G10 Swap and Exchange Rates*

Jeremy J. Graveline† and Scott Joslin‡

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Abstract

This paper develops a simple no-arbitrage model of foreign exchange rates and interest rates that can easily be applied to an arbitrary number of foreign currencies. The model has the appealing feature that it reduces to a standard two or three factor model for pricing yields in each currency, yet it can still accommodate a small number of globally priced risk factors. We use the model to analyze the joint dynamics of exchange rates and the term structures of swap rates for the G10 currencies. Using both in- and out-of-sample measures of fit we conclude that there is one priced risk factor in G10 swap rates. We estimate the risk premium for exposure to this single factor and show that a U.S. fixed income investor can more than triple the Sharpe ratio of her portfolio if she is willing/able to invest in any yield in any G10 currency.

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†University of Minnesota Carlson School of Management, jeremy@umn.edu

‡MIT Sloan School of Management, sjoslin@mit.edu

1 Introduction

This paper develops a simple no-arbitrage model of foreign exchange rates and interest rates that can easily be applied to an arbitrary number of foreign currencies and term structures. The model we develop has two particularly appealing features. First, it reduces to a standard two or three factor affine term structure model for pricing yields with different maturities in each currency. Second, although the number factors in the model can grow large with the number of foreign currencies and term structures, the number of priced risk factors can remain small.

We use weekly data from 1993 to 2001 to estimate our multi-currency model of the joint dynamics of exchange rates and the term structures of swap rates for the G10 currencies. Using the period from 2002 until March 2009 as an out-of-sample test, we find that our ten currency model with a single priced risk factor is better able to predict changes in yields compared with ten single-currency models that are each solely designed to explain changes in that currency's yields. Our empirical estimates illustrate that the information in global swap markets can be used in a parsimonious manner to improve our predictions of changes in the yield curves of individual the countries.

We also use the model to examine the risk/reward tradeoff that is available to global fixed income investors who are able to invest in any zero-coupon bond in any of the G10 currencies. An investor who only holds U.S. zero-coupon bonds (but can use the yields on zero-coupon bonds in the other nine currencies to help predict changes in U.S. interest rates) can more than triple her expected excess return for a given level of volatility if she is able to hold foreign currencies and zero-coupon bonds in her portfolio.

Our paper provides a significant extension to the growing literature on two-currency term structure models.[1] To our knowledge, Hodrick and Vassalou (2002) was the first paper to empirically analyze a multi-currency term structure model. They examine whether short-term interest rates in the U.S., Germany, Japan, and Britain can help to predict changes in the exchange rates and short-term interest rates (up to one year). Our paper differs along a couple of important dimensions. Most importantly, Hodrick and Vassalou (2002) assume that exchanges rates and interest rates are both driven by the same sources of variation.[2] However, Brandt and Santa-Clara (2002), Graveline (2006), and Leippold and Wu (2007) all emphasize that changes in exchange rates are largely independent of changes in interest rates, and our model accommodates this well-documented feature of the data. Second, we consider the full term structure of interest rates out to ten years in maturity so that our model must simultaneously account for risk premia on both long-term bonds *and* currencies.

Our model uses an affine framework (e.g., see Dai and Singleton (2000)) which differs slightly from the multi-currency quadratic term structure model developed Leippold and Wu

[1] Recent examples of two-currency term structure models include Backus et al. (2001), Brandt and Santa-Clara (2002), Ahn (2004), and Graveline (2006). Bansal (1997) and Brennan and Xia (2006) discuss multi-currency and two-currency term structure models respectively, but empirically they analyze single-currency models and do not consider the joint dynamics of interest rates in different currencies.

[2] Bansal (1997), Backus et al. (2001), and Brennan and Xia (2006) also assume that exchange rates and interest rates are both driven by the same sources of variation so that currencies are spanned by bonds.

(2007).[3] Again, our paper differs along a couple of more important dimensions. First, we draw a clear distinction between the total number of risk factors (sources of uncertainty) and the number of linear combinations of those factors that are priced. This distinction is quite relevant because the number of sources of uncertainty must necessarily grow large when the model is used to analyze the joint dynamics of many currencies and yield curves, however basic economic intuition suggests that each risk factor should not be independently priced. The second important distinction in our paper is that the model remains tractable so that we are able to empirically analyze the joint dynamics of the G10 currencies and yields curves in a no-arbitrage framework. By contrast, Leippold and Wu (2007) empirically implement a two-currency version of their model to analyze the U.S. and Japan.

More recently, Diebold et al. (2008) and Jotikasthir et al. (2010) have also analyzed the joint dynamics of yield curves in multiple currencies. Diebold et al. (2008) provide a statistical analysis of the joint dynamics of yield curves in Germany, Japan, the U.K. and the U.S., and use a Nelson and Siegel (1987) approach to parameterize the cross-section of yields. They do not model the relevant exchange rates and do not use a pricing kernel (i.e. they do not enforce no-arbitrage). Jotikasthir et al. (2010) also model the joint dynamics of yields in the U.S., the U.K., and Germany. They do not explicitly analyze exchange rates, but implicitly their model assumes that exchanges rates and interest rates are both driven by the same sources of variation (i.e. exchange rates are spanned by bonds).

Finally, our paper is related to papers such as Ang and Chen (2010) and Lustig et al. (2011) who analyze the returns to portfolios of currencies. Lustig et al. (2011) form currency portfolios based on the difference in short-term interest rates across currencies, while Ang and Chen (2010) form currency portfolio based on changes in interest rates and term spreads. We use our model to analyze the optimal growth portfolio when investors can invest in both currencies *and* swaps (bonds) with different maturities across the G10. Moreover, our framework imposes a small number of priced portfolios with time-varying prices of risk across both currencies and swaps. Our analysis of the optimal growth portfolio also relates to work by Glen and Jorion (1993), Campbell et al. (2003), and Campbell et al. (2010) who analyze the diversification benefits of currencies of foreign bonds.

The remainder of the paper is organized as follows. Section (2) develops our arbitrage-free econometric model, Section (3) discusses our estimation method and empirical results, and Section (4) concludes.

2 Econometric Model

Our econometric model for exchange rates and interest rates in multiple currencies draws heavily on the insights from single-currency affine term structure models, so we begin with a review of these models in a continuous-time diffusion setting as described in Duffie and Kan (1996) and Dai and Singleton (2000).

[3] Bakshi et al. (2008) also develop and analyze a dynamic multi-currency model but they do not model interest rate dynamics.

2.1 Single-Currency Affine Term Structure Models

Let M_t be the minimum variance pricing kernel and r_t be the (instantaneous) risk-free short interest rate. The standard single-currency affine term structure model described by Duffie and Kan (1996) is

$$
\begin{aligned}
r_t &= \rho_0 + \rho_1 \cdot X_t, && \text{(1a)}\\
d\left\langle X, X^\top \right\rangle_t &= \left(H_0 + H_1 \cdot X_t\right) dt, && \text{(1b)}\\
\underbrace{\mathbb{E}_t\left[dX_t\right] + d\left\langle X, \log M\right\rangle_t}_{\mathbb{E}_t^{\mathbb{Q}}[dX_t]} &= \left(\theta + \mathcal{K} X_t\right) dt, && \text{(1c)}
\end{aligned}
$$

where $d\langle \cdot, \cdot \rangle_t$ denotes quadratic variation (i.e. instantaneous covariance) and $H_0 + H_1 \cdot X_t$ is symmetric and positive semi-definite. Let $P_t(T) = \mathbb{E}_t\left[\frac{M_T}{M_t} \cdot 1\right]$ be the price at time t of a zero-coupon bond that pays 1 at time T and let $Y_t(T) = -\left[\log P_t(T)\right] / (T-t)$ be its continuously-compounded yield. Yields are affine (and prices are exponential affine) in the state vector X_t since

$$
P_t(T) = e^{A_{T-t} + B_{T-t} \cdot X_t} \quad \Leftrightarrow \quad Y_t(T) = \underbrace{-A_{T-t} / (T-t)}_{\tilde{A}_{T-t}} \underbrace{- \left[B_{T-t} / (T-t)\right]}_{\tilde{B}_{T-t}} \cdot X_t, \tag{2a}
$$

where

$$
\begin{aligned}
B_\tau &= -\int_0^\tau \left[\rho_1 - \mathcal{K}^\top B_u - \frac{1}{2} B_u^\top H_1 \cdot B_u\right] du, && \text{(2b)}\\
A_\tau &= -\int_0^\tau \left[\rho_0 - \theta \cdot B_u - \frac{1}{2} B_u^\top H_0 B_u\right] du, && \text{(2c)}
\end{aligned}
$$

Let P_t be a vector of zero-coupon bond prices with different maturities and let A and B be the model-implied vector and matrix from equation (2) such that $\log P_t = A + B X_t$. Then expected returns on zero-coupon bonds are

$$
\mathbb{E}_t\left[dP_t\right] = P_t\left[r_t\, dt - d\left\langle \log P, \log M\right\rangle_t\right] dt = P_t\left[r_t\, dt - B\, d\left\langle X, \log M\right\rangle_t\right]. \tag{3}
$$

Researchers typically model the risk premium, or quadratic variation of the state vector with the pricing kernel, as also being affine in the state vector

$$
-\,d\left\langle X, \log M\right\rangle_t = \left(\Lambda_0 + \Lambda_1 X_t\right) dt.^{4} \tag{4}
$$

[4] This specification uses an extended affine price of risk from Cheridito et al. (2007) which is an extension of the essentially affine price of risk in Duffee (2002) and the completely affine price of risk in Dai and Singleton (2000). With this specification, the true drift (and not just the risk-neutral drift in equation (1c)) is affine since

$$
\mathbb{E}_t\left[dX_t\right] = \mathbb{E}_t\left[dX_t\right] + d\left\langle X, \log M\right\rangle_t - d\left\langle X, \log M\right\rangle_t .
$$

Only the risk-neutral drift is required to be an affine function of the states for the yields to also be affine in the states.

Together, equations (1) and (4) constitute a single-currency term structure model. Any affine transformation of the state vector produces a model of the same form, so all of the parameters are not uniquely identified. Dai and Singleton (2000) provide a canonical representation (affine transformation) of single-currency affine term strucuture models and also discuss the parameter restrictions that are required to ensure that the elements of the state vector that govern stochastic volatility (i.e. those elements of X_t with non-zero weights in H_1) remain positive. Our empirical implementation uses a Gaussian model (i.e. $H_1 = 0$) so we do not discuss the necessary parameter restrictions for specifications with stochastic volatility. Also, we do not discuss the various canonical representations because we follow Duffie and Kan (1996) and fix the affine transformation of the model so that the factors are specific linear combinations of zero-coupon yields.

