matrix unchanged.

To make this more visual, one can think of the operation of multiplication with -1 as spraying dots on the matrix, or erasing them if they are already there. Since the dots treat the whole matrix indiscriminately, without ruining the underlying colors from Step 1, and since we proved that the final matrix must have no dots if the initial matrix had none, it is clear that this operation can not cause any problems for us.

This concludes our proof that the number of quadrants which are unrelated by a symmetry of our class of potentials grows exponentially fast in the number of variables for large N.

3.3.2 Proof that for any N, one can construct a potential such that any two minima, which are unrelated by a symmetry of our class of potentials, always have distinct depth

We assume that we start with a Coca-cola bottle potential, in which all quadrants look the same. We describe an algorithm which tells us how to modify the potential so that once we are done, any two minima unrelated by a symmetry of our class of potentials have distinct depth. Since for any set of minima, one can always follow this algorithm, the existence of this algorithm proves the hypothesis we are trying to prove.

Algorithm: First, we define a tiny parameter λ. We work perturbatively, analyzing each order of λ separately, working to higher and higher orders in λ. We want λ to be pretty small, so that the analysis at a particular order is not influenced by what is done later, at higher orders in λ. Furthermore, we want λ to be small enough, so that during the process, we do not turn something that was a minimum before we started into a not-minimum; in addition, we do not want to move any of the minima outside of the quadrants where they started.
1^{st}: First, examine all the minima. Pick the two which have distinct depth, but such that $|V(\mathbf{M}_{min1}) - V(\mathbf{M}_{min2})| \equiv \alpha$ is smallest. Then, pick the smallest $k \in \mathcal{N}$ such that $\lambda^k \leq \alpha$. If all minima have the same depth, say $k = 0$.
2^{nd}: Denote $p \equiv k + 1$.
3^{rd}: Pick any two minima that have exactly the same depth, but are not related by any of the symmetries of our class of potentials. If there are no such minima, we are done.
4^{th}: Add additional allowed terms to the potential in order to break the degeneracy of the two minima from the 3^{rd}:, but in such a manner as not to move the depth, or the position of any of the existing minima by more than λ^p; this is always doable according to the lemma stated below. The reason we do it to this precise order is that we want to prevent accidentally bringing to same depth some other two minima which were distinct before.
5^{th}: Go back to 1^{st} till there are no more minima of the same depth which are not related by a symmetry of our class of potentials.

This algorithm clearly separates into distinct depths all the minima we wanted to separate. However, note that p gets exponentially small in the process, thereby making our construct look a bit artificial. We explain in Subsection 3.4 why we do not find this worrisome.

Lemma: If one has two minima which are unrelated by a symmetry of our class of potentials, one can always separate them into having different depths by adding some of the allowed terms to the original potential. Furthermore, for any $p \in \mathcal{N}$ one can do it in such a manner so that neither the depth, nor the position of any of the other existing minima moves by more than λ^p. Note that we made no statement about how much will the two minima in question actually move apart.

Proof: We are still working on this proof. However, we strongly believe that the statement is provable. Our intuition about it goes essentially as follows. We focus our attention on the two particular minima, and observe how their positions and depths move as we vary the allowed parameters randomly. We denote the position of the first minima with $\mathbf{M}_1(\{\lambda_k\})$, and the position of the second minima with $\mathbf{M}_2(\{\lambda_k\})$, where λ_k's are the small coefficients we are varying. These positions are given as solutions of $N(N+1)/2$ equations in $N(N+1)/2$ variables, namely:

$$\left(\frac{\partial V(\mathbf{M})}{\partial M_{ij}}\right)_{\mathbf{M_1}} = 0, \text{ and } \left(\frac{\partial V(\mathbf{M})}{\partial M_{ij}}\right)_{\mathbf{M_2}} = 0, \tag{10}$$

for all $i \leq j$. Note that the solution always exists, and it is unique according to our assumptions; i.e. we can always find exactly one minima at the place where we are looking for it.

If in addition to (10), we require that $V(\mathbf{M}_1) = V(\mathbf{M}_2)$ for all $\{\lambda_k\}$'s, we are over-determining the system, since we are introducing one more constraint than necessary. It would be a remarkable coincidence if the system satisfies this additional constraint automatically unless the system has a good reason to satisfy it; i.e. there is a symmetry relating the two minima. However, if such a symmetry does not exist, it seems reasonable to expect that one is allowed to pick from an uncountable set of $\{\lambda_k\}$'s which all break the degeneracy of the two minima.

Note that in what we said above, no statement is made about how big are the coefficients multiplying the terms we are adding. Consequently, we can always pick these coefficients so that we do not move either the depth, or the position of any existing minima by more than λ^p, for any $p \in \mathcal{N}$. How small exactly do these coefficients have to be clearly depends on how many terms we intend to add, and also on N. Nevertheless, it is clear that we can always pick them small enough not to move the position or the depth of any minima in the potential by more than an arbitrarily small amount.

