PRICING EXOTIC OPTIONS USING LIMITS AND INFINITE SERIES

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0. Rhetorical Context

This paper takes the form of an email to PhD students in finance and economics (as well as MFin and advanced MBA students), who are interested in the subfield of quantitative finance—specifically, derivative pricing—as well as more general applications of models to financial markets. These students may be familiar with black-boxed tools such as the Black-Scholes model, and this paper will address the context of applying such models to more complex derivative products using the first principles of real analysis.

1. Introduction

In the world of quantitative finance, few problems are as interesting and impactful as derivative pricing. While financial derivatives in the form of forwards have existed for centuries, their popularity has exploded since the early 1970's [1]. With the advent of greater market access and participation, research into the field of derivative pricing increased as well; most notably, Fischer Black, Myron Scholes, and Bob Merton published their model ("Black-Scholes") in 1973, the first practical options pricing model [2, p. 76].

1.1. Exotic Derivatives.
Due to the intricate risks and nuanced positions modern financial institutions often face or wish to take on, respectively, companies were historically unsatisfied with the limited positions linear combinations of "vanilla" options offer. In response, investment banks developed derivative contracts with more complicated payout structures—or "exotic derivatives"—in a booming market over the past 20 years [3]. Surprisingly, however, no widespread exotics pricing model exists to this day; but, the tools of real analysis inspire derivations of great approximations.

2. Preliminary

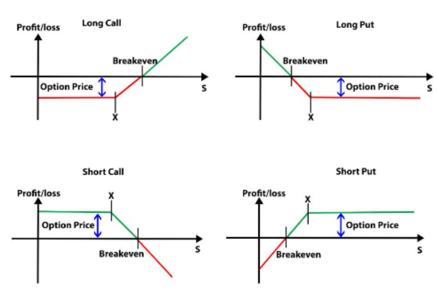

Figure 1. Payoff diagrams of buying and selling vanilla options.

2.1. Vanilla Options.
To price exotics, we revisit their building blocks: calls and puts. A *call* option is a contract which yields the right, but not obligation, to buy an underlying security for a predetermined price (the "strike," K) at a predetermined time (the "expiry," T); similarly, a *put* option yields the right, but not obligation, to sell [2, p. 16]. Payoff diagrams for these derivatives can be seen in Figure 1; the Black-Scholes model, in essence, determines a probability distribution for a contract's underlying security's future price (at time T) given relevant inputs and outputs the resulting expected profit calculated from these payoffs[1] [2, pp. 76-79]. For the purposes of this exposition, we let the function $V(K, T, S, r, \sigma)$ represent the value of a call option with standard inputs via Black-Scholes (S represents current price of underlying, r risk-free rate, and σ volatility). As the call and put payoffs in Figure 1 are visually continuous and uniquely defined at every point, it is intuitive why Black-Scholes can price vanilla options with ease.

[1]These inputs are: the option's exercise price, time to expiry, the underlying's current price, the risk-free interest rate, and volatility of the underlying security [2, p. 77].

2.2. **Exotic Payoffs.** We now consider a common exotic known as the "digital." A *digital* option is a contract which has no payoff if the underlying's value at expiry is below the strike, but has fixed payoff p otherwise. Much like in § 2.1, pricing this derivative requires knowing the probability distribution of the underlying's forward price; however, since the payoff diagram in Figure 2 has a discontinuity at K, we also run into another issue: what should the contribution to V be at K [5]? To answer this question, market makers borrow the real analysis concepts of limits and series.

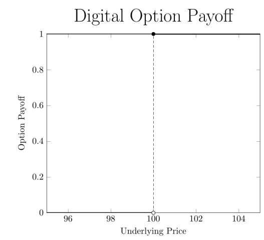

FIGURE 2. Payoff diagram of a long digital option [4].

3. Limits & Infinite Series

In mathematics, a *sequence* $\{x_n\}$ is a list of integers $x_1, x_2, \ldots$.

Definition 1 ([6, Def. 3.1 (p. 47)])**.** A sequence where each $x_i \in \mathbb{R}$ is said to *converge* to x if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $|x_n - x| < \varepsilon$; we call x the limit of $\{x_n\}$.