To be specific, let Y_t be a K-dimensional vector of zero-coupon yields with different maturities and define the state vector as $X_t = L_0 + L_1 Y_t$, where L_0 is $N \times 1$ and L_1 is $N \times K$ for $N \leq K$. We can write $Y_t = \tilde{A} + \tilde{B} X_t$, where $\tilde{A}$ and $\tilde{B}$ are $K \times 1$ and $K \times N$ matrices that are determined by the model parameters (ρ_0, ρ_1, θ, $\mathcal{K}$, H_0, and H_1) according to equation (2). No arbitrage provides $N \times (N+1)$ parameter restrictions since

$$X_t = L_0 + L_1 Y_t = L_0 + L_1 \left(\tilde{A} + \tilde{B} X_t \right) \quad \Rightarrow \quad L_0 + L_1 \tilde{A} = 0 \quad \text{and} \quad L_1 \tilde{B} = \mathcal{I}. \tag{5}$$

In the empirical implementation it is important to choose the state vector to be a linear combination of yields that can account for much of the cross-sectional variation in yields. We know from previous literature (e.g., Litterman and Scheinkman (1991)) that the first 3 principal components of changes in each country's yield curve account for most of the variation along the yield curve. Therefore, in our estimations we choose the state vector to be the first 2 or 3 principal components (depending upon data availability) of yields in each currency.[5]For the yield curves of the G10 countries that we model in this paper, the first 3 principal components of changes in each country's yield curve are very closely related to changes in the level, slope, and curvature of yields in that currency.

We make a small but important modification to the standard model to reflect the empirical work of Cochrane and Piazzesi (2005). They find that one linear combination of yields (equivalently, forward rates) does a good job of predicting excess returns of bonds with different maturities. Moreover, this linear combination is not well-spanned by the first 3 principal components of yields. We incorporate this finding into the risk premia specification as follows

$$-d \left\langle X, \log M \right\rangle_t = \left(\Lambda_0 + \Lambda_1 Y_t \right) dt, \quad \text{such that } \operatorname{rank}\left([\Lambda_0, \Lambda_1] \right) = 1.^{6} \tag{6}$$

The rank restriction in equation (6) means that

$$\Lambda_0 + \Lambda_1 Y_t = \Lambda \left(\tilde{L}_0 + \tilde{L}_1 \cdot Y_t \right) \tag{7}$$

[5] A common approach to estimating term structure models is to follow Chen and Scott (1993) and assume that the model prices a subset of yields exactly and the remaining yields are priced with error. With this estimation approach, the state vector is an affine function of the yields that are assumed to be priced exactly. Our approach is a slight generalization of this technique in that we effectively assume that the model correctly prices an affine function of yields rather than a specific subset of yields. See also Joslin et al. (2010b) for a discussion of this approach.

[6] Joslin et al. (2010a) also use a rank restriction on risk premia.

for some $N \times 1$ vector Λ, constant $\tilde{L}_0$, and $K \times 1$ vector $\tilde{L}_1$.

In theory, if our model was perfect then it would correctly price all of the yields and there would be no new information in $\tilde{L}_0 + \tilde{L}_1 \cdot Y_t$ that was not already contained in the state vector X_t,

$$\tilde{L}_0 + \tilde{L}_1 \cdot Y_t = \tilde{L}_0 + \tilde{L}_1 \left(\tilde{A} + \tilde{B} X_t \right) = \left(\tilde{L}_0 + \tilde{L}_1 \tilde{A} \right) + \tilde{L}_1 \tilde{B} X_t. \tag{8}$$

In practice, the model is not perfect and only prices the state vector $X_t = L_0 + L_1 Y_t$ without error. Therefore all of the yields outside of this span, including $\tilde{L}_0 + \tilde{L}_1 \cdot Y_t$, will be priced with error,

$$Y_t = \tilde{A} + \tilde{B} X_t + \varepsilon_t \quad \Rightarrow \quad \tilde{L}_0 + \tilde{L}_1 \cdot Y_t = \left(\tilde{L}_0 + \tilde{L}_1 \tilde{A} \right) + \tilde{L}_1 \tilde{B} X_t + \tilde{L}_1 \varepsilon_t. \tag{9}$$

Although the model does not perfectly price the entire cross-section Y_t of yields with different maturities, they are still observable at time t and therefore we can include them in the model. Intuitively, it is convenient to reduce the state space to $X_T - X_t$ when constructing the model-implied distribution of changes yields $Y_T - Y_t$ at a future dates $T > t$. However, at time t we observe the contemporaneous cross-section Y_t of yields so there no benefit to conditioning down the state space to X_t and ignoring the information in the model's cross-sectional pricing errors.

To conclude, the multi-currency model that we develop in the next section draws heavily on the following single-currency model,

$$\begin{aligned} r_t &= \rho_0 + \rho_1 \cdot X_t, && (10a) \\ d \left\langle X, X^\top \right\rangle_t &= H_0 \, dt, && (10b) \\ \mathbb{E}_t \left[dX_t \right] + d \left\langle X, \log M \right\rangle_t &= \left(\theta + \mathcal{K} X_t \right) dt, && (10c) \end{aligned}$$

where

$$X_t = L_0 + L_1 Y_t, \tag{10d}$$

and L_0 and L_1 are the weights from the first 2 or 3 principal components of changes in yields. Also, in our single-currency model risk premia are governed by

$$\begin{aligned} - d \left\langle X, \log M \right\rangle_t &= \left(\Lambda_0 + \Lambda_1 Y_t \right) dt, \quad \text{such that } \operatorname{rank}\left(\left[\Lambda_0, \Lambda_1 \right] \right) = 1, && (11a) \\ &= \Lambda \tilde{X}_t dt, && (11b) \end{aligned}$$

where

$$\tilde{X}_t = \tilde{L}_0 + \tilde{L}_1 \cdot Y_t \tag{11c}$$

and $\tilde{L}_0$ is a constant and $\tilde{L}_1$ is a vector with the same length as the vector of yields Y_t.

We apply this same style of single-currency model to the term structure of interest rates in foreign currencies. Let $S_t^{(i)}$ be country i's exchange rate expressed in units of domestic currency per unit of foreign currency i and let $r_t^{(i)}$ be the risk-free short interest rate in that currency. If $Y_t^{(i)}$ is a $K^{(i)}$-dimensional vector of zero-coupon yields in foreign currency i then the equivalent term structure model is

$$
\begin{aligned}
r_t^{(i)} &= \rho_0^{(i)} + \rho_1^{(i)} \cdot X_t^{(i)}, && (12a) \\
d\left\langle X^{(i)}, X^{(i)\top} \right\rangle_t &= H_0^{(i)}\, dt, && (12b) \\
\mathbb{E}_t\left[dX_t^{(i)}\right] + d\left\langle X^{(i)}, \log MS^{(i)} \right\rangle_t &= \left(\theta^{(i)} + \mathcal{K}^{(i)} X_t^{(i)}\right) dt,^{7} && (12c)
\end{aligned}
$$

where

$$
X_t^{(i)} = L_0^{(i)} + L_1^{(i)} Y_t^{(i)}, \tag{12d}
$$

and $L_0^{(i)}$ and $L_1^{(i)}$ are the weights from the first 2 or 3 principal components of changes in yields in foreign currency i. Similarly, risk premia are governed by

$$
\begin{aligned}
-\, d\left\langle X^{(i)}, \log MS^{(i)} \right\rangle_t &= \left(\Lambda_0^{(i)} + \Lambda_1^{(i)} Y_t^{(i)}\right) dt, \quad \operatorname{rank}\left(\left[\Lambda_0^{(i)}, \Lambda_1^{(i)}\right]\right) = 1, && (15a) \\
&= \Lambda^{(i)} \tilde{X}_t^{(i)} dt, && (15b)
\end{aligned}
$$

where

$$
\tilde{X}_t^{(i)} = \tilde{L}_0^{(i)} + \tilde{L}_1^{(i)} \cdot Y_t^{(i)} \tag{15c}
$$

and $\tilde{L}_0^{(i)}$ is a constant and $\tilde{L}_1^{(i)}$ is a vector with the same length as the vector of yields $Y_t^{(i)}$.

2.2 Multi-Currency Extension

The most general multi-currency affine model with I foreign currencies and term structures has the same form as the single-currency model in equations (1) and (4),

$$
\begin{aligned}
r_t &= \rho_0 + \rho_1 \cdot X_t, && (16a) \\
r_t^{(i)} &= \rho_0^{(i)} + \rho_1^{(i)} \cdot X_t, \quad i = 1, \ldots, I, && (16b) \\
d\left\langle \begin{bmatrix} X \\ \log S \end{bmatrix}, \begin{bmatrix} X \\ \log S \end{bmatrix}^\top \right\rangle_t &= H_0 + H_1 \cdot X_t, && (16c) \\
\mathbb{E}_t\left[dX_t\right] + d\left\langle X, \log M \right\rangle_t &= \left(\theta + \mathcal{K} X_t\right) dt,^{8} && (16d) \\
-d\left\langle \begin{bmatrix} X \\ \log S \end{bmatrix}, \log M \right\rangle_t &= \Lambda_0 + \Lambda_1 \begin{bmatrix} X \\ \log S \end{bmatrix}_t, && (16e)
\end{aligned}
$$

[7] If M is the pricing kernel denominated in the domestic currency then

$$
M_t Z_t = \mathbb{E}_t\left[M_T Z_T\right], \tag{13}
$$

for any asset price Z in the domestic currency. If $S^{(i)}$ is country i's exchange rate expressed in units of domestic currency per unit of foreign currency i then $Z/S^{(i)}$ is the asset's price in foreign currency i and $MS^{(i)}$ is the associated pricing kernel since

$$
M_t S_t^{(i)} Z_t / S_t^{(i)} = M_t Z_t = \mathbb{E}_t\left[M_T Z_T\right] = \mathbb{E}_t\left[M_T S_T^{(i)} Z_T / S_T^{(i)}\right]. \tag{14}
$$

where $\log S_t = \left[\log S^{(1)}, \ldots, \log S^{(I)}\right]_t^\top$ is a vector of the log exchange rates. From a practical standpoint, the dimension of the state vector must grow with the number of exchange rates and term structures the model is designed to capture. The size of the parameter space of the most general multi-currency term structure model is proportional to the cubed length of the state vector (without stochastic volatility it's proportional to the squared length of the state vector). Therefore, as the number of currencies and term structures in a multi-currency model grows, the most general model can quickly become impractical to estimate. We develop a simple and tractable multi-currency model that draws on two key insights. First, single-currency term structure models typically have low cross-sectional pricing errors and multi-currency models should inherit this feature. Second, the number of priced risk factors in the model does not need to grow with the size of the state vector.

