Self-Similarity and Fractals Driven by Soliton Dynamics

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Abstract

We have discovered recently that nonlinear systems that support solitons can often, (under proper conditions,) give rise to Self-Similarity and Fractals on successively smaller scales. Such fractals can be observed in most soliton-supporting systems in nature. As an example of our general idea, we present particular examples that theoretically predict Optical Fractals evolving dynamically from a single input beam (or pulse).

It has long been claimed that fractals, being one of the most fundamental concepts in nature, can characterize (and in some cases, describe) natural phenomena [1]. In other words, there is a large number of objects and processes in nature that can be described in terms of their fractal properties. Indeed, fractals have been identified in material structures [e.g., polymers, aggregation, interfaces, etc.], in biology, medicine, electric circuits, computer interconnects, galactic clusters, and many other non-related and surprising areas, including stock markets [2]. In Optics, Fractals have been recently identified in conjunction with Talbot effect [3] and with unstable cavity modes [4]. In both cases the system is fully linear and responds in a passive manner to illumination by constructing fractals through linear diffraction. Here, we show that nonlinear systems that support solitons can, under proper non-adiabatic conditions, **evolve** and give rise to fractals on successively smaller scales. In the most general case, the end result are statistical fractals, that is, a structure that exhibits self-similarity in its statistical properties on successively smaller scales. However, for specific choices of parameters, the same system can give rise to exact (regular) fractals. This is a unique case where exact fractals are a real physical entity and not just a mathematical example. This capability of soliton-supporting systems to generate fractals is universal and seems to hold for all systems in nature in which solitons exist, irrespective of the type of waves (EM, density, etc.), the medium in which they propagate, and the nonlinearity involved. To illustrate this general idea, we present a specific example that theoretically predicts optical fractals that evolve from a single input pulse (or beam).

As an example of a soliton-supporting system, consider a system described by the normalized Nonlinear Schrodinger Equation (NLSE:)

$$i\frac{\partial \Psi}{\partial z} + \frac{1}{2}\nabla_T^2\Psi + f(|\Psi|^2)\Psi = 0 \tag{1}$$

where the nonlinear term $f(|\Psi|^2)$ is specific to the physical system involved, and ∇_T^2 is the Laplacian transverse to the direction of propagation z (the primary direction along which the

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carrier propagates, for envelope waves). For waves in a single transverse dimension [(1+1)D NLSE], $\nabla_T^2 = \partial^2 / \partial x^2$. Equation (1) describes many physical systems, primarily those in which nonlinear waves propagate in centrosymmetric media [5], where Ψ describes the slowly varying envelope that modulates a fast underlying carrier wave. In particular, NLSE describes several optical systems, [6,7], in which case Ψ is the slowly varying amplitude of the electric field, superimposed on an underlying single **k** vector carrier plane wave. We focus on the (1+1)D NLSE and on two particular forms of nonlinearity that are common in optics [7]: the Kerr-type where $f(|\Psi|^2) = |\Psi|^2$, and the saturable type where $f(|\Psi|^2) = |\Psi|^2 /(1+|\Psi|^2)$. Extending our ideas to other forms of nonlinearities is straightforward, and extending them to higher dimensions maintains all the main results while adding beauty and complexity to the fractals generated. The particular systems we discuss are just examples of the general principle we propose.

Since $df(|\Psi|^2)/d|\Psi|^2 > 0$, both the Kerr, and the saturable nonlinearity are of the self-focusing type, i.e., the nonlinearity has a tendency to shrink a pulse. In optical systems, this happens because the presence of the light pulse increases the local index of refraction, which, in turn, tends to shrink the pulse. The tendency to shrink competes with diffraction, which tries to expand the pulse, and, for some NLSEs, these two tendencies can exactly cancel each other, producing a localized pulse whose shape is stationary as it propagates: a soliton [8]. Solitons are universal nonlinear phenomena, and they have fascinated scientists of many different fields for more than 150 years now [9]. They have been described in many systems: on the surface of shallow water [9], in deep sea water [5], in plasma [10], on the surface of black holes [11], for torsional waves on DNA molecules [12], for sound waves in superfluid ^{3}He [13], to name a few, and of course in nonlinear optics, primarily as temporal [6] and as spatial solitons [7]. The

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solitons of the particular NLSEs we discuss in this article are very robust creatures. Even if one perturbs them slightly from their equilibrium shape, they soon evolve into stable solitons again.

Consider first the (1+1)D Kerr NLSE, of which a fundamental soliton solution is $\Psi(x,z)$. One can obtain a whole family of solitons of Eq. (1) by a simple re-scaling: $\Psi(x,z) \rightarrow q\Psi(qx,q^2z)$ for any real q. According to the definition of self-similarity, this means that all solitons of the same order of this equation are self-similar to each other [14]. That is, a simple re-scaling of solitons' coordinates, and their amplitudes relates (and 'maps') all solitons to one another; one can take any soliton of this equation, and by this simple rescaling map it onto any other soliton, point by point. The physical base for this self-similarity is the fact that the (1+1)D Kerr NLSE does not have any natural scale built into it. Thereby the physics of this equation looks the same on all scales.

