Cantor Set Fractals from Solitons

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We show how a nonlinear system that supports solitons can be driven to generate exact (regular) Cantor set fractals. As an example, we use numerical simulations to demonstrate the formation of Cantor set fractals by temporal optical solitons. This fractal formation occurs in a cascade of nonlinear optical fibers through the dynamical evolution from a single input soliton.

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A fractal, as defined by Mandelbrot, "is a shape made of parts similar to the whole in some way" [1]. Fractals can be classified in numerous manners, of which one stands out rather distinctly: *exact* (regular) fractals versus *statistical* (random) fractals. An exact fractal is an "object which appears self-similar under varying degrees of magnification ... in effect, possessing symmetry across scale, with each small part replicating the structure of the whole" [1]. Taken literally, when the same object replicates itself on successively smaller scales, even though the number of scales in the physical world is never infinite, we call this object an "exact fractal." When, on the other hand, the object replicates itself in its statistical properties only, it is defined as a "statistical fractal." Statistical fractals have been observed in many physical systems, ranging from material structures (polymers, aggregation, interfaces, etc.), to biology, medicine, electric circuits, computer interconnects, galactic clusters, and many other surprising areas, including stock market price fluctuations [1]. In optics, fractals were identified in conjunction with the Talbot effect and diffraction from a binary grating [2] and with unstable cavity modes [3]. Exact fractals, on the other hand, such as the Cantor set, occur rarely in nature except as mathematical constructs. In this Letter we describe how a Cantor set of exact fractals can be constructed, under proper nonadiabatic conditions, in systems described by the $(1 + 1)$D cubic self-focusing nonlinear Schrödinger equation (NLSE). We demonstrate exact Cantor set fractals of temporal light pulses in a sequence of nonlinear optical fibers. We calculate their fractal similarity dimensions and explain how these results can be produced experimentally.

A Cantor set is best characterized by describing its generation [1]. Starting with a single line segment, the middle third is removed to leave behind two segments, each with length one-third of the original. From each of these segments, the middle third is again removed, and so on, *ad infinitum*. At every stage of the process, the result is *self-similar* to the previous stage, i.e., identical upon rescaling. This "triplet set" is not the only possible Cantor set: any arbitrary cascaded removal of portions of the line segment may form the repetitive structure.

This Letter is based on a recent idea [4] that nonlinear soliton-supporting systems can evolve under nonadiabatic conditions to give rise to self-similarity and fractals. Such fractals should be observable in many systems, and their existence depends on two requirements: (i) the system does not possess a natural length scale; i.e., the physics is the same on all scales (or, any natural scale is invisible in the parameter range of interest) and (ii) the system undergoes abrupt, nonadiabatic changes in at least one of its properties [4].

To illustrate generating fractals from solitons, Ref. [4] showed optical fractals evolving dynamically from a single input pulse or beam. The idea is to repetitively induce the breakup of the pulse (beam) into smaller pulses by abruptly modifying the balance between dispersion (diffraction) and nonlinearity. Consider a broad pulse launched into a nonlinear dispersive medium. The pulse is broad in the sense that its width is much larger than that of the characteristic fundamental soliton, given the peak intensity. This fundamental soliton width is determined by properties of the medium such as the dispersion and nonlinearity coefficients as well as by the soliton peak power. A broad pulse will always break up, either due to modulation instability [5] when random noise dominates or by soliton dynamics-induced breakup [4] when the noise is weak. The result of the breakup is a number of smaller pulses or "daughter solitons," which propagate stably in the medium in which the "mother pulse" broke up. The daughter solitons are self-similar to one another in the sense that they can be mapped (by change of scale only) onto one another, because they all have the same shape (hyperbolic secant for the Kerr-type nonlinearity). Now, if an adequately abrupt change is made to a property of the medium (e.g., the dispersion or the nonlinear coefficient [6]), then each of the daughter solitons seems broad and therefore unstable in the "modified" medium. The daughter solitons undergo the same instability-induced breakup experienced by the initial mother pulse and generate even smaller "granddaughter solitons." Successive changes to the medium properties thus create successive generations of solitons

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on successively smaller scales. *The resultant structure after every breakup is self-similar with the products of the first breakup.* The successive generations of breakups of each soliton into many daughter solitons leads to a structure which is self-similar on widely varying scales, and each part breaks up again in a structure replicating the whole. The entire structure is therefore a fractal.