Our empirical implemenation works with a domestic term structure plus nine foreign currencies and term structures, but the approach is easy to generalize to an arbitrary number. Let $X_t = L_0 + L_1 Y_t$ be the affine function of yields that serves as the state vector for the domestic single-currency term structure model. Similarly, let $X_t^{(i)} = L_0^{(i)} + L_1^{(i)} Y_t^{(i)}$ be the state vectors for the single-currency term structure models in foreign currencies $i = 1, \ldots, I$. Again, $S_t^{(i)}$ is foreign currency i's exchange rate expressed as units of domestic currency per unit of foreign currency i.

The state vector in our multi-currency model consists of X_t, $X_t^{(1)}$, ..., $X_t^{(I)}$ and $\log S_t^{(1)}$, ..., $\log S_t^{(I)}$. We model the second moments (quadratic variation) of this state vector as constants (Gaussian) so that

$$d\left\langle \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \\ \log S^{(1)} \\ \vdots \\ \log S^{(I)} \end{bmatrix}, \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \\ \log S^{(1)} \\ \vdots \\ \log S^{(I)} \end{bmatrix}^\top \right\rangle_t = \begin{bmatrix} H_0 & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & H_0^{(1)} & \cdots & \cdot & C_0^{(1)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \cdot \\ \cdot & \cdot & \cdots & H_0^{(I)} & \cdot & \cdots & C_0^{(I)} \\ \cdot & C_0^{(1)\top} & \cdots & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & C_0^{(I)\top} & \cdot & \cdots & \cdot \end{bmatrix} dt, \quad (17a)$$

The single-currency models also require $H_0 := d\left\langle X, X^\top \right\rangle_t / dt$ and $H_0^{(i)} := d\left\langle X^{(i)}, X^{(i)\top} \right\rangle_t / dt$. To conserve notation, we have used a simple dot to denote most of the other parameters of the matrix of second moments. As we'll see shortly, these parameters do not affect the cross-section of yields in our model so they are free to capture co-variation in the elements of the state vector. Although it does not restrict the cross-section of yields, we have explicitly labeled $C_0^{(i)} = d\left\langle X^{(i)}, \log S^{(i)} \right\rangle_t / dt$ because it is relevant below when changing from the risk-neutral measure in foreign currency i to the risk-neutral measure in the domestic currency.

[8]The risk-neutral drift of foreign currency i's exchange rate must be $\mathbb{E}_t\left[dS_t^{(i)}\right] = S_t^{(i)}\left[r_t - r_t^{(i)}\right] dt$. Therefore one can model either the short-interest rate in foreign currency i or the risk-neutral drift of that currency's exchange rate, but not both.

To complete the cross-sectional pricing implications for the model, we also need to specify the dependence of the short interest rates on the state vector,

$$\begin{bmatrix} r \\ r^{(1)} \\ \vdots \\ r^{(I)} \end{bmatrix}_t = \begin{bmatrix} \rho_0 \\ \rho_0^{(1)} \\ \vdots \\ \rho_0^{(I)} \end{bmatrix} + \begin{bmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_1^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \rho_1^{(I)} \end{bmatrix}^\top \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \end{bmatrix}_t, \tag{17b}$$

as well as the risk-neutral drift of the state vector

$$\begin{aligned} &\mathbb{E}_t \left[d \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \end{bmatrix}_t \right] + d \left\langle \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \end{bmatrix}, \log M \right\rangle_t \\ &= \left(\begin{bmatrix} \theta \\ \theta^{(1)} - C_0^{(1)} \\ \vdots \\ \theta^{(I)} - C_0^{(I)} \end{bmatrix} + \begin{bmatrix} \mathcal{K} & 0 & \cdots & 0 \\ 0 & \mathcal{K}^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{K}^{(I)} \end{bmatrix} \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \end{bmatrix}_t \right) dt. \end{aligned} \tag{17c}$$

Our simple restricted setup in equation (17) has the appealing feature that the cross-section of zero-coupon yields in each currency are the same as the individual single-currency models which typically capture the cross-section of yields with different maturities very well. Moreover, the only new parameters in equation (17) are the additional elements of the matrix of second moments that capture co-variation in the combined state vector.

We have chosen X_t as an affine function of domestic yields and each $X_t^{(i)}$ as an affine function of yields in foreign currency i. For the domestic currency and seven of the foreign currencies we choose the state vector to be the first three principal components of changes in yields. Due to data restrictions, the state vectors for the remaining two foreign currencies are the first two principal components of changes in yields. Therefore, when we include the no-arbitrage restrictions from equation (5), our restricted model has $8 \times 3 \times 4 + 2 \times 2 \times 3 = 108$ free parameters in equations (17b) and (17c).

One could instead use the most general model and allow each element of the state vector (excluding the log exchange rates) to be an affine function of yields in all of the currencies rather than just a single currency. This approach would allow greater flexibility in matching the cross-section of yields in each currency at the expense of adding many more parameters. For example, if we relaxed the zero restrictions in equations (17b) and (17c) but maintained a 28-dimensional state vector (excluding the log exchange rates) then the unrestricted model would have $28 \times 29 = 812$ free parameters in equations (17b) and (17c). Single-currency models typically capture the cross-section of yields with different maturities very well, so there is little room for benefit from the large number of additional parameters. One could also use the unrestricted model with a lower dimensional state vector. For example, if we chose the state vector (excluding the log exchange rates) to be the first ten PCs of changes in yields of all currencies then the unrestricted model would have $10 \times 11 = 110$ free parameters in equations (17b) and (17c).[9] The clear benefit of our approach is that all of the parameters in equations (17b) and (17c) have direct counterparts in the single-currency term structure models, so they are very easy to estimate. Secondly, it is very easy to add additional currencies and term structures to the model because we simply expand the state vector to include affine functions of yields in those currencies.

To complete the model, we need to specify risk premia, or equivalently, the quadratic variation of the state vector with the pricing kernel. The second key insight of our model is that the size of the state vector can grow large as the model incorporates more foreign currencies and term structures, but the number of priced risk factors does not need to increase. We use the same approach as the single-currency models and restrict the number of priced risk factors in the model by constraining the rank of the risk premia parameters,

$$-d\left\langle \begin{bmatrix} X \\ X^{(1)} \\ \vdots \\ X^{(I)} \\ \log S^{(1)} \\ \vdots \\ \log S^{(I)} \end{bmatrix}, \log M \right\rangle_t = \left(\Lambda_0 + \Lambda_1 \begin{bmatrix} \tilde{X} \\ \tilde{X}^{(1)} \\ \vdots \\ \tilde{X}^{(I)} \end{bmatrix}_t\right) dt, \quad \operatorname{rank}\left([\Lambda_0, \Lambda_1]\right) = R. \tag{18}$$

$\tilde{X}_t = \tilde{L}_0 + \tilde{L}_1 \cdot Y_t$ is from the specification in equation (11) of the single-currency risk premia for the domestic currency and $\tilde{X}_t^{(i)} = \tilde{L}_0^{(i)} + \tilde{L}_1^{(i)} \cdot Y_t^{(i)}$ is from the specification in equation (15) of the single-currency risk premia for foreign currency i. Intuitively, risk premia in our multi-currency model depend on the individual affine functions of yields in each currency that best predict excess bond returns in that currency. In our empirical analysis, we examine the performance of the model both in- and out-of-sample with different numbers of priced risk factors. The rank restrictions in equation (18) have both an economic and statistical role. From an economics standpoint, it is logical to work with a relatively small number of priced risk factors. Intuitively, although one may need to model the joint dynamics of hundreds of prices, it makes little economic sense to all each of the source of uncertainty to be priced.

From a statistical standpoint, risk premia are notoriously difficult to estimate so again it is instructive to examine the number of free parameters under different specifications. When we model the G10 currencies and term structures, the state vector on the left hand side of equation (18) is 37-dimensional and the state vector on the right hand side is 10-dimensional. If there is one priced risk factor (i.e. $R = 1$) then there are $37 + 10 = 47$ free parameters in equation (18). If there are two priced risk factors (i.e. $R = 2$) then there

[9] Note that the number of free parameters in equation (17a) would be reduced from $37 \times 38/2 = 703$ to $19 \times 20/2 = 190$ if we reduced the dimension of the state vector (including the log exchange rates of the nine foreign currencies) from 37 to 19. However, this reduction in free parameters can be misleading because one still has to estimate the full matrix of second moments in order to determine the state vector. For example, if we were to choose the state vector to be the first ten PCs of changes in yields of all currencies then we would still need to estimate the full matrix of second moments of yields in order to determine the PC weights on each yield.

are $2 \times (37 + 9) = 92$ free parameters. If there are three priced risk factors then there are $3 \times (37 + 8) = 135$ free parameters, and so on. If we used the same 37-dimensional state vector on the right hand side of (18) then with one priced risk factor there are $37 + 37 = 74$ free parameters. With two priced risk factors there are $2 \times (37 + 36) = 146$ free parameters, with three are $3 \times (37 + 35) = 216$ free parameters, and so on. With no restrictions on the number of priced risk factors there are $37 \times 38 = 1406$ free parameters! We could again consider the possibility of reducing the dimension of the state vector. For instance, if we chose an unrestricted model with a 12-dimensional state vector (excluding the log exchange rates) then there would be $12 \times 13 = 156$ free parameters in equations (17b) and (17c) but with three priced risk factors there would be only $3 \times (21 + 8) = 87$ free parameters in equation (18) for a total of $156 + 87 = 243$. The number of free parameters in these equations is the same as our restricted model with a 28-dimensional state vector (excluding the log exchange rates) which has $108 + 135 = 243$. Regardless of the dimension of the chosen state vector, it is clear from both an economic and statistical standpoint that one needs to restrict the number of priced risk factors in the model.