In contrast with the Kerr nonlinearity, the (1+1)D saturable NLSE does have a natural scale, because of the number 1 in the denominator of the nonlinear term. However, for $|\Psi|^2 \ll 1$, the nonlinearity reduces to the Kerr nonlinearity, so self-similarity can also exist in the saturable case. Furthermore, if a soliton of (1+1)D saturable NLSE satisfies $|\Psi(x=0,z)|^2 \gg 1$ in the regions where most of the energy of the soliton is contained, we can capture most of the physics by approximating $|\Psi|^2/(1+|\Psi|^2) \approx 1-(1/|\Psi|^2)$. Of course, this approximation does not hold at the tails of the soliton. Still, most of the interesting properties can be captured by studying (1+1)D NLSE with a nonlinearity given by $1-(1/|\Psi|^2)$; we call this nonlinearity the "deep-saturation nonlinearity". We have checked numerically that if the condition $|\Psi(x=0,z)|^2 \gg 1$ is satisfied, then indeed most of the soliton's physics is captured by

studying the (1+1)D Deep-saturation NLSE. If $\Psi(x,z)$ is a solution of the (1+1)D deep-saturation NLSE, then a whole family of solutions can be obtained by re-scaling $\Psi(x,z) \to e^{iz(1-q^2)}\Psi(qx,q^2z)/q$, for any real q. Therefore, all solitons of the same order of (1+1)D saturable NLSE are related by this simple re-scaling, as long as most of the energy of the solitons is in the regions where $|\Psi|^2 \gg 1$; all of these solitons are self-similar to one another in their physical properties, such as intensity shape etc. This is because the natural scale in the saturable NLSE is visible only in the margins of the intensity profile of the soliton, and its effect on the shape is tiny.

We now introduce the concept of the soliton existence curve [15], a two-dimensional curve that gives the FWHM of the soliton intensity in normalized units, as a function of the peak amplitude of the corresponding soliton, $\Psi_0 \equiv \Psi(x=0,z)$. The curve is drawn for the set of all solitons of the same order of a given NLSE, where each soliton is represented by a point on the graph. This yields a curve, which describes all possible same order soliton solutions in parameter space: the soliton existence curve. Different NLSEs have different existence curves, and solitons of different orders of the same NLSE lie on different existence curves. According to the scaling relation described above, all existence curves (of solitons of all orders) of Kerr NLSEs are parallel lines of slope -1 on a log-log plot. The existence curves of saturable NLSEs are also parallel lines of slope -1 on a log-log plot (which coincide with the Kerr curves) in the region $\Psi_0 \ll 1$. On the other hand, in deep-saturation where $\Psi_0 \gg 1$, the existence curves are parallel lines of slope 1 on a log-log plot like in Fig. 1. The region in between these two regimes, i.e., where $\Psi_0 \sim 1$, we call the valley. All solitons of the same order of a saturable NLSE are to a large extent self-similar to each other as long as they are all on the same side of the valley.

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The existence curves are a useful physical tool for NLSEs whose solitons are known to be stable. These curves are helpful in predicting the outcome of collisions in non-integrable systems [16] and in identifying the spectrum of solutions for multi-mode (vector) solitons [17]. The existence curves can also provide information about the evolution of arbitrary input pulses into solitons. Consider a pulse of width w and peak amplitude Ψ_0, and assume that this pulse does not have the stationary soliton shape. For example, in optics, one would typically have Gaussian beams at the input, which is never the same as the stationary lowest order soliton shape. This pulse is represented by a point with coordinates (Ψ_0, w) on the existence curve plot. If this point is close to the curve, then the pulse soon evolves into a stable soliton shape (while shedding some power in the form of radiation modes or smaller scale solitons). Since the solitons of the NLSEs we study here are stable, this happens even though their initial shape only approximates a soliton.

As discussed above, Kerr solitons are all exactly self-similar, and deep-saturation solitons are all approximately self-similar; hence, it is now compelling to ask: "Can solitons of various scales coexist in the same nonlinear medium simultaneously?" If the answer is positive, can they coexist within one another in a fractal structure? And, if the answer to this question is also positive, then how can a nonlinear system be driven to generate solitons organized in a fractal structure? The answers to all of these questions are in the response of the nonlinear system to a non-adiabatic change in one (or more) of its properties. As described below, an abrupt change in the nonlinear coefficient, or in the saturation coefficient (in saturable systems), or in the dispersion coefficient (for temporal solitons), or in almost any parameter that leads to a large deviation of the pulse from the soliton existence curve, will lead to the appearance of a fractal structure driven by soliton dynamics. In our simulations, we observe self-similarity and fractals

both in the Kerr regime, and in the deep-saturation regime of Eq. (1).