In the general case, this method of generating fractals from solitons gives rise to statistical fractals. In the fractal which results from each breakup, the amplitudes of the individual solitons, the distances between them, and their relation to the solitons of a different "layer" are random. Thus, the self-similarity between the structures at different scales is only in their statistical properties. Here we show that the principle of "fractals from solitons" can be applied to create exact (regular) fractals, in the form of an exact Cantor set. The requirement is that after every breakup stage, all of the "daughter pulses" must be identical to one another. In this case, *all* the daughter pulses can be rescaled from one breakup stage to the next by the same constant, and the entire propagation dynamics repeats itself in an exact rescaled fashion. The resulting scaling on all length scales constitutes an exact Cantor set. *In this manner, one can obtain exact Cantor set fractals from solitons.* This represents one of the rare examples of a physical system that supports exact (regular), as opposed to statistical (random), fractals [1].

To illustrate the idea of generating Cantor set fractals from solitons, we analyze the propagation of a temporal optical pulse in a sequence of nonlinear fiber stages with dispersion coefficients and lengths specifically chosen to impose a constant rescaling factor between consecutive breakup products. We solve the $(1 + 1)$D cubic self-focusing NLSE, vary the dispersion coefficient in a manner designed to generate doublet- and quadruplet-Cantor set fractals, and show the formation of temporal optical soliton Cantor set fractals (Fig. 1).

The nonlinear propagation and breakup process in fiber segment "i" is described by the $(1 + 1)$D cubic NLSE:

$$i\,\frac{\partial\psi}{\partial z} - \frac{\beta^{(i)}}{2}\,\frac{\partial^2\psi}{\partial T^2} + \gamma|\psi|^2\psi = 0\,, \qquad (1)$$

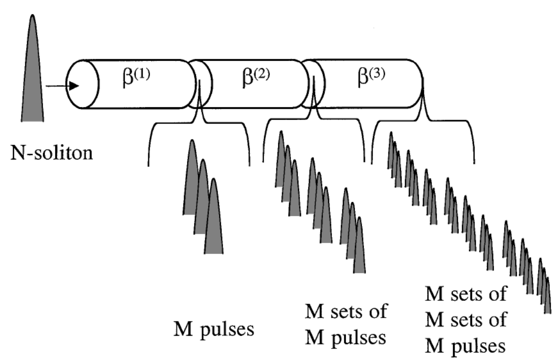

FIG. 1. Illustration of a sequence of nonlinear optical fiber segments with their dispersion constants and lengths specifically chosen to generate exact Cantor set fractals.

where $\psi(z, T)$ is the slowly varying envelope of the pulse, $T = t - z/v_g$ is the time in the propagation frame, v_g is the group velocity, $\beta^{(i)} < 0$ is the (anomalous) group velocity dispersion coefficient of segment i, and $\gamma > 0$ is proportional to the nonlinearity ($n_2 > 0$); z is the spatial variable in the direction of propagation and t is time. Equation (1) has a fundamental soliton solution of the form

$$\psi(z, T) = \sqrt{\frac{|\beta^{(i)}|}{\gamma(T_0^{(i)})^2}}\,\mathrm{sech}(T/T_0^{(i)}) \times \exp\{iz|\beta^{(i)}|/[2(T_0^{(i)})^2]\}\,, \qquad (2)$$

where $1.762\,74T_0^{(i)}$ is the temporal full width half maximum of $|\psi(z, T)|^2$ and $z_0^{(i)} = \pi(T_0^{(i)})^2/(2|\beta^{(i)}|)$ is the soliton period for fiber segment *i*. The *N*-order soliton (at $z = 0$) of Eq. (1) can be obtained by multiplying $\psi(z, T = 0)$ from Eq. (2) by a factor of N. A higher order soliton of a given $N > 1$ propagates in a periodic fashion. In the first half of the soliton period $(z_0^{(i)}/2)$, the pulse splits into two pulses, then into three, then into four, etc., up to $N - 1$ pulses [5]. In the second half of the period the process reverses itself until all the pulses have recombined into a single pulse identical to the original one. While attempting to generate Cantor set fractals from solitons, we observed that, if we start with an N-order soliton, it splits into $M < N$ pulses, each of which reaches an approximately hyperbolic secant shape. Furthermore, there is always a region in the evolution where all the M daughter pulses are almost fully identical and possess the same height. The breakup can be reproduced if we cut the fiber at this point and couple the pulses into a new fiber with a dispersion coefficient chosen such that each of the pulses launched into the second fiber is an N-order soliton. Each of the daughter pulses generated in the first fiber *exactly* replicates the breakup of the "mother soliton," on a smaller scale. Because Eq. (1) is the same on all scales, the entire second breakup process of each daughter pulse is a rescaled replica of the initial mother-pulse breakup. In fact, we can redefine the coordinates in the second fiber by simple rescaling, so that in the new coordinates the equation is identical to the equation (including all coefficients) describing the pulse dynamics in the first fiber. In this manner, we can continue the process recursively many times, resulting in an exact fractal structure that reproduces, on successively smaller scales, not only the final "product" (the pulses emerging from each fiber segment), but also the entire breakup evolution.