3 Data and Estimation Results

Our empirical implementation uses swap and exchange rate data from Bloomberg on the G10 currencies which are the U.S. dollar (USD), British pound (GBP), Japanese yen (JPY), Australian dollar (AUD), Euro (EUR), Canadian dollar (CAD), Swiss franc (CHF), New Zealand dollar (NZD), Swedish krona (SEK), and Norwegian krone (NOK). We use weekly (Wednesday) data from January 6, 1993 until March 28, 2009 and use data on the German mark (DEM) before the Euro was introduced. We use 3- and 6-month Libor rates and 2-, 3-, 5-, 7-, and 10-year swap rates in each currency to bootstrap a zero-coupon yield curve under the assumption that forward rates are constant between observations. For the early part of our sample we are missing data on 7- and 10-year Norwegian swap rates and 6-month Libor and 7- and 10-year New Zealand swaps rates.

We begin by estimating the single-currency term structure model in equations (10) and (11) for the U.S. dollar and the model equations (12) and (15) for the remaining nine currencies in the G10. Table 1 provides the yield loadings for the first two (NZD and NOK) or three (the remaining eight currencies) principal components of changes in each currency's yield. We use these principal component loadings to construct the state vectors $X_t = L_1 Y_t$ for the U.S. dollar and $X_t^{(i)} = L_1^{(i)} Y_t^{(i)}$ for the other nine currencies. L_1 is equal to the three rows in Table 1 for the USD, $L_1^{(1)}$ is the three rows for GBP, $L_1^{(2)}$ is the three rows for JPY, and so on. We set the constant terms in equations (10d) and (12d) to zero for each currency (i.e. $L_0 = L_0^{(i)} = 0$).

We use maximum likelihood to estimate the domestic single-currency model parameters ρ_0, ρ_1, $\mathcal{K}$, θ, H_0, Λ_0, and Λ_1 from equations (10) and (11). Let $\tilde{A}$ and $\tilde{B}$ be the yield weightings for these parameters calculated from equation (2) so that $Y_t = \tilde{A} + \tilde{B} X_t$. The parameters must satisfy the no-arbitrage restrictions from equation (5) that $X_t = L_1 Y_t \Rightarrow L_1 \tilde{A} = 0$ and $L_1 \tilde{B} = \mathcal{I}$. We also add the additional restriction from equation (4) that

Currency	Principal Component	3M	6M	2Y	3Y	5Y	7Y	10Y
		Loadings on Yields						
USD	**1**	-0.15	-0.21	-0.42	-0.44	-0.44	-0.44	-0.42
	2	0.67	0.63	0.09	-0.01	-0.14	-0.23	-0.26
	3	-0.30	-0.09	0.55	0.42	0.01	-0.25	-0.60
GBP	**1**	0.14	0.20	0.42	0.45	0.45	0.45	0.40
	2	0.55	0.55	0.29	0.14	-0.14	-0.32	-0.42
	3	0.54	0.27	-0.44	-0.45	-0.04	0.22	0.44
JPY	**1**	-0.11	-0.15	-0.39	-0.46	-0.49	-0.45	-0.41
	2	-0.51	-0.48	-0.35	-0.25	0.09	0.35	0.44
	3	0.59	0.36	-0.36	-0.43	-0.09	0.23	0.39
AUD	**1**	-0.16	-0.21	-0.43	-0.43	-0.43	-0.44	-0.43
	2	-0.60	-0.63	-0.10	-0.09	0.07	0.25	0.39
	3	0.26	0.31	-0.42	-0.43	-0.28	0.34	0.53
EUR	**1**	0.16	0.20	0.43	0.45	0.45	0.43	0.40
	2	0.64	0.56	0.21	0.06	-0.16	-0.28	-0.35
	3	0.44	0.17	-0.47	-0.43	-0.09	0.27	0.54
CAD	**1**	0.35	0.41	0.44	0.41	0.38	0.35	0.29
	2	-0.62	-0.52	0.07	0.17	0.28	0.33	0.36
	3	0.26	0.09	-0.55	-0.47	0.11	0.28	0.56
CHF	**1**	-0.35	-0.36	-0.45	-0.44	-0.38	-0.35	-0.31
	2	-0.66	-0.48	0.02	0.12	0.27	0.33	0.38
	3	-0.40	0.01	0.52	0.41	0.00	-0.34	-0.53
NZD	**1**	0.61		0.51	0.47	0.40		
	2	0.78		-0.25	-0.36	-0.45		
SEK	**1**	-0.21	-0.25	-0.40	-0.43	-0.44	-0.42	-0.42
	2	-0.63	-0.60	-0.12	0.00	0.18	0.25	0.36
	3	-0.32	-0.18	0.52	0.44	0.11	-0.20	-0.59
NOK	**1**	-0.55	-0.52	-0.40	-0.38	-0.35		
	2	-0.56	-0.34	0.35	0.42	0.52		

Table 1: Principal Component Loadings on Yields

This table provides the first two or three principal components weights for weekly changes in zero-coupon yields of each currency ranging from 3 months to 10 years. The data period used for calculation was January 6, 1993 to December 26, 2001.

$\operatorname{rank}([\Lambda_0, \Lambda_0]) = 1$ so that there is only one linear combination of yields that is priced. From equations (10) and (11),

$$\mathbb{E}_t \left[d \begin{bmatrix} X \\ \Lambda_0 + \Lambda_1 Y \end{bmatrix}_t \right] = \begin{bmatrix} \mathcal{I} \\ \Lambda_1 \tilde{B} \end{bmatrix} \left(\theta + \begin{bmatrix} \mathcal{K} & \mathcal{I} \end{bmatrix} \begin{bmatrix} X \\ \Lambda_0 + \Lambda_1 Y \end{bmatrix}_t \right) dt, \quad (19)$$

$$d \left\langle \begin{bmatrix} X \\ \Lambda_0 + \Lambda_1 Y \end{bmatrix}, \begin{bmatrix} X \\ \Lambda_0 + \Lambda_1 Y \end{bmatrix}^\top \right\rangle = \begin{bmatrix} \mathcal{I} \\ \Lambda_1 \tilde{B} \end{bmatrix} H_0 \begin{bmatrix} \mathcal{I} \\ \Lambda_1 \tilde{B} \end{bmatrix}^\top dt. \quad (20)$$

Since the quadratic variation is constant, changes in the state vector $X_{t+\Delta t} - X_t = L_1 (Y_{t+\Delta t} - Y_t)$ over a discrete time interval Δt (in our case, 1 week) have a multivariate Gaussian distribution. Appendix A shows that the conditional mean is

$$\mu_t^X = \mathbb{E}_t [X_{t+\Delta t} - X_t] = \left(e^{\Delta t (\mathcal{K} + \Lambda_1 \tilde{B})} - \mathcal{I} \right) \left(\mathcal{K} + \Lambda_1 \tilde{B} \right)^{-1} \left(\theta + \begin{bmatrix} \mathcal{K} & \mathcal{I} \end{bmatrix} \begin{bmatrix} X \\ \Lambda_0 + \Lambda_1 Y \end{bmatrix}_t \right), \quad (21a)$$

$$= L_1 \mathbb{E}_t [Y_{t+\Delta t} - Y_t] = \left(e^{\Delta t (\mathcal{K} + \Lambda_1 \tilde{B})} - \mathcal{I} \right) \left(\mathcal{K} + \Lambda_1 \tilde{B} \right)^{-1} \left(\theta + \begin{bmatrix} \mathcal{K} & \mathcal{I} \end{bmatrix} \begin{bmatrix} L_1 Y \\ \Lambda_0 + \Lambda_1 Y \end{bmatrix}_t \right), \quad (21b)$$

and the conditional variance is

$$V_X = \mathbb{E}_t \left[(X_{t+\Delta t} - \mathbb{E}_t [X_{t+\Delta t}]) (X_{t+\Delta t} - \mathbb{E}_t [X_{t+\Delta t}])^\top \right], \quad (22a)$$

$$= \int_0^{\Delta t} e^{u (\mathcal{K} + \Lambda_1 \tilde{B})} H_0 e^{u (\mathcal{K} + \Lambda_1 \tilde{B})^\top} du = P \Omega (\Delta t) P^\top, \quad (22b)$$

where $P \tilde{K} P^\top = \mathcal{K} + \Lambda_1 \tilde{B}$ is the eigenvalue decomposition of $\mathcal{K} + \Lambda_1 \tilde{B}$ (i.e. $\tilde{K}$ is a diagonal matrix with elements $\tilde{K}_j$) and

$$\Omega_{jk} (\Delta t) = \Sigma_{jk} \frac{e^{(\tilde{\mathcal{K}}_j + \tilde{\mathcal{K}}_k) \Delta t} - 1}{\tilde{\mathcal{K}}_j + \tilde{\mathcal{K}}_k}, \quad (22c)$$

$$\Sigma = P^{-1} H_0 \left(P^{-1} \right)^\top. \quad (22d)$$

The likelihood of changes in the state vector is,

$$\mathcal{L}_X = \frac{1}{N} \sum_{n=1}^{N} \ell_{t_n}, \quad (23a)$$

where

$$\ell_t^X = -\frac{3}{2} \ln (2\pi) - \frac{1}{2} \ln |V_X| - \frac{1}{2} \left(Y_{t+\Delta t} - Y_t - \mu_t^X \right)^\top L_1^\top V_X^{-1} L_1 \left(Y_{t+\Delta t} - Y_t - \mu_t^X \right). \quad (23b)$$

We assume that the cross-sectional pricing errors for yields with different maturities are independent with mean zero and a Gaussian distribution. The K-dimensional vector of pricing errors is

$$\varepsilon_t = Y_t - \left[\tilde{A} + \tilde{B} X_t \right] = Y_t - \left[\tilde{A} + \tilde{B} L_1 Y_t \right], \quad (24a)$$

and let $\varepsilon_k = \left[\varepsilon_k(t_1), \ldots, \varepsilon_k(t_{N+1})\right]^\top$ be the $N+1$-dimensional time-series of pricing errors for the kth element of ε (i.e. the kth yield). The log-likelihood of the pricing errors is

$$\mathcal{L}_Y = \sum_{k=1}^{K} \left\{ -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\sigma_k^2 - \frac{1}{2}\frac{1}{N+1}\frac{\varepsilon_k^\top \varepsilon_k}{\sigma_k^2} \right\}, \tag{24b}$$

$$= \sum_{k=1}^{K} \left\{ -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\left(\frac{\varepsilon_k^\top \varepsilon_k}{N+1}\right) - \frac{1}{2}\underbrace{\frac{N+1}{N+1}\frac{\varepsilon_k^\top \varepsilon_k}{\varepsilon_k^\top \varepsilon_k}}_{1} \right\}. \tag{24c}$$

We choose the single-currency domestic term-structure model parameters ρ_0, ρ_1, $\mathcal{K}$, θ, H_0, Λ_0, and Λ_1 to maximize the log-likelihood of the changes in the state vector plus the cross-sectional pricing errors for yields with different maturities, $\mathcal{L}_X + \mathcal{L}_Y$, subject to the no-arbitrage restrictions in equation (5) and the risk premia rank restriction $\operatorname{rank}([\Lambda_0, \Lambda_0]) = 1$.