Our goal is to design a physical system that can support many solitons all of different sizes simultaneously. Since the solitons of the (1+1)D Kerr and saturable NLSEs are robust, if one starts with a pulse that is close to a soliton shape, the pulse evolves into a stable soliton. However, if one starts with a pulse whose shape is very far off the existence curve, this pulse is not able to evolve smoothly into a soliton. Under proper conditions, this input pulse breaks up into smaller pieces and radiation. Quite often, the pieces resulting from this "explosion" include many small solitons, all of different sizes. If the nonlinearity is such that these solitons are self-similar to each other, one can claim to have observed self-similarity.

We distinguish between two scenarios that produce such breakup. The first is driven by noise and is easier to realize experimentally. Consider a pulse whose initial width is far above the existence curve launched into a nonlinear medium in the regime that can support self-similar solitons; the width of the pulse is much larger than the soliton of such peak intensity. Therefore, small perturbations (initiated by noise) of large wavelengths grow on top of the pulse as it propagates. After some distance, the energy in these perturbations becomes significant, and the pulse breaks up into smaller pulses. This phenomenon is known [6] as Modulational Instability (MI). In many systems, the products of this breakup include many solitons of different sizes. We call it "MI-induced breakup". An example of such a breakup in saturable NLSE is given in Fig. 2.

The second breakup scenario is "Dynamics-induced breakup". It is observable in numerical calculations that inherently have no or very little noise. It is also observable in "clean"

experimental systems, like temporal solitons in optical fibers [22]. Consider a pulse above the existence curve launched into a self-focusing medium. The local intensity increases the local index of refraction, thereby creating a waveguide structure that attracts light to the center of the induced waveguide. If the initial width of the pulse is much larger than the width of the lowest guided mode of this induced waveguide, the light coalesces towards the center trying to reach the solitonic shape, in which diffraction is exactly balanced by self-focusing. However, once the equilibrium is reached the pulse keeps shrinking because of its inertia. Since the equilibrium could not have been reached smoothly (the pulse is initially far above the existence curve), the pulse explodes into smaller pieces, which form smaller solitons of different sizes. For an example in saturable NLSE, please see the top plot of Fig. 3. Since both the underlying equation and the initial pulse obey left-right symmetry, the output multi-soliton pulses also obey this symmetry (in contrast with an MI-induced breakup, since noise obeys no symmetry). At any rate, Fig. 2, and the top plot in Fig. 3 clearly demonstrate self-similarity in systems that allow for existence of solitons.

In order to create fractals, one can apply the logic that has caused such breakups in a repetitive manner. The breakups into self-similar solitons were caused by an abrupt change from a medium in which the input pulse was stable into a second medium that tried to force the pulse to be much narrower. In the above examples, the initial medium is free space and the second medium is a self-focusing medium that supports solitons that are much narrower than the input pulse, when both have the same peak intensity. The same logic can be applied when the initial medium is nonlinear, when the abrupt transition is from a medium in which the input pulse is a stable soliton into a medium that breaks it up into much narrower solitons. In other words, if after the abrupt transition, the pulse is much wider than the soliton of the same intensity (i.e., the pulse is far above the existence curve), it cannot evolve smoothly into a single soliton. Instead, it

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explodes into solitons of different scales. In this spirit one can take the output "daughter solitons" at the end of the top plot in Fig. 3, and make an abrupt change in the nonlinear medium in which they propagate, and force each one of the "daughter solitons" to break up into a train of smaller solitons. This happens if the change moves the position of the "daughter-solitons" far above the existence curve. Such a change in the nonlinear medium can be realized either by altering the intensity of the pulses abruptly, or by changing the coefficient in front of ∇_T^2 in Eq. (1), or by changing the properties of the nonlinearity (magnitude, saturation, etc.). It is important that the change in the conditions is abrupt; an adiabatic change does not cause a breakup, but instead the pulse adapts and evolves smoothly into a narrower soliton, as shown in Fig. 5. When the change is abrupt and large enough, each of the pulses undergoes a self-similar breakup as illustrated in the middle plot of Fig. 3, or top plot of Fig. 4. This process can be repeated, in principle, an infinite number of times, thereby creating a fractal structure. Of course, all the resulting solitons after each breakup have to be in the regime where they are all self-similar to each other.