3.4 Epilogue of the proof

We succeeded to prove that for any N, we can construct at least one potential that has $2^{N(N+1)/2}$ minima which can be separated into at least a^{N^2} classes for at least some $a > 1$, such that each class has distinct depth of the potential. We start with a Coca-cola bottle potential, and using the procedure from Subsection 3.3.2, separate all the minima unrelated by the symmetries of our class of potentials into having distinct depths. Since the number of the elements of the subset from Subsection 3.3.1 grows exponentially with the number of variables, and all of these are unrelated by the symmetries, the number of distinct minima classes grows at least as fast as the number of elements of this subset, thereby exponentially.

A bit worrisome thing about our proof is that the parameter p from the algorithm in Subsection 3.3.2 gets exponentially small during the process described, thus making

the potential we constructed in that subsection look a bit artificial. However, as it is transparent from the lemma of Subsection 3.3.2, and also from $\underline{4^{th}}$ step of the same subsection, p gets exponentially small only because we were trying to be overly careful. It is clear from the lemma that one has a choice from an uncountable set of possibilities when separating the minima in $\underline{4^{th}}$ step. However, only some finite number of choices will bring some other two minima together; if we pick at random the coefficient of the term we are adding in $\underline{4^{th}}$ step, the probability that we bring some other two minima to be the same depth is of measure zero. Nevertheless, in order for our analysis to apply, one should make sure during the process not to make something that is a minimum initially into something that is not a minimum, and also not to move any of the minima outside of their original quadrants; but, this requirement by no means implies that one has to go to exponentially small orders of λ.

In fact, almost any potential one constructs using most of the allowed terms satisfies the requirement that the number of distinct minima classes grows exponentially in the number of variables. If one starts say from a Coca-cola bottle potential, and adds tiny allowed terms to the potential, picking their coefficients at random, the probability not to end up with a potential that satisfies the requirement is of measure zero.

The physical reason for this is transparent from the lemma; unless two minima have a very good reason to be exactly the same, (i.e. a symmetry of the potential,) they are most likely not going to have the same depth, unless we specifically construct them so. The symmetries of the most general potential of our class allow for the number of quadrants not related by a symmetry to grow exponentially fast in the number of variables. Therefore, most potentials of our class have the property that the number of distinct minima grows exponentially in the number of variables.

In fact, in the numerical experiments we performed, only two terms $\lambda_1 M_{ij}M_{jk}M_{kl}M_{li}$, and $\lambda_2 M_{ii}M_{ij}M_{jk}M_{kl}$, where λ_1 and λ_2 are small, seemed to create an exponentially large number of distinct classes of minima, when added to a Coca-cola bottle potential. So, already some extraordinarily simple potentials of our kind satisfy the original requirement, thereby making our class of potentials very suitable for analysis.

4 Evaluation of some expectation values of our potential

In this section we show how to evaluate expectation values of some operators using some particular examples of our class of potentials in order to study their properties. To make things simpler, we add just a tiny perturbation to the Coca-cola bottle potential. Clearly, we miss quite a few physically interesting potentials by limiting ourselves to tiny perturbations only; we hope to study other possibilities in the future. However, as we describe below, this subset of our class of potentials is extremely easy to analyze.

Because we are adding tiny perturbations, we are justified in evaluating the changes in the potential only to the first order; we say that the depth of each minimum moves by whatever the perturbation we are adding evaluates to at the original position of the minimum in question; these positions are given in (6). To the first order, the degeneracy can not be broken into an exponentially large number of minima classes; for example, a quartic term that involves 8 indices can assume at most $\mathcal{O}(N^8)$ different values when evaluated at the positions given in (6). Even if we add all the allowed terms, each multiplied by an arbitrary coefficient, we still get at best a power law breaking of the degeneracy.

Nevertheless, the number of distinct minima one can in principle get by analyzing only to the first order is quite large, especially if we include many allowed terms to create the perturbation. Furthermore, for small perturbations, the expectation values of different operators will typically not differ significantly if we evaluate the changes in the depths only to the first order, versus evaluating them exactly. Finally, we sort the minima into energy bins of finite width in our plots. If our perturbation breaks the degeneracy to the first order into say $\mathcal{O}(N^8)$ distinct minima classes and $N = 6$, we have in principle up to $\mathcal{O}(10^6)$ distinct minima. Since our plots typically involve 200 bins, it does not matter for the plots whether we evaluate the depth changes to the first order only instead of calculating them exactly.