What remains to be specified is how we choose the sequence of fibers and the relations between their dispersion coefficients and lengths. Consider a sequence in which the ratio between the dispersion coefficients of every pair of consecutive segments is fixed $\beta^{(i+1)}/\beta^{(i)} = \eta$, where $\eta < 1$. This implies that the periods of the fundamental solitons in consecutive segments are related through $z_0^{(i+1)}/z_0^{(i)} = [(T_0^{(i+1)})^2/(T_0^{(i)})^2][1/\eta]$.

Numerically, we launch an N-order soliton into the first fiber segment and let it propagate until it breaks into M

hyperbolic-secant-like pulses of almost identical heights and widths. At this location we terminate the first fiber and label the distance propagated in it $L^{(1)}$. From the simulations we find the peak power $P_M^{(1)}$ and the temporal width $T_M^{(1)}$ of the M almost-identical pulses emerging from the first segment. The M pulses are then launched into the second segment. Our goal is to have, in the second segment, a rescaled replica of the evolution in the first segment. To achieve this, we require that each of these M pulses will become an N-order soliton in the second segment. Thus we equate the peak power in each of the M pulses in the first fiber to the peak power of an N-order soliton in the second fiber:

$$P_M^{(1)} = P_N^{(2)} = \frac{|\beta^{(2)}|}{\gamma (T_0^{(2)})^2} N^2, \tag{3}$$

where $T_M^{(1)} = T_0^{(2)}$ since it is the width of the input pulse to the second fiber. From Eq. (3) we find the dispersion coefficient in the second fiber, $\beta^{(2)}$. The ratio η between the dispersion coefficients in consecutive fibers determines the scaling of the similarity transformation. Using η and $T_0^{(2)}$ we calculate the period $z_0^{(2)}$. Requiring that the evolution in the second fiber is a rescaled replica of that in the first fiber, we get $L^{(2)}/z_0^{(2)} = L^{(1)}/z_0^{(1)}$. Each of the M pulses in the second fiber exactly reproduces the dynamics of the original soliton in the first fiber but on a smaller scale. At the end of the second stage, each of the M pulses transforms into M pulses, resulting in M sets of M pulses. The logic used to calculate the second stage parameters is used repeatedly to create many successive stages, each producing a factor of M pulses more than the previous stage.

We provide examples of Cantor set fractals from solitons by numerically solving Eq. (1). The order of the soliton used and the fraction of a soliton period propagated vary depending on the desired number of pulses, M. Figure 2 shows a quadruplet Cantor set fractal. We launch an $N = 8$ soliton into the first fiber characterized by $\gamma = 1$ and $\beta^{(1)} = -1$ and let it propagate for $0.1261 z_0^{(1)}$. At this point the pulse has separated into four nearly identical hyperbolic secant shaped pulses. We launch the emerging four pulses into the next fiber, characterized by $\beta^{(2)} = -0.012\,85$ and $\gamma = 1$. Each of the four solitons is an $N = 8$ soliton in the second fiber. We let the four soliton set propagate for $0.1261 z_0^{(2)}$, which is identical to $0.032\,90 z_0^{(1)}$. The scaling factor η is 0.012 85. We repeat this procedure with the third fiber and let the four sets of four solitons propagate for $0.1261 z_0^{(3)}$, so there are three stages total. The output consists of four sets of four sets of four solitons. This evolution is shown in Fig. 2(a), where the degree of darkness is proportional to $|\psi(z,T)|^2$. In Fig. 2(b), we show a magnified version of the lowermost branch of the fractal of Fig. 2(a). Figure 2(c) shows a magnified version of the lower branch of Fig. 2(b).