A three stage fractal is presented in Figs. 3, and 4. These breakups are dynamics-induced. At the input, self-focusing is much stronger than diffraction for a pulse of that width, so the pulse contracts and eventually breaks up into many self-similar solitons observed at the output of the upper-left figure. At the plane of the output of the upper-left figure, we change the denominator in the nonlinear term from $1+|\psi|^2$ to $1+(|\psi|^2/8)$. This makes all the pulses at the output of the upper-left figure have amplitude 8 times smaller than solitons of the same widths have. Then, we propagate the output of the upper-left figure for a few more diffraction lengths, resulting in a self-similar breakup of every pulse, as shown in the middle plot of Fig 3. At the output of the middle plot of Fig. 3, we change the nonlinear term into $1+(|\psi|^2/64)$, and propagate the pulse further. As shown in the bottom plot of Fig. 3, we observe one more stage of self-similar breakup. In these simulations, we use the saturable $\dfrac{|\psi|^2}{1+|\psi|^2}$ nonlinearity, to show when we expect the

fractal generation process to end in a real system. In this case, the third stage shown by bottom plot is the final breakup, because most of the end solitons at this stage are of peak intensities on the order of unity, which cease to be self-similar. As for the other end of this process, i.e., the first breakup, there is no upper limit: one can start this fractal generation process by literally breaking up plane waves. In practical systems in nonlinear optics, we expect to observe at least 3-4 breakup stages.

Let us go back now to the definition of a fractal [2] as "an object which appears self-similar under varying degrees of magnification. In effect, possessing symmetry across scale, with each small part replicating the structure of the whole". It is obvious that the structures described in Fig. 3 are fractals: they are self-similar **within** each scale (i.e., the "daughter solitons" after each breakup), and they are also self-similar over at least three widely separated scales. Also, as we show by the increasing magnification (Fig. 3), each part breaks up again and again in a structure replicating the whole. It is obvious that we have found a way to generate fractals, which are driven by soliton dynamics.

Several topics merit discussion. First, we note that fractals generated through the Kerr NLSE are **exact fractals** as they possess an **exact and strict self-similarity** on all scales. They have no upper or lower limit on the scale, as long as the underlying equation still describes the physical process within its approximations. They are **self-similar everywhere**. On the other hand, fractals generated through the saturable NLSE are more like **natural fractals**, because they are self-similar only at points where the intensity is much greater than unity and are not self-similar at the tails of each pulse. We want to emphasize the resemblance of our fractals to Cantor Sets [2]. The fractal structures in Figs. 3, and 4, are actually *Randomized Cantor Sets*, because at any given stage self-similar structures (soliton-like pulses) of various scales coexist,

with the distances between these pulses varying in what looks like a random manner, especially in the presence of significant noise.

Nevertheless, we have shown recently that the principle we described can be applied to create exact (regular) fractals as well. We simulated (1+1)D cubic self-focusing NLSE, and varied the dispersion coefficient in a prescribed manner, so that before each breakup we always start from the same point above the existence curve. This gives rise to the exact (regular) fractals shown in Fig 6. In Fig 7, we show the excellent overlap between the outputs of the various stages in the process (when they are all magnified and stretched to be on the same scale), thereby demonstrating that our fractals are indeed **exact Cantor Sets**. This is one of very rare physical systems in nature that supports exact (regular), as opposed to statistical (random) fractals.

One more issue that merits discussion is the question of the **fractal dimensions**. In the case of random fractals, like in Figs. 3, and 4, one can estimate the fractals dimension using box-counting method. However, in case of exact fractals, like in Fig. 6, one can calculate various fractal dimensions exactly. For example, one can take a fractal from Fig. 6, and construct a line-shaped Cantor set from it by drawing lines connecting the two half-maximum points for each soliton; these lines form a Cantor Set in the usual sense. The fractal in the left column of Fig. 6 would hence have "similarity dimension" of 0.2702, while the fractal in the right column of Fig. 6 would have a "similarity dimension" of 0.43175. Of course, the fractal dimension is specific to the system and the initial conditions used. Another topic has to do with **fractals in higher Euclidian dimensions**, e.g., in full 3D. Since _the ideas presented are not restricted by dimensionality,_ they should be easy to translate into higher dimensions and we envision fractals that have two different transverse scales, that is, when the width of every pulse in x is much

different than that in *y*. However, caution must be taken to make sure that the system can support (2+1)D soliton structures first. For example, "bright" 2D fractals cannot be generated through the Kerr NLSE because bright (2+1) D Kerr solitons are highly unstable unless they are organized in particular "necklace" structures [19]. The saturable NLSE, on the other hand, is known to support stable bright (2+1)D solitons [20], so we expect it to support 2D fractals as well. There are many other new ideas under research, but the main challenge is experimental: to demonstrate fractals.

In conclusion, we have proposed a general scheme of generating fractals by the dynamics of solitons that undergo abrupt changes along their propagation paths [21]. This method of fractal generation is universal and should exist in any nonlinear system that can support solitons. Interestingly, this is one of the very few cases in which one can generate fractals experimentally and investigate them theoretically knowing all the physics involved. Furthermore, this is one of very rare physical systems that can support exact fractals as well. In fact, experiments are currently carried out in our laboratory to demonstrate fractals from spatial and temporal solitons.