We use the same maximum likelihood approach to estimate the parameters $\rho_0^{(i)}$, $\rho_1^{(i)}$, $\mathcal{K}^{(i)}$, $\theta^{(i)}$, $H_0^{(i)}$, $\Lambda_0^{(i)}$, and $\Lambda_1^{(i)}$ for the single-currency term structure model in equations (12) and (15) for the remaining nine currencies in the G10. For each of the ten single-currency term structure models we use the period from January 6, 1993 to December 26, 2001 to estimate the model parameters and the period from January 2, 2002 to March 28, 2009 for out-of-sample testing. As our measure of model fit to the cross-section of yields we compute the square root of the mean squared pricing errors $\sqrt{\frac{1}{N+1}\sum_{n=1}^{N+1}\varepsilon_k(t_n)}$ for each yield maturity and each currency.

Table 2 provides the in- and out-of-sample pricing errors for each currency's single-currency term structure model. In general, the in- and out-of-sample fits are comparable in magnitude. We use two factor models for New Zealand and Norway because we are missing longer dated swap rates for much of our sample and the pricing errors are larger for these currencies. The out-of-sample pricing errors are actually smaller for the Australian currency which we think reflects relatively noisy data for this currency over the period that we used for estimation.

For the domestic single-currency term structure model we have an estimate of the 7×1 vectorΛ_0 and the 7×3 matrix Λ_1 such that $\operatorname{rank}([\Lambda_0, \Lambda_1]) = 1$. From these estimates we can compute the singular value decomposition $[\Lambda_0, \Lambda_1] = U * S * V$ where U is 3×1 with $\|U\|^2 = U^\top U = 1$, S is 1×1, and V is 1×8 with $\|V\|^2 = V^\top V = 1$. From equation (11) we define $\Lambda = U * S$, $\tilde{L}_0 = V_1$, and $\tilde{L}_1 = V_{2\to 8}$ so that we can write our estimates of risk premia in the single-currency domestic term structure model as

$$-\,d\langle X, \log M\rangle_t = (\Lambda_0 + \Lambda_1 Y_t)\,dt = \Lambda\Big(\underbrace{\tilde{L}_0 + \tilde{L}_1 Y_t}_{\tilde{X}_t}\Big)\,dt. \tag{25}$$

Similarly, for each of the other nine single-currency term structures models we can compute the singular value decomposition $\left[\Lambda_0^{(i)}, \Lambda_1^{(i)}\right] = U^{(i)} * S^{(i)} * V^{(i)}$. Again, from (15) we can define $\Lambda^{(i)} = U^{(i)} * S^{(i)}$, $\tilde{L}_0^{(i)} = V_1^{(i)}$, and $\tilde{L}_1^{(i)} = V_{2\to 8}^{(i)}$ so that we can write our estimates of

Currency		Zero-Coupon Maturity 3M	6M	2Y	3Y	5Y	7Y	10Y
USD	**In**	4.2	4.6	2.2	1.9	2.3	1.9	2.5
	Out	7.2	8.3	3.9	1.8	3.6	1.7	3.2
GBP	**In**	4.1	4.3	4.4	3.3	5.3	3.0	5.2
	Out	4.5	6.1	4.6	1.7	3.6	1.7	3.2
JPY	**In**	1.7	2.0	2.2	1.5	3.0	2.3	3.0
	Out	1.7	2.4	3.0	0.4	3.6	2.3	4.1
AUD	**In**	10.9	10.6	13.9	9.8	6.7	2.8	2.9
	Out	6.6	5.9	8.2	4.5	7.5	4.7	5.0
EUR	**In**	3.0	3.7	2.6	1.9	3.5	1.9	3.0
	Out	3.5	4.8	3.8	1.4	3.2	1.9	3.2
CAD	**In**	5.3	5.8	4.1	4.3	4.2	5.3	4.3
	Out	7.0	8.0	2.9	2.2	3.6	2.5	3.9
CHF	**In**	3.1	4.5	2.8	2.3	3.4	2.2	3.2
	Out	4.1	5.7	4.3	2.2	5.8	3.3	6.6
NZD	**In**	2.1	10.8	8.7	3.2	9.5		
	Out	2.4	9.4	9.4	3.6	11.4		
SEK	**In**	6.1	5.8	6.5	2.4	7.2	5.9	7.2
	Out	5.6	5.8	9.6	1.6	10.7	7.8	10.5
NOK	**In**	6.3	6.2	13.0	4.9	14.4		
	Out	8.4	8.3	17.3	6.1	19.9		

Table 2: Root Mean Squared Zero-Coupon Swap Rate Pricing Errors in Basis Points
In-sample pricing errors are from January 6, 1993 to December 26, 2001 and out-of-sample pricing errors are from January 2, 2002 to March 28, 2009.

Currency	Constant ($\tilde{L}_0$ and $\tilde{L}_0^{(i)}$)	Yield Maturity ($\tilde{L}_1$and $\tilde{L}_1^{(i)}$) 3M	6M	2Y	3Y	5Y	7Y	10Y
USD	0.00	-0.02	-0.02	0.54	-0.67	-0.09	0.46	-0.21
GBP	-0.00	-0.42	0.64	-0.51	0.37	-0.01	-0.10	0.05
JPY	0.01	0.15	-0.05	-0.02	-0.03	-0.40	0.78	-0.45
AUD	0.00	-0.17	0.21	-0.09	-0.15	0.57	-0.69	0.30
EUR	-0.00	-0.21	0.20	0.20	-0.22	0.34	-0.73	0.43
CAD	-0.01	-0.02	-0.07	0.52	-0.04	-0.80	0.13	0.26
CHF	-0.00	-0.33	0.37	0.27	-0.46	0.40	-0.50	0.23
NZD	0.00	-0.24		0.80	-0.12	-0.54		
SEK	0.00	0.17	-0.36	0.78	-0.46	-0.01	-0.15	-0.01
NOK	0.00	-0.04	0.09	-0.56	0.77	-0.29		

Table 3: Single-Currency Risk Premia Factor Weights

This table provides our estimates from equations (25) and (26) of the vector of yields in each currency that determines risk premia in that currency's single-currency term structure. The estimates were obtained using weekly data from January 6, 1993 to December 26, 2001.

risk premia in the single-currency term structure model for foreign currency i as

$$-\,d\left\langle X^{(i)}, \log MS^{(i)}\right\rangle_t = \left(\Lambda_0^{(i)} + \Lambda_1^{(i)} Y_t^{(i)}\right) dt = \Lambda^{(i)} \Big(\underbrace{\tilde{L}_0^{(i)} + \tilde{L}_1^{(i)} Y_t^{(i)}}_{\tilde{X}_t^{(i)}} \Big)\, dt. \tag{26}$$

Table 3 provides our estimates of $\tilde{L}_0$ and $\tilde{L}_1$ for the domestic single-currency model and $\tilde{L}_0^{(i)}$ and $\tilde{L}_1^{(i)}$ for each of the nine foreign currency term structure models.

With our estimates of the ten single-currency term structure models in hand, we move to estimating the multi-currency model in equations (17) and (18). We use the same maximum likelihood technique described in equations (23) and (24) to estimate the model parameters. To our knowledge, there is no existing research on the number of priced risk factors so we vary specification of risk premia in equation (18) to allow anywhere from one to ten priced risk factors. Regardless of the number of priced risk factors in the multi-currency model, the root mean squared pricing errors for each country's zero-coupon swap rates to do not change by more than one tenth of a basis point from the single-currency results reported in Table (2), so in the interest of space we do not repeat those results.

Our multi-currency model provides an ideal framework for answering a number of interesting questions about risk premia in global swap markets. How many priced risk factors are there in global swap (interest rates) markets? Does information in global swap markets improve our ability to predict changes in the yield curves of individual countries? What is the risk/reward tradeoff available to global fixed income investors?

To answer these questions, we look at how well different specifications of the model (i.e. different numbers of priced risk factors) predict changes in zero-coupon swap rates. We compare the squared prediction errors from the model to the squared prediction errors when there are no risk premia and the expectations hypothesis holds (i.e. the expected future zero-coupon swap rate is today's forward rate), and compute the following statistics

$$R_k^2 = 1 - \frac{\sum \left(Y_{t+\Delta t} - \mathbb{E}_t\left[Y_{t+\Delta t}\right]\right)_k^2}{\sum \left(Y_{t+\Delta t} - \mathbb{E}_t^{\mathbb{Q}}\left[Y_{t+\Delta t}\right]\right)_k^2} \quad \text{and} \quad R_k^{(i)2} = 1 - \frac{\sum \left(Y_{t+\Delta t}^{(i)} - \mathbb{E}_t\left[Y_{t+\Delta t}^{(i)}\right]\right)_k^2}{\sum \left(Y_{t+\Delta t}^{(i)} - \mathbb{E}_t^{\mathbb{Q}_i}\left[Y_{t+\Delta t}^{(i)}\right]\right)_k^2}, \tag{27}$$

where $\mathbb{Q}$ is the risk-neutral measure for the domestic currency and $\mathbb{Q}_i$ is the risk-neutral measure for currency i (under which the expectations hypothesis holds). Our multi-currency model is constructed to exactly match the prices of the interest rate factors $X_t = L_1 Y_t$, $X_t^{(1)} = L_1^{(1)} Y_t^{(1)}$, ..., and $X_t^{(9)} = L_1^{(9)} Y_t^{(9)}$.