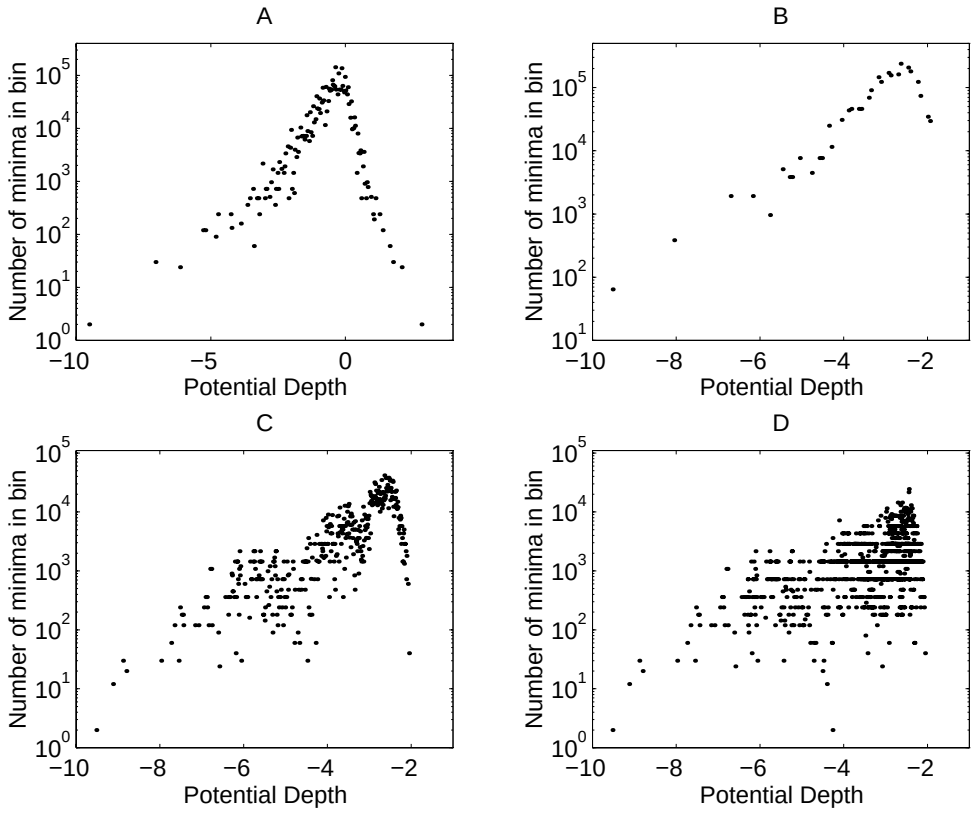

Figure 2: Plots of bin occupation numbers versus the changes in minima depths, evaluated to the first order in the small perturbations. All plots are for $N = 6$. The x-axes are in arbitrary units. The width of the bins in plots A,B, and C is 10^{-2} energy units, and in plot D is $2.5 * 10^{-4}$ energy units. The term added in plot A was $-M_{ii}M_{ij}M_{jk}M_{kl}$, while in plot B it was $-M_{ij}M_{jk}M_{kl}M_{li}$. To create plot C, we add 20 different terms, with random coefficients multiplying them. Plot D is exactly the same as plot C, except with much higher bin resolution.

Some of our plots are presented in Figure 2. Plots A and B from that figure demonstrate that one can get quite a rich structure by using only a few of the allowed terms. Furthermore, the breaking of degeneracy is quite large even to the first order only. In some cases, there are quite a few minima that all have the same energy depth; physical intuition is that they probably all belong to a large family in which all members are related by symmetries.

When we include more than only one perturbative term, the degeneracy breaking is even bigger, producing quite a rich structure even only to the first order. This is visible

in plots C and D of Figure 2, where we included 20 of the allowed terms, with random coefficients multiplying them. The plot D has a very high resolution of almost 40000 bins for the whole plot; both plots are for exactly the same potential.

Suppose that the potentials from Figure 2 represent some spin glasses; spin glasses are an example of a physical system that has the characteristic that the number of distinct minima is exponentially large in the number of the order parameters [1]. One can see from the plots in Figure 2 that the occupation number of the lowest energy bin which is still occupied is typically very low. Consequently, for high temperatures, the states of lowest energy are not the most likely set of states. Instead, one is much more likely to observe these particular spin glass systems to be in one of the minima that belongs to a large family in which all minima are mutually related by symmetries. As we lower the temperature, the probability to observe the systems in some of the lower energy states increases.

We can make this more quantitative [2] if we assume that the probability of the system to be in the minimum denoted by α is given by:

$$P_\alpha = \frac{e^{-F_\alpha/\tau}}{\sum_\beta e^{-F_\beta/\tau}}, \tag{11}$$

where $\tau \equiv k_B T$, and F_α is the value of the free energy evaluated at the position of the minimum denoted by α.