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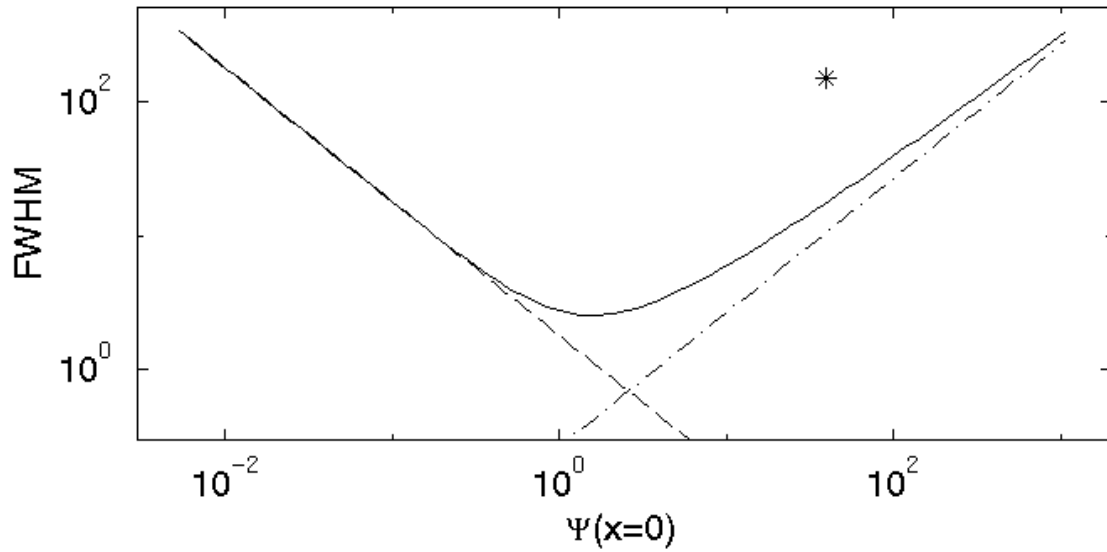
Figure 1: Existence Curves of Kerr-type solitons (dashed line), solitons in a saturable nonlinear medium (solid curve), and deep-saturation solitons (dashed-dotted line), all in (1+1) D. The vertical axis gives the normalized width of the intensity of the soliton, in the *x*-units of Eq. 1. The point indicated by * describes the input pulse to the fractal-generating process of Fig. 3.
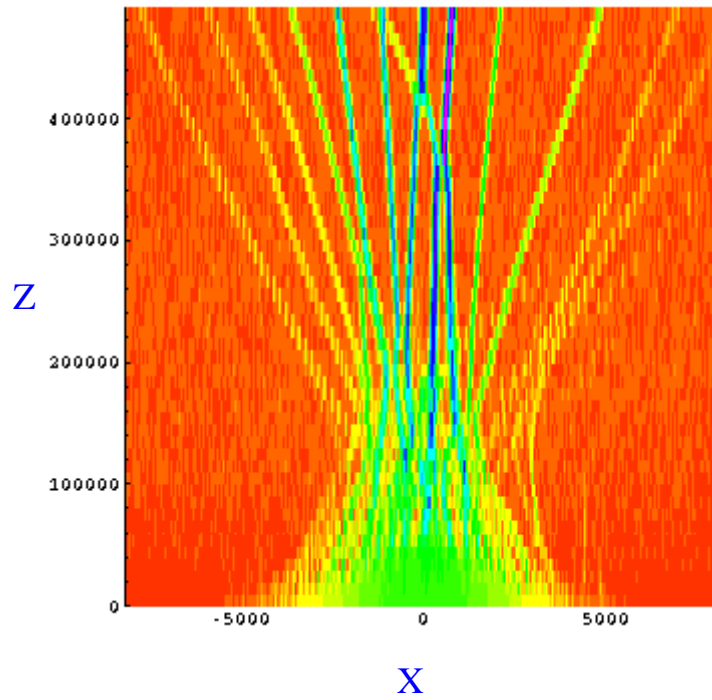
Figure 2: An example of an MI driven breakup, starting in the deep-saturation regime of (1+1)D saturable NLSE. The initial pulse was 80 times wider than the soliton of the same peak amplitude, and there was a significant amount of background noise.
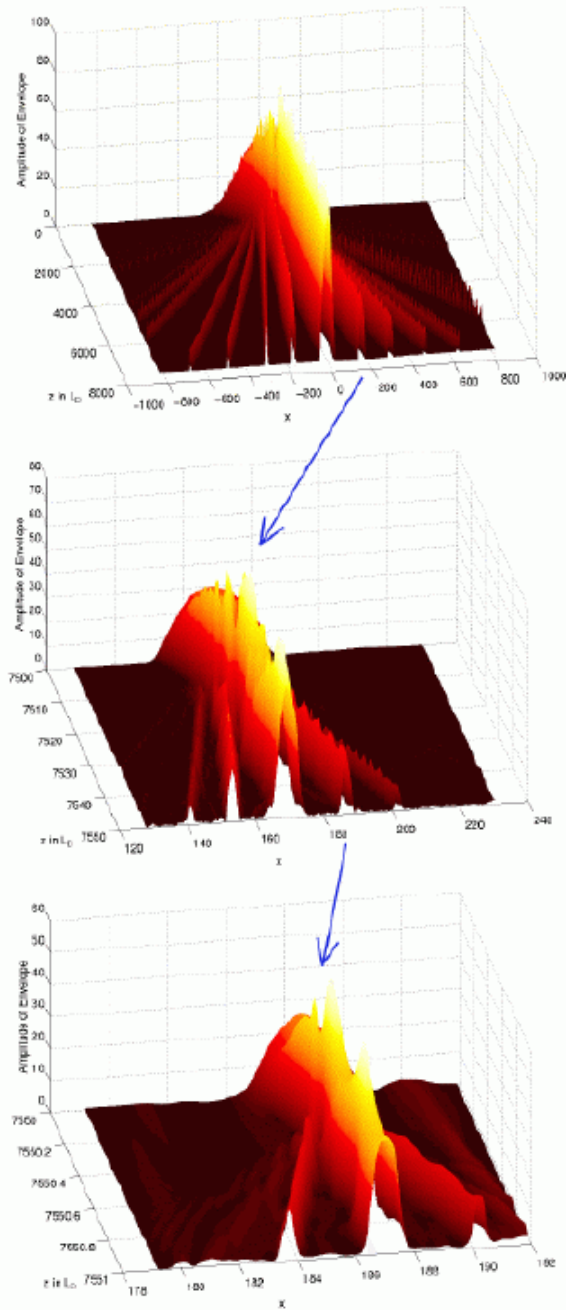
Figure 3: A 3-stage self-similar dynamics-driven breakup creating a fractal structure **starting** in the deep-saturation regime of the (1+1)D saturable NLSE. The initial pulse is given