As we highlighted earlier, the model correctly prices the factors, but the individual zero-coupon swap rates in each currency are priced with error. Therefore the measure of fit in equation (27) depends on both the time-series prediction errors, as well as the cross-sectional pricing errors. The pricing errors turn out to be virtually the same across all specifications of the model, but we subtract these pricing errors from (27) so as to not contaminate our measure of the time-series fit. The resulting statistics that we use to compare the time-series fit of the different model specifications are

$$R_k^2 = 1 - \frac{\sum \left\{Y_{t+\Delta t} - \mathbb{E}_t\left[Y_{t+\Delta t}\right] - \left(Y_{t+\Delta t} - Y_t - \tilde{B}L_1\left[Y_{t+\Delta t} - Y_t\right]\right)\right\}_k^2}{\sum \left\{Y_{t+\Delta t} - \mathbb{E}_t^{\mathbb{Q}}\left[Y_{t+\Delta t}\right] - \left(Y_{t+\Delta t} - Y_t - \tilde{B}L_1\left[Y_{t+\Delta t} - Y_t\right]\right)\right\}_k^2}, \tag{28a}$$

$$= 1 - \frac{\sum \left\{\tilde{B}L_1\left(Y_{t+\Delta t} - \mathbb{E}_t\left[Y_{t+\Delta t}\right]\right)\right\}_k^2}{\sum \left\{\tilde{B}L_1\left(Y_{t+\Delta t} - \mathbb{E}_t^{\mathbb{Q}}\left[Y_{t+\Delta t}\right]\right)\right\}_k^2}, \tag{28b}$$

and

$$R_k^{(i)2} = 1 - \frac{\sum \left\{Y_{t+\Delta t}^{(i)} - \mathbb{E}_t\left[Y_{t+\Delta t}^{(i)}\right] - \left(Y_{t+\Delta t}^{(i)} - Y_t^{(i)} - \tilde{B}^{(i)}L_1^{(i)}\left[Y_{t+\Delta t}^{(i)} - Y_t^{(i)}\right]\right)\right\}_k^2}{\sum \left\{Y_{t+\Delta t}^{(i)} - \mathbb{E}_t^{\mathbb{Q}_i}\left[Y_{t+\Delta t}^{(i)}\right] - \left(Y_{t+\Delta t}^{(i)} - Y_t^{(i)} - \tilde{B}^{(i)}L_1^{(i)}\left[Y_{t+\Delta t}^{(i)} - Y_t^{(i)}\right]\right)\right\}_k^2}, \tag{29a}$$

$$= 1 - \frac{\sum \left\{\tilde{B}^{(i)}L_1^{(i)}\left(Y_{t+\Delta t}^{(i)} - \mathbb{E}_t\left[Y_{t+\Delta t}^{(i)}\right]\right)\right\}_k^2}{\sum \left\{\tilde{B}^{(i)}L_1^{(i)}\left(Y_{t+\Delta t}^{(i)} - \mathbb{E}_t^{\mathbb{Q}_i}\left[Y_{t+\Delta t}^{(i)}\right]\right)\right\}_k^2}. \tag{29b}$$

Tables 6 and 7 in Appendix B provide the in-sample (January 6, 1993 to December 26, 2001) R^2 statistics computed from the weekly (overlapping) errors in the predicted change of the zero-coupon swap rates four weeks ahead. By construction, as we increase the number of priced risk factors in the model, the in-sample R^2 also increases. For comparison, we also include the R^2 statistic for each currency's single-currency term structure model. We focus our analysis on changes in longer maturity yields (i.e. two to ten years) where risk premia play a larger role. For all of the currencies, the in-sample R^2 from the ten currency model with a single priced risk factor is comparable in magnitude to the in-sample R^2 from the individual single-currency models. In fact, for most of the currencies, the in-sample R^2's from the ten currency model with only two priced risk factors are higher than those from the individual single-currency. The two notable exceptions are the U.S. dollar and the Canadian dollar where the in-sample R^2's from those single-currency models are as high as the R^2's from the ten currency model with three priced risk factors. We conclude from this in-sample evidence that there are a relatively small number of priced risk factors in G10 swap rates. Moreover, the information in global swap markets can be used in a fairly parsimonious manner (i.e. no more than three priced risk factors) to improve our predictions of changes in the yield curves of individual the countries.

As a further test of our model, in Tables 4 and 5 we compare the prediction results from the ten currency model with those from each of the single-currency models over the out-of-sample period from January 2, 2002 to March 28, 2009. The out-of-sample evidence in these tables favors the ten currency model with a single priced risk factor. For every currency but the U.S., the out-of-sample R^2's from the ten currency model with a single priced risk factor are higher (often much higher) than, or at worst virtually the same as, the R^2's from the individual single-currency models. The U.S. is the only currency where the single-currency model out-performed the ten currency model with a single priced risk factor out-of-sample. Moreover, for every currency but Japan, the out-of-sample R^2's from the ten currency model actually decrease as the number of priced risk factors in the model increase from one to three (Japan is the one currency where none of the models is very successful at predicting out-of-sample changes in yields). Our view is that the out-of-sample evidenc ine Tables 4 and 5 strongly favors the joint ten currency model with a single priced risk factor.[10]

Next we use our favored ten currency model with a single priced risk factor to compute the risk/reward tradeoff for a global investor who is free to invest in any combination of zero-coupon bonds in any of the G10 currencies. Figure (1) plots the maximum Sharpe ratio implied by this model. For completeness, we also include the Sharpe ratio from the full-sample estimate. The mean Sharpe ratio using the estimates from January 3, 1993 to December 26, 2001 is 0.98 and the standard deviation is 0.71. If we use the full sample from January 3, 1993 to March 28, 2009 to estimate the model parameters then the mean Sharpe ratio is reduced to 0.56 and the standard deviation is 0.46.[11]

We can also use the model to examine how much a U.S. investor could improve his or her risk/reward tradeoff by holding a portfolio of zero-coupon bonds in the G10 currencies rather than restricting her portfolio to just U.S. zero-coupon bonds. Figure 2 compares the full sample estimate of the Sharpe ratio shown in Figure 1with the Sharpe ratio (computed from the same model) for an investor who only holds U.S. zero-coupon bonds (but can use the yields on zero-coupon bonds in the other nine currencies to help predict changes in U.S. interest rates). The mean Sharpe ratio is 0.15 (standard deviation is 0.10), which indicates

[10] Although we do not discuss the results in detail, Table 8 in Appendix B shows that the ten currency model with a single priced risk factor is also best for predicting changes in the exchange rates.

[11] It is interesting to note that the time-series variation in the Sharpe ratio over the out-of-sample period for the model that is estimated with data from 1993 to 2001 is very similar to the variation when the model is estimated using the full sample, but the level of the Sharpe ratio from 2002 to 2009 is essentially shifted down with the full sample estimates.

Currency	# of Priced Factors	3M	6M	2Y	3Y	5Y	7Y	10Y
		Yield Maturity						
USD	**Single-Cur**	-4.0	4.5	14.5	12.2	7.7	4.6	1.8
	1	-2.3	2.2	5.4	4.1	2.7	1.9	1.3
	2	-4.7	1.3	4.4	2.2	-0.3	-1.5	-2.4
	3	-1.5	4.8	4.9	2.0	-1.0	-2.2	-2.0
GBP	**Single-Cur**	-13.0	-7.0	-0.4	-1.7	-2.8	-2.5	-1.8
	1	10.5	10.7	5.0	3.8	3.6	3.7	3.7
	2	11.4	11.2	2.4	0.6	0.2	0.8	1.6
	3	10.8	10.9	2.6	0.8	0.4	0.9	1.6
JPY	**Single-Cur**	-25.7	-58.1	-76.3	-52.3	-25.8	-15.1	-9.0
	1	-17.2	-49.3	-73.6	-49.4	-22.6	-12.1	-6.3
	2	1.2	-19.1	-47.4	-33.0	-15.6	-8.6	-4.8
	3	-11.0	-45.2	-73.5	-47.7	-20.2	-10.2	-5.2
AUD	**Single-Cur**	1.9	0.7	-1.3	-1.7	-2.3	-2.7	-3.0
	1	3.9	5.0	3.4	2.7	2.4	2.2	2.1
	2	6.9	7.0	2.6	0.6	-2.0	-3.7	-4.9
	3	4.4	2.9	-4.5	-6.8	-9.3	-10.3	-10.8
EUR	**Single-Cur**	13.3	13.8	7.5	6.2	5.4	4.9	4.2
	1	16.6	16.7	7.5	6.1	5.4	4.9	4.2
	2	21.5	19.3	5.6	3.7	2.4	1.7	0.9
	3	26.6	22.7	5.7	3.6	2.0	0.9	-0.4

Table 4: Out-of-Sample Weekly Overlapping 4 Week Prediction Errors R^2 in %

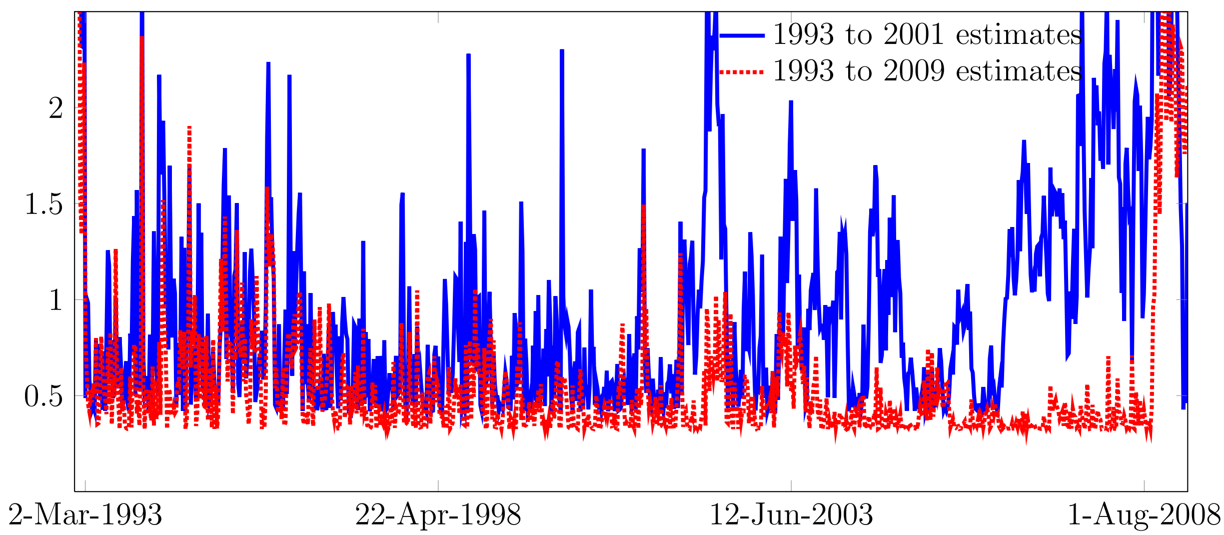
This table provides our estimates of the in-sample R^2 statistic from equations (28) and (29). We use weekly (overlapping) errors in the predicted change of the zero-coupon swap rates four weeks ahead. The first row for each currency are the prediction results using the estimates of the single-currency term structure model in that currency. The next three rows are the prediction results from the multi-currency model with one, two, or three priced risk factors across all currencies (i.e. $R = 1, 2, 3$ in equation (18)). The model parameters were estimated using weekly data from January 6, 1993 to December 26, 2001. The out-of-sample period is from January 2, 2002 to March 28, 2009.