The same method is used to generate the doublet Cantor set fractals in Fig. 3, where an $N = 5$ soliton is propagated for $0.1623 z_0^{(i)}$ in each segment. Figure 3(a) shows

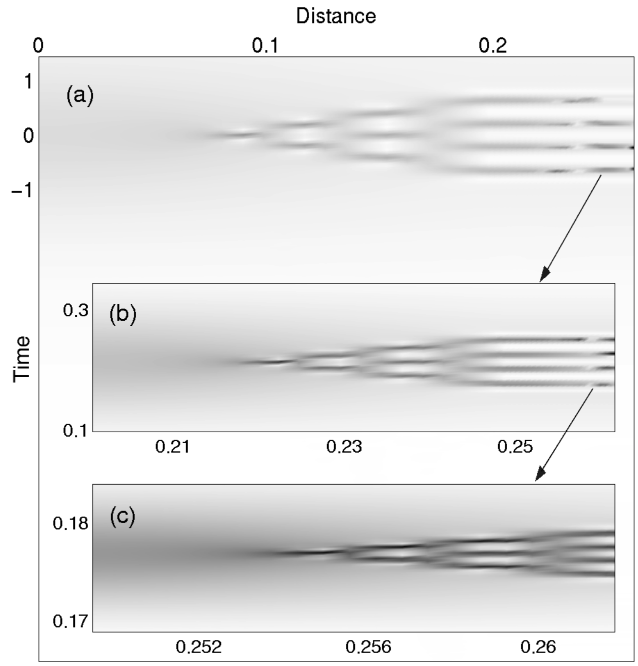

FIG. 2. Evolution of pulse envelope during the generation of a quadruplet Cantor set. The darkness is proportional to the pulse intensity. (a) shows the entire process. An $N = 8$ soliton is propagated for $0.3112 z_0^{(1)}$ and then propagated in a rescaled environment so that the input to that stage is four $N = 8$ solitons. The procedure is repeated for one more stage. (b) shows a magnification of the second stage; (c) shows the third stage. Units are normalized: $T_0 = 1$, peak power = 1, and 1 unit of distance = $T_0^2/|\beta^{(i)}|$.

the two output pulses emerging from the first segment. The two pulses are then fed into the rescaled environment, where they mimic the original $N = 5$ soliton, each breaking up into two more pulses [Fig. 3(b)]. Figure 3(c) shows the output after the third segment. At this stage we have two sets of two sets of two pulses, which is a Cantor set prefractal. If one could construct an infinite number of fiber segments, then it would be an exact regular Cantor set fractal in the mathematical sense. In physical systems, limitations such as high order dispersion, dissipation, and Raman scattering place a bound on the number of stages. As with any physical fractal, the breakups are prefractals rather than fractals; yet, we expect at least three stages in a real fiber sequence. To prove the generation of an exact Cantor set fractal, we choose *random* selections from each of the three panels of Fig. 3 and plot them on the same scale in Fig. 4: They fully coincide with one another. The exactness of the overlap in Fig. 4 indicates that this indeed is an exact Cantor set fractal. Similarly, we verify that the quadruplet fractals from Fig. 2 are exact. We have also generated a triplet Cantor set fractal from an $N = 6$ soliton, propagated for $0.1649 z_0^{(i)}$.

One can design an experiment of Cantor set fractals in a fiber optic system. For example, a doublet Cantor set can be generated from the breakup of an $N = 3$ soliton. In the first stage a 50 ps FWHM pulse of 0.88 W

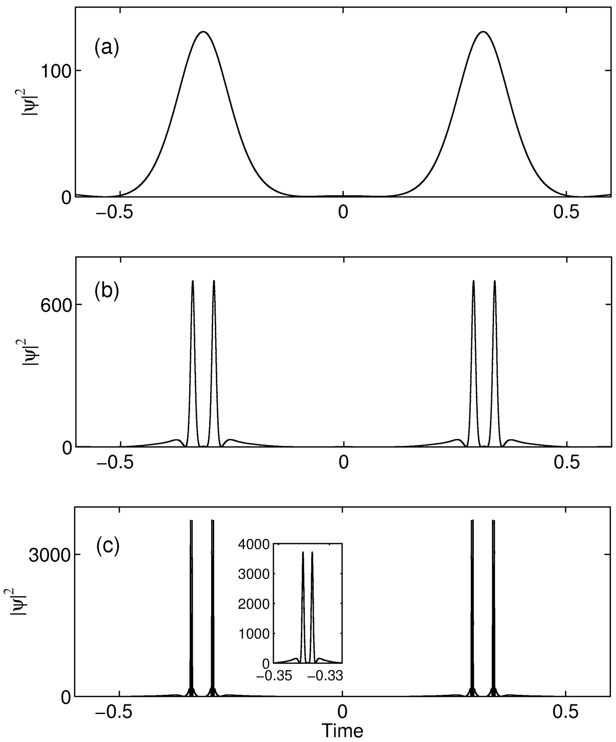

FIG. 3. Temporal pulse envelope after each of the three stages for doublet Cantor set. (a) shows the output from the first stage, (b) shows the results from the second, and (c) shows those from the third. The inset in (c) shows a magnification of one of the four sets of two. Units are normalized so that $T_0 = 1$, peak power $= 1$, and 1 unit of distance $= T_0^2/|\beta^{(i)}|$.