by point * in Fig. 1. The 1ˢᵗ stage is given in the top plot. A detail of the 2ⁿᵈ stage is given in the middle plot. Finally, a detail of the 3ʳᵈ stage is given in the bottom plot.
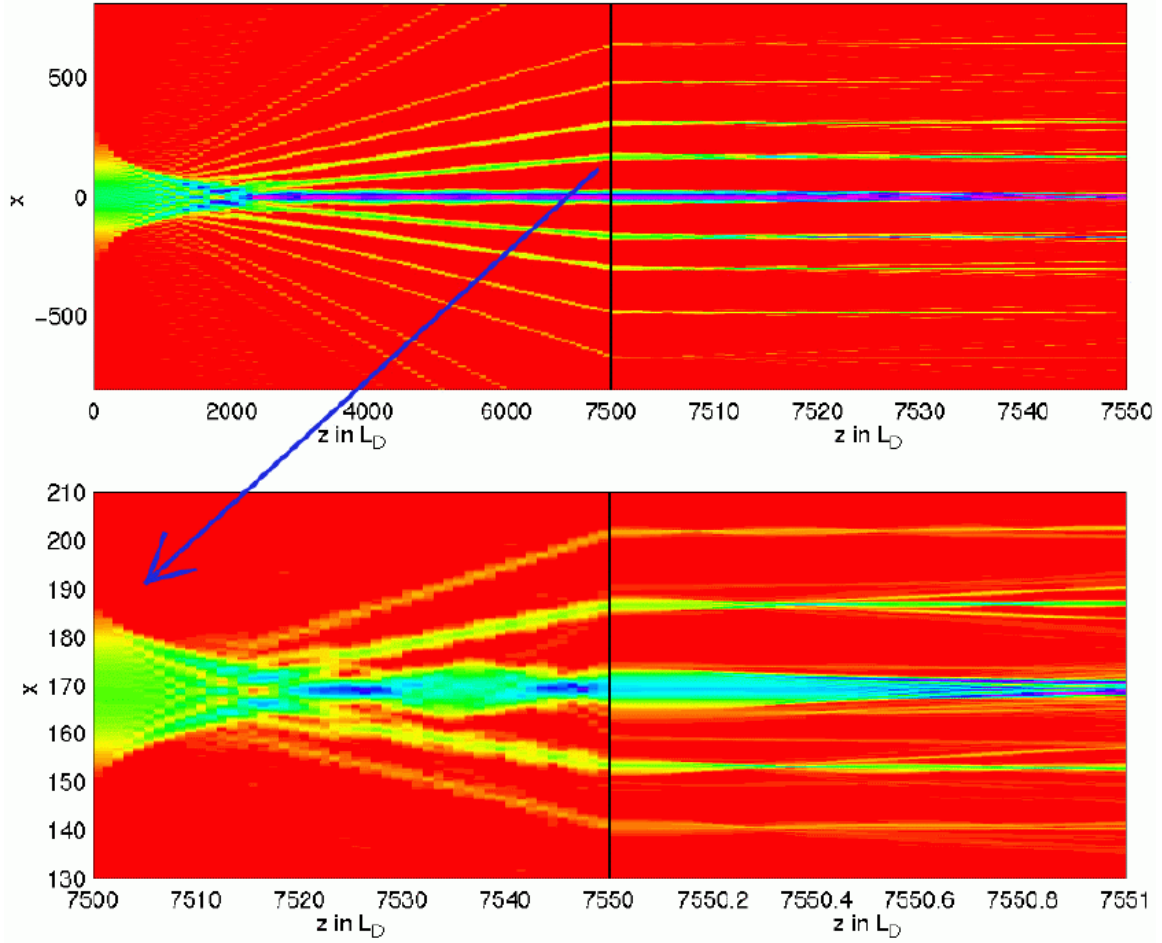
Figure 4: Top view of a 3-stage self-similar breakup creating a fractal structure **starting** in the deep-saturation regime of the (1+1)D saturable NLSE. The 1st stage is given in the upper-left plot. The 2nd stage is given in the upper-right plot. A magnified detail of the 2nd stage is shown in the lower-left plot. The continuation of evolution of that detail into the 3rd stage is given in the lower-right plot.
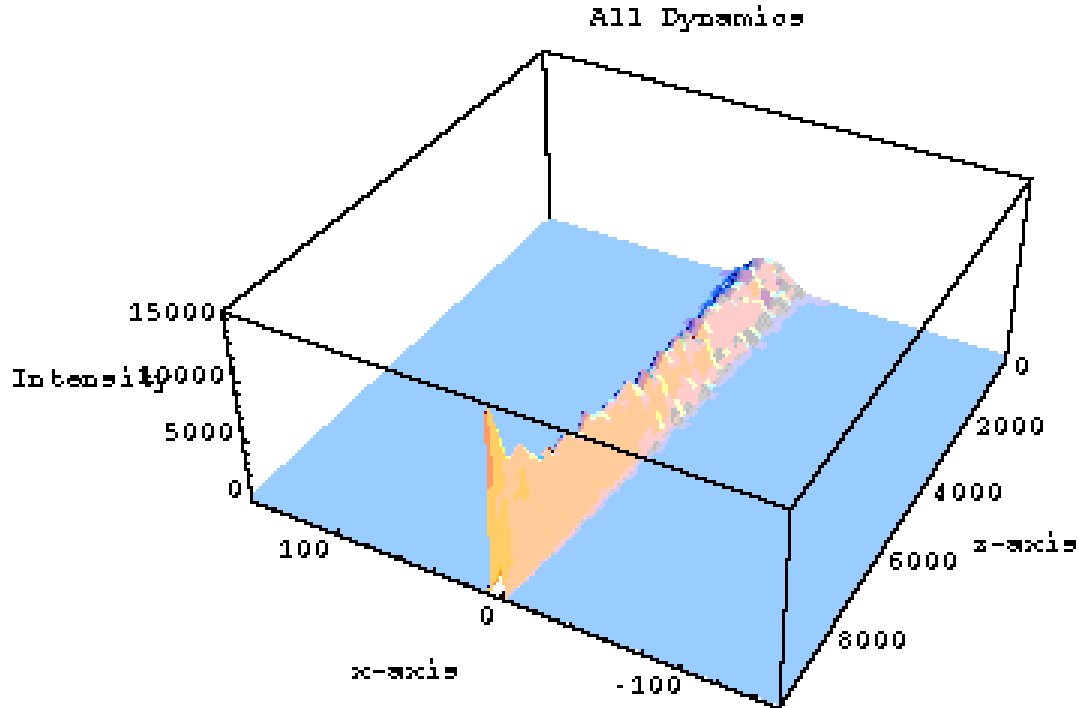
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Figure 5: An adiabatic change of the physical system in which the pulse propagates does not cause the required breakup. We start with a pulse that is initially a soliton of the (1+1)D cubic NLSE, and vary the saturation coefficient **adiabatically** (i.e. the