Currency	# of Priced Factors	3M	6M	2Y	3Y	5Y	7Y	10Y
		Yield Maturity						
CAD	**Single-Cur**	-29.1	-18.7	-2.5	-1.2	0.1	0.9	1.5
	1	-92.0	-45.1	5.4	7.7	9.2	9.4	8.7
	2	-77.9	-34.7	5.7	6.5	6.4	5.8	4.7
	3	-66.7	-31.2	2.3	3.0	3.2	3.1	2.6
CHF	**Single-Cur**	30.3	26.2	-1.0	-1.6	1.9	4.1	4.9
	1	28.6	25.7	10.9	8.6	6.9	5.9	5.1
	2	33.7	27.3	5.5	2.8	1.3	0.9	0.8
	3	34.8	28.2	5.7	2.9	1.1	0.5	0.2
NZD	**Single-Cur**	6.0		-32.0	-34.4	-34.7		
	1	4.7		1.5	0.2	-1.0		
	2	4.8		-2.3	-4.9	-7.0		
	3	-39.1		-31.5	-28.3	-25.6		
SEK	**Single-Cur**	45.8	43.7	12.8	6.4	5.4	7.4	9.4
	1	35.5	34.6	14.8	9.7	7.1	6.9	7.1
	2	36.0	34.8	9.5	3.1	0.0	0.5	2.0
	3	40.5	37.9	4.9	-2.4	-4.7	-3.0	0.0
NOK	**Single-Cur**	-74.4	-87.6	-103.5	-91.3	-74.9		
	1	12.0	13.0	10.8	8.2	5.5		
	2	8.3	8.0	1.8	-0.7	-2.7		
	3	6.5	4.7	-6.9	-9.6	-11.0		

Table 5: Out-of-Sample Weekly Overlapping 4 Week Prediction Errors R^2 in %

This table provides our estimates of the in-sample R^2 statistic from equations (28) and (29). We use weekly (overlapping) errors in the predicted change of the zero-coupon swap rates four weeks ahead. The first row for each currency are the prediction results using the estimates of the single-currency term structure model in that currency. The next three rows are the prediction results from the multi-currency model with one, two, or three priced risk factors across all currencies (i.e. $R = 1, 2, 3$ in equation (18)). The model parameters were estimated using weekly data from January 6, 1993 to December 26, 2001. The out-of-sample period is from January 2, 2002 to March 28, 2009.

Figure 1: **Maximum Sharpe Ratio for G10 Term Structure Model**



This figure plots the maximum Sharpe ratio from the ten currency model in equations (17) and (18). For the line labeled "1993 to 2001 estimates", the model parameters were estimated using weekly data from January 6, 1993 to December 26, 2001. For the line labeled "1993 to 2009 estimates", the parameters were estimated using weekly data from January 6, 1993 to March 28, 2009.

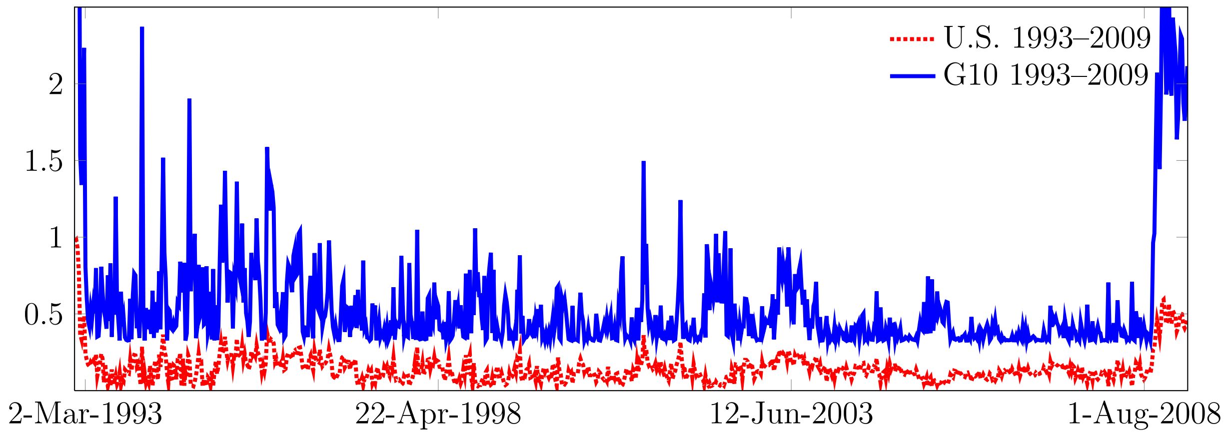
that an investor can more than tripe her expected excess return for a given level of volatility if she is willing/able to hold foreign zero-coupon bonds and currencies in her portfolio.

Finally, we examine how our estimates of risk premia change when we move from a single-currency model to a multi-currency model that must predict the changes in more than one currency's yields. Figure (3) plots the the maximum Sharpe ratio estimated from the single-currency model for the U.S. term structure of interest rates versus the estimate of the same quantity from the ten currency model with a single priced risk factor. The mean Sharpe ratio for the single-currency model is 0.38 and the standard deviation is 0.54 as compared to a mean of 0.15 and standard deviation of 0.10 in the multi-currency model. The take away from this plot is that the estimated maximum Sharpe ratio (equivalently, the estimated market prices of risk) in the multi-currency model is in general lower, less volatile, and arguably more econonomically plausible than the estimate from the single-currency model.

4 Conclusions

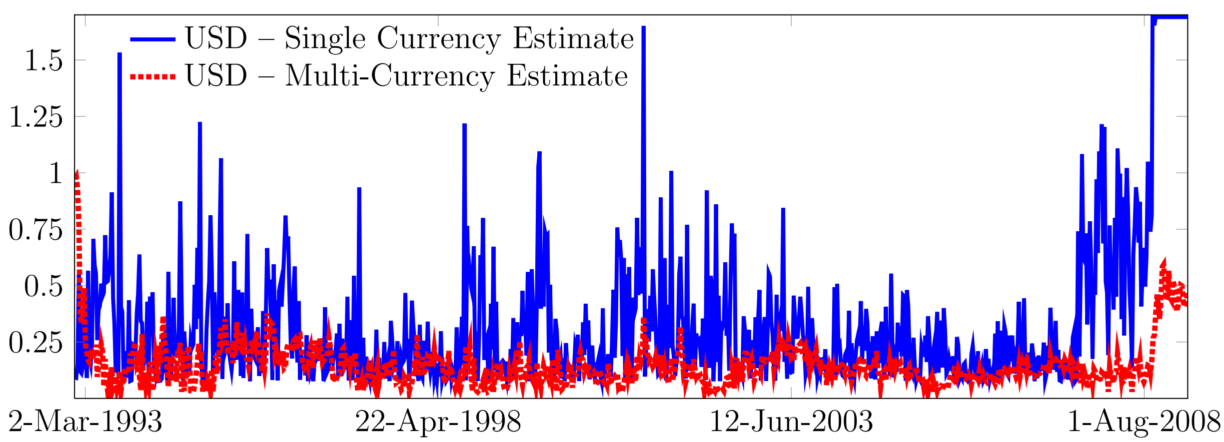
We develop a simple no-arbitrage model of foreign exchange rates and interest rates that can easily be applied to an arbitrary number of foreign currencies. The model has the appealing feature that it reduces to a standard two or three factor model for pricing yields in each currency, yet still maintains a small number of globally priced risk factors. We use the model to analyze the joint dynamics of exchange rates and the term structures of

Figure 2: **Maximum Sharpe Ratio for a U.S Investor vs. a Global Investor**



The line labeled "G10" is the maximum Sharpe ratio from the ten currency model in equations (17) and (18)with a single priced risk factor where the investor is free to invest in any combination of zero-coupon bonds in any of the G10 currencies. The line labeled "US" is the maximum Sharpe ratio using the same model applied to an investor who is restricted to invest in U.S. zero-coupon bonds but can use the yields on zero-coupon bonds in the other nine currencies to help predict changes in U.S. interest rates. The model was estimated with weekly data from January 6, 1993 to March 28, 2009.

Figure 3: **Maximum Sharpe Ratio in Single- versus Multi-Currency Model**



The line labeled "Single Currency Estimate" is the maximum Sharpe ratio estimated from the single-currency model for the U.S. term structure of interest rates. The line labeled "Multi-Currency Estimate" is the maximum Sharpe ratio from the ten currency model with a single priced risk factor where the investor is restricted to invest U.S. zero-coupon bonds but can use the yields on zero-coupon bonds in the other nine currencies to help predict changes in U.S. interest rates. Both models were estimated with weekly data from January 6, 1993 to March 28, 2009.

swap rates for the G10 currencies. Using both in- and out-of-sample measures of fit we conclude that there is one priced risk factor in G10 swap rates. Our ten currency model with a single priced risk factor is better able to predict changes in yields compared with ten single-currency models that are each solely designed to explain changes in that currency's yields. These empirical estimates illustrate that the information in global swap markets can be used in a parsimonious manner to improve our predictions of changes in the yield curves of individual the countries. We estimate the risk premium for exposure to this single factor and show that a U.S. fixed income investor can more than triple the Sharpe ratio of her portfolio if she is willing/able to invest in any yield in any G10 currency.

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A Model Moments

Our model uses a continuous-time affine process with constant quadratic variation so changes in the state vector over discrete intervals are distributed multivariate Gaussian. Fisher and Gilles (1996) show how to calculate the mean and covariance of general affine processes over discrete intervals. Our risk premia specifications in equations (6), (15), and (18) vary slightly from the standard model, so for completeness we show how to compute the first and second moments of the Gaussian distribution.