Using (11), we can evaluate the expectation value of any operator in a spin glass:

$$\langle \mathcal{O} \rangle = \sum_\alpha P_\alpha \langle \mathcal{O} \rangle_\alpha, \tag{12}$$

where $\langle \mathcal{O} \rangle_\alpha$ is the mean value of the operator $\mathcal{O}$ in the valley α.

As a demonstration of some further calculations in our potentials, we use the equation (12) to evaluate the expectation value of the free energy of the spin glass in the potential from Figure 2, plot D, for different values of temperature. The result is displayed in Figure 3. As expected, at low temperatures, one is most likely to observe the system in one of the states of the lowest energies. Therefore, for low temperatures, the expectation value saturates at the energy of the lowest states. In contrast, for high temperatures the term $e^{-F_\alpha/\tau}$ evaluates essentially to 1 for all minima indiscriminately, thereby causing a saturation of the expectation value in the high temperature limit. Physically most interesting region is in between these two extremes.

Using the same methods, we could in principle evaluate expectation value of any operator in a spin glass, but the point of this section was just to demonstrate how one would calculate some values of possible interest using our potentials; so, we stop the discussion here till we decide what exactly do we want to use our potentials for.

5 Conclusion

Our goal in this project was to come up with a class of potentials that one can in principle use to model large N broken permutation symmetry systems. Consequently, our main goal was to engineer as simple potential as possible whose number of minima grows exponentially in the number of variables.

First, we show that the most naive logical attempt which treats all the variables indiscriminately does not work; the number of distinct minima grows only linearly in

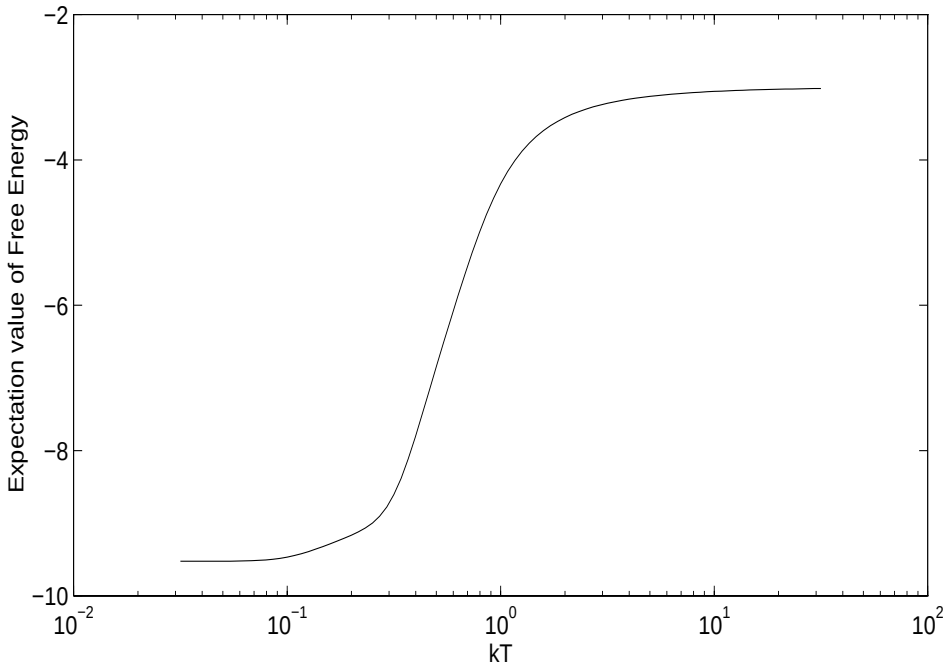

Figure 3: We used one of our potentials to represent an imaginary spin glass system. Dependence of $\langle F \rangle$ on $k_B T$; both of these are in same, arbitrary units.

the number of variables for such a potential. Then, we show that already the next logical step satisfies the requirement. Furthermore, it seems that almost any potential of the class we construct has the number of distinct minima growing exponentially in the number of variables. Finally, we analyze some properties of our class of potentials in more detail to see how they behave, and also to demonstrate how one would use our potentials. For this purpose, we analyze the potentials to the first order in small perturbations only, which is an extremely easy thing to do. Nevertheless, our potentials displays quite a rich structure even when analyzed only to the first order. So, maybe some applications of our class of potentials will involve analyzing them to the first order only.

In the second phase of the project, which we hope to pursue during the spring term, we plan to look for some possible applications of our class of potentials to real physical systems; we plan to study some of the real systems to see if they can be modeled using our class of potentials.

References

[1] For general review about spin glasses, see Marc Mezard, Giorgio Parisi, and Miguel Angel Virasoro: *Spin Glass Theory and Beyond*: World Scientific Lecture Notes in Physics Vol. 9: Singapore, 1987.

[2] D.J. Gross and M. Mezard, Nuclear Physics **B240** (1984) 431.