characteristic propagation length scale on which the coefficient varies is much longer than the characteristic length scale of the pulse propagation given by the diffraction length of the pulse at any given moment.) Instead of breaking up, the soliton keeps evolving smoothly into whatever the instantaneous solitonic shape is. Physically this is expected because solitons of (1+1)D cubic NLSE are very stable creatures.
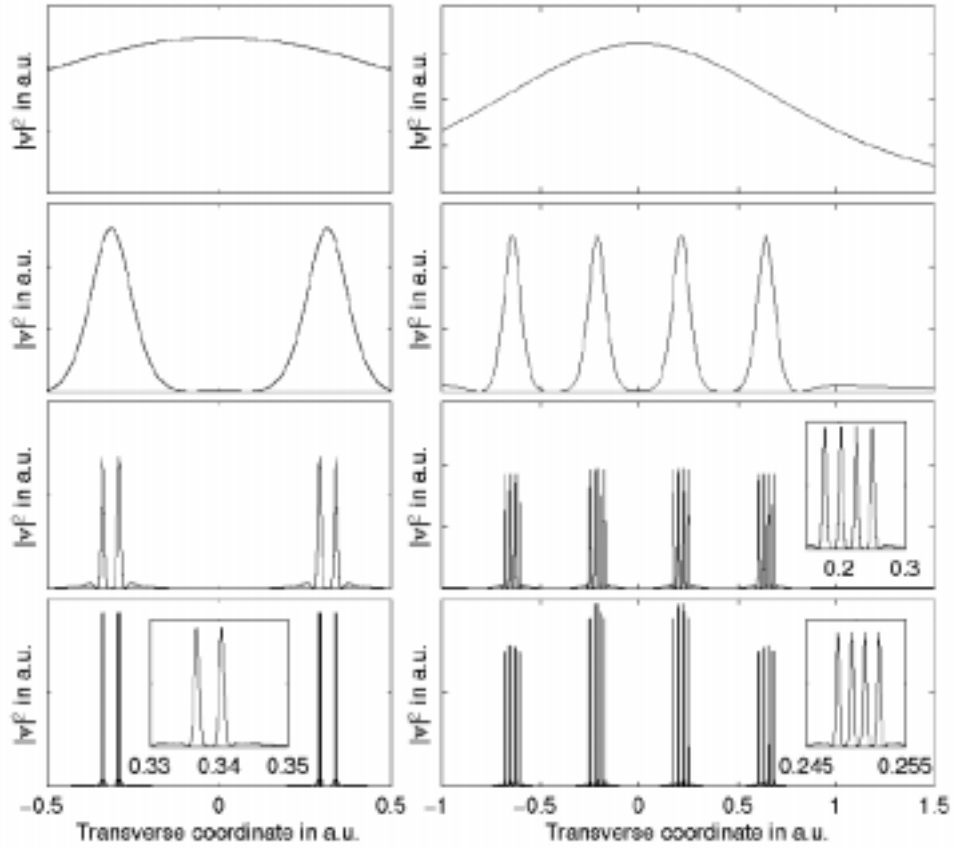
Figure 6: The two columns of the figure represent exact (regular) Cantor set fractal generation for two particular cases in the (1+1)D cubic self-focusing NLSE. In each case, we start with a sech-shaped pulse that is significantly higher than the soliton of the same width, as shown in the first row. The pulses start breaking up, as shown in the second row. When they acquire the shape depicted in the second row, we abruptly significantly decrease the diffraction coefficients in the underlying equations. This induces the second stage