power is launched into a 6 km long fiber with $\beta^{(1)} = -127.6\ \mathrm{ps}^2/\mathrm{km}$ (assuming $\gamma = 1.62\ \mathrm{W}^{-1}\ \mathrm{km}^{-1}$ for all fibers). At the end of this fiber, which corresponds to the midpoint of the soliton period, the input pulse has broken into two pulses of peak power 1.2 W and width 13.2 ps spaced 42 ps apart. These pulses are then coupled into a second 4.1 km long fiber characterized by a dispersion parameter of $\beta^{(2)} = -12.2\ \mathrm{ps}^2/\mathrm{km}$. The pulses exiting this second stage are each 3.3 ps in duration and peak power 1.9 W. They are grouped in pairs separated by 9.9 ps. Finally, the pulses are propagated in a third 2.7 km long stage with $\beta^{(3)} = -1.2\ \mathrm{ps}^2/\mathrm{km}$. This results in two sets of two sets of two pulses, each of width 816 fs and peak power 3 W, grouped in pairs separated by 2.4 ps. These results have been confirmed through simulations including third order dispersion, fiber loss, and Raman scattering. The inclusion of these additional terms in Eq. (1) limits the number of stages which may be realistically obtained experimentally. The example system given above is consistent with readily available fibers. One may use specialty fibers (dispersion flattened or dispersion decreasing fibers) to combat effects of third order dispersion and loss to expand the number of experimentally realizable stages.

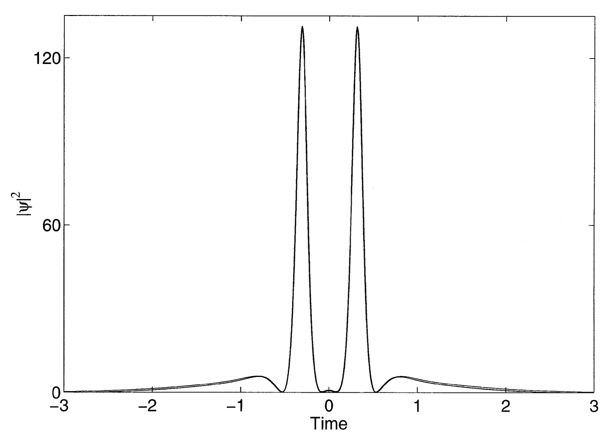

FIG. 4. Illustration of exact self-similarity of pulse envelopes after each of the stages of the doublet Cantor set. The three panels shown in Fig. 3 have been appropriately rescaled, shifted, and overlapped. Units are $T_0 = 1$, peak power $= 1$, and 1 unit of distance $= T_0^2/|\beta^{(i)}|$.

The Cantor set fractals in the fiber optic system are robust to a variety of perturbations in the fiber parameters and variations in the initial pulse conditions. We simulated the evolution of the Cantor set fractals under 5% deviations in the pulse peak power, pulse width, fiber length, and dispersion. We also added 2% (of the power) of excess Gaussian white noise and launched a Gaussian initial pulse shape. Under all these variations, the resulting Cantor set fractals exhibit excellent similarity to the ideal case.

Although we generate only prefractals, we can calculate the fractal dimension for an equivalent infinite number of stages. There are various definitions of fractal dimensions; here we calculate the *similarity dimension* D_s. In the construction of a fractal an original object is replicated into many rescaled copies. If the length of the original object is unity, ε is the length of each new copy, and N is the number of copies. The similarity dimension is [1]: $D_s = \log(N)/\log(1/\varepsilon)$. For the doublet Cantor set fractal from Fig. 3, $D_s = 0.2702$, and for the quadruplet Cantor set fractal from Fig. 2, $D_s = 0.4318$.

In conclusion, we have shown how a nonlinear soliton-supporting system can be driven to generate exact (regular) Cantor set fractals and have demonstrated theoretically optical temporal Cantor set fractals in nonlinear fibers. The next challenge is to observe Cantor set fractals experimentally.

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