If

$$\mathbb{E}_t\left[d\begin{bmatrix} X \\ Z \end{bmatrix}_t\right] = \begin{bmatrix} \mathcal{I} \\ G \end{bmatrix}\left(\theta + \begin{bmatrix} \mathcal{K} & F \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}_t\right)dt, \tag{30}$$

then

$$\begin{aligned}
\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t} - \begin{bmatrix} X \\ Z \end{bmatrix}_t &= \int_t^{t+\Delta t} d\begin{bmatrix} X \\ Z \end{bmatrix}_u, \\
&\Downarrow \\
\mathbb{E}_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right] - \begin{bmatrix} X \\ Z \end{bmatrix}_t &= \mathbb{E}_t\left[\int_t^{t+\Delta t} d\begin{bmatrix} X \\ Z \end{bmatrix}_u\right] = \int_t^{t+\Delta t}\mathbb{E}_t\left[d\begin{bmatrix} X \\ Z \end{bmatrix}_u\right], \\
&= \begin{bmatrix} \mathcal{I} \\ G \end{bmatrix}\int_t^{t+\Delta t}\left(\theta + \begin{bmatrix} \mathcal{K} & F \end{bmatrix}\mathbb{E}_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_u\right]\right)du.
\end{aligned}$$

The solution to this ordinary differential equation is

$$\mathbb{E}_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right] - \begin{bmatrix} X \\ Z \end{bmatrix}_t = \begin{bmatrix} \mathcal{I} \\ G \end{bmatrix}\left(e^{\Delta t(\mathcal{K}+FG)} - \mathcal{I}\right)(\mathcal{K}+FG)^{-1}\left(\theta + \begin{bmatrix} \mathcal{K} & F \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}_t\right). \tag{31}$$

Similarly, if

$$d\left\langle \begin{bmatrix} X \\ Z \end{bmatrix}, \begin{bmatrix} X \\ Z \end{bmatrix}^\top \right\rangle_t = \begin{bmatrix} \mathcal{I} \\ G \end{bmatrix} H_0 \begin{bmatrix} \mathcal{I} & G^\top \end{bmatrix} dt, \tag{32}$$

then

$$\begin{aligned}
\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t} - \mathbb{E}_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right] &= \int_t^{t+\Delta t} d\,\mathbb{E}_s\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right], \\
&\Downarrow \\
V_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right] &= \begin{bmatrix} \mathcal{I} \\ G \end{bmatrix}\int_0^{\Delta t} e^{u(\mathcal{K}+FG)} H_0 e^{u(\mathcal{K}+FG)^\top} du \begin{bmatrix} \mathcal{I} & G^\top \end{bmatrix},
\end{aligned} \tag{33}$$

where

$$V_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right] := \mathbb{E}_t\left[\left(\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t} - \mathbb{E}_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right]\right)\left(\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t} - \mathbb{E}_t\left[\begin{bmatrix} X \\ Z \end{bmatrix}_{t+\Delta t}\right]\right)^\top\right].$$

If we take the eigenvalue decomposition of $\mathcal{K} + FG = P\tilde{K}P^{-1}$ where $\tilde{K}$ is a diagonal matrix with elements $\tilde{K}_i$, then the variance in equation (33) is

$$\begin{aligned}
V_t\left[\left[\begin{array}{c} X \\ Z \end{array}\right]_{t+\Delta t}\right] &= \left[\begin{array}{c} \mathcal{I} \\ G \end{array}\right] P \int_0^{\Delta t} e^{u\tilde{\mathcal{K}}} P^{-1} H_0 P^{-1\top} e^{u\tilde{\mathcal{K}}} du\, P^\top \left[\begin{array}{cc} \mathcal{I} & G^\top \end{array}\right], \\
&= \left[\begin{array}{c} \mathcal{I} \\ G \end{array}\right] P\Omega\left(\Delta t\right) P^\top \left[\begin{array}{cc} \mathcal{I} & G^\top \end{array}\right], \qquad (34a)
\end{aligned}$$

where

$$\begin{aligned}
\Omega_{ij}\left(\Delta t\right) &= \Sigma_{ij} \frac{e^{\left(\tilde{\mathcal{K}}_i + \tilde{\mathcal{K}}_j\right)\Delta t} - 1}{\tilde{\mathcal{K}}_i + \tilde{\mathcal{K}}_j}, \quad \text{and} \qquad (34b) \\
\Sigma &= P^{-1} H_0 \left(P^{-1}\right)^\top. \qquad (34c)
\end{aligned}$$

B Additional Tables

Currency	# of Priced Factors	Yield Maturity 3M	6M	2Y	3Y	5Y	7Y	10Y
USD	**Single-Cur**	25.0	20.0	5.7	3.5	1.7	0.9	0.3
	1	11.7	7.9	1.2	0.7	0.3	0.2	0.2
	2	15.1	9.0	1.7	1.7	2.3	2.7	3.0
	3	25.0	18.0	5.1	3.9	3.3	3.2	3.2
GBP	**Single-Cur**	16.4	7.9	0.2	0.7	0.8	0.5	0.2
	1	13.7	8.6	0.7	0.4	0.5	0.8	1.2
	2	14.8	8.8	1.9	2.4	3.3	3.7	3.9
	3	15.6	9.2	1.8	2.3	3.3	3.7	3.9
JPY	**Single-Cur**	18.9	21.0	16.1	14.1	11.2	8.9	6.7
	1	10.7	13.0	11.7	10.5	8.4	6.6	4.9
	2	14.4	17.1	15.1	13.6	11.2	9.0	6.8
	3	19.0	22.4	18.6	16.2	12.4	9.5	7.0
AUD	**Single-Cur**	28.4	22.3	7.5	6.1	5.6	5.6	5.4
	1	26.3	18.4	3.2	1.9	1.3	1.1	1.1
	2	26.9	19.7	6.5	6.5	7.9	8.8	9.3
	3	27.6	20.4	6.5	6.3	7.4	8.2	8.7
EUR	**Single-Cur**	10.7	11.7	8.0	6.8	5.4	4.4	3.3
	1	2.8	3.9	4.1	3.5	2.6	1.8	1.1
	2	4.7	4.1	5.9	6.2	5.8	4.9	3.7
	3	5.2	4.4	5.9	6.2	5.8	4.9	3.7

Table 6: In-Sample Weekly Overlapping 4 Week Prediction Errors R^2 in %

This table provides our estimates of the in-sample R^2 statistic from equations (28) and (29). We use weekly (overlapping) errors in the predicted change of the zero-coupon swap rates four weeks ahead. The first row for each currency are the prediction results using the estimates of the single-currency term structure model in that currency. The next three rows are the prediction results from the multi-currency model with one, two, or three priced risk factors across all currencies (i.e. $R = 1, 2, 3$ in equation (18)). The model parameters were estimated using weekly data from January 6, 1993 to December 26, 2001.

Currency	# of Priced Factors	3M	6M	2Y	3Y	5Y	7Y	10Y
		Yield Maturity						
CAD	**Single-Cur**	17.6	15.3	7.5	6.6	6.6	6.8	6.8
	1	16.8	14.0	5.3	3.9	3.0	2.6	2.1
	2	17.3	14.9	7.2	6.2	5.8	5.6	5.3
	3	20.5	17.7	8.3	6.9	6.2	6.9	5.4
CHF	**Single-Cur**	2.5	4.0	5.4	4.8	3.4	2.2	1.3
	1	4.1	3.9	2.6	2.4	2.3	2.2	2.0
	2	5.0	4.0	4.7	5.5	6.3	6.4	6.0
	3	4.5	3.6	4.7	5.5	6.3	6.4	6.0
NZD	**Single-Cur**	-0.1		5.3	7.9	10.1		
	1	0.4		-0.0	-0.1	-0.0		
	2	0.6		2.5	3.0	3.3		
	3	8.3		6.5	5.4	4.5		
SEK	**Single-Cur**	17.5	12.3	0.9	0.1	-0.0	0.1	0.2
	1	22.3	18.3	4.6	2.7	1.8	1.6	1.7
	2	22.0	18.1	9.0	8.3	7.6	7.0	6.1
	3	23.0	18.6	8.8	8.1	7.6	7.0	6.2
NOK	**Single-Cur**	14.2	15.1	14.8	12.4	9.3		
	1	3.2	3.3	2.6	1.9	1.3		
	2	12.3	13.9	17.8	16.9	14.7		
	3	12.3	14.0	18.4	17.7	15.7		

Table 7: In-Sample Weekly Overlapping 4 Week Prediction Errors R^2 in %

This table provides our estimates of the in-sample R^2 statistic from equations (28) and (29). We use weekly (overlapping) errors in the predicted change of the zero-coupon swap rates four weeks ahead. The first row for each currency are the prediction results using the estimates of the single-currency term structure model in that currency. The next three rows are the prediction results from the multi-currency model with one, two, or three priced risk factors across all currencies (i.e. $R = 1, 2, 3$ in equation (18)). The model parameters were estimated using weekly data from January 6, 1993 to December 26, 2001.

	# of Priced Risk Factors					
	In-Sample			Out-of-Sample		
Currency	**1**	**2**	**3**	**1**	**2**	**3**
GBP	0.4	0.4	0.5	-0.7	-0.7	-1.0
JPY	0.1	1.4	1.3	-1.1	-3.7	-3.9
AUD	0.1	0.2	0.4	0.4	0.3	-0.0
EUR	0.4	0.1	1.3	-1.7	-2.6	-2.3
CAD	0.2	0.2	0.3	-0.7	-0.8	-0.8
CHF	0.2	0.1	1.0	-1.3	-1.9	-2.9
NZD	1.0	3.4	3.3	0.7	0.0	0.2
SEK	0.0	0.0	2.6	-0.1	-0.0	-4.8
NOK	0.0	-0.1	0.0	0.2	0.1	-0.1

Table 8: Weekly Overlapping 4 Week Prediction Errors R^2 in % for Exchange Rates
The table provides our estimates of the in and out-of-sample R^2 statistic

$$R^{(i)2} = 1 - \frac{\sum \left(\log S_{t+\Delta t} - \mathbb{E}_t \left[\log S_{t+\Delta t}\right]\right)^2}{\sum \left(\log S_{t+\Delta t} - \mathbb{E}_t^{\mathbb{Q}} \left[\log S_{t+\Delta t}\right]\right)^2},$$

with one, two, or three priced risk factors in the ten currency model. The in-sample period from January 6, 1993 to December 26, 2001 was used to estimate the model parameters and the out-of-sample period is from January 2, 2002 to March 28, 2009.