of the breakup and results in the shapes shown in the third row. Then, once again we decrease the diffraction coefficient abruptly, and induce the third stage of the breakup shown in the fourth row. In some cases, the pulses were too tiny to see at the given magnification, so the insets in the plots show magnified details of their corresponding plots.
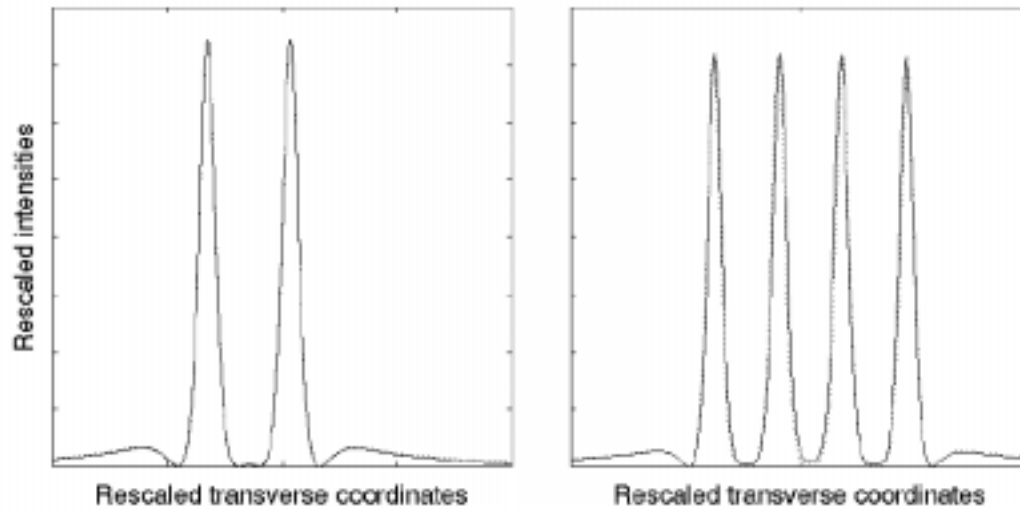
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Figure 7: This illustrates the fact that the fractals we generated in Figure 6 are indeed **exact**. We take the shapes of the second row of Figure 6, shift the shapes, rescale them with the appropriate factors, and superimpose them on the insets of the last row of Figure 6. The doted-lines represent the rescaled versions of the shapes of the second row of Figure 6, while the full-lines represent the shapes of the insets of the last row of Figure 6. The overlap in the left plot is so excellent that the two curves can hardly be distinguished.