Intuitively, Definition 1 implies a sequence converges if, over time, it gets arbitrarily closer to a certain fixed point which is its limit. It is also possible for a sequence to not converge—or *diverge*—if it is unbounded, oscillates, or varies seemingly at random (as is often for financial time series).

Example. The sequence $x_n = r^n$ converges to 0 for $|r| < 1$ and diverges for $|r| > 1$.

Quite related to sequences is the concept of an infinite series, which sums terms of a sequence.

Definition 2 ([6, Def. 3.21 (p. 59)])**.** Given a sequence $\{a_n\}$, we define $\sum_{n=1}^{\infty} a_n$ to be an *infinite series*, with *partial sums* $s_n = \sum_{k=1}^{n} a_k$.

If $\{s_n\}$ converges to s, we say that the series itself converges, with $\sum_{n=1}^{\infty} a_n = s$. There likewise exists the concept of *divergent* series; to identify whether a series is convergent (and determine its limit, if so) mathematicians have devised numerous tests [6, pp. 59-72]. These are useful when formulating pricing models for variance swaps and Asian (or path-dependent) options, but we will focus on a different class of exotics known as "digitals."

4. Pricing Exotics

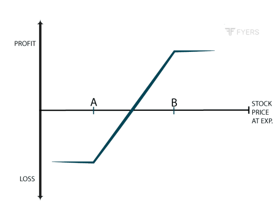

FIGURE 3. Payoff diagram of a bull call spread with strikes $0 < A < B$.

4.1. **Digital Options as Limits.** Looking at Figure 2, one quickly recognizes its similarity with the payoff diagram of a bullish call spread as shown in Figure 3[2]. As a result, we will attempt to approximate a digital using a replicating portfolio of calls, which we can easily price; then the efficient markets hypothesis states financial instruments which behave equivalently must be priced equally[3]. Our initial guess for $V_D\left(K,T,S,r,\sigma\right)$, the value of our digital option as a call spread, is

$$v_1 = p \cdot \left(V\left(K+1,T,S,r,\sigma\right) - V\left(K,T,S,r,\sigma\right)\right), \tag{1}$$

where $V : \mathbb{R}^5 \to \mathbb{R}$ is the Black-Scholes pricing model defined in § 2.1 and p is the digital's payoff in the positive case. A gap of 1 between our strikes is not the best we can do; however, suppose we attempt a gap of $\frac{1}{2}$. In this case each call spread has a max profit of $\frac{1}{2}$, so we must purchase $2p$ for our portfolio to match the digital's max payoff. Thus, inspired by Equation 1, we say

$$v_2 = 2p \cdot \left(V\left(K+\frac{1}{2},T,S,r,\sigma\right) - V\left(K,T,S,r,\sigma\right)\right).$$

Proceeding in this fashion, we see v_n requires np call-spreads or

$$v_n = np \cdot \left(V\left(K+\frac{1}{n},T,S,r,\sigma\right) - V\left(K,T,S,r,\sigma\right)\right).$$

It is then clear that V_D, the value of our digital option, is $V_D = \lim_{n\to\infty} v_n$. In this specific case, we are comfortable that this series does in fact converge; it is easy to see that

$$V_D = p \cdot \frac{\partial V\left(K,T,S,r,\sigma\right)}{\partial K},$$

but this will not always be true in the general case [4]. Finally, Figure 4 visualizes $\{v_n\}$.

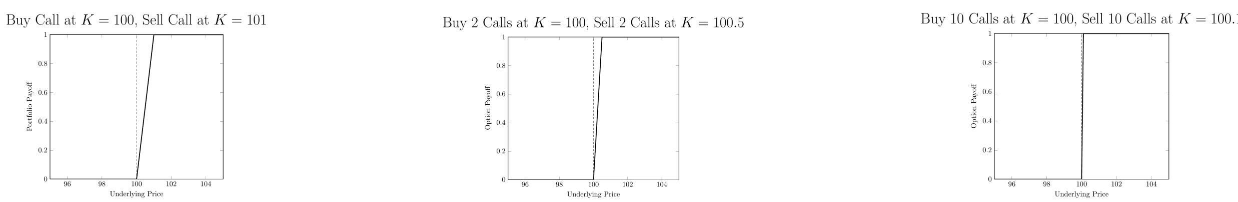

FIGURE 4. Payoff diagrams of digital option replicating portfolios [4].

Remark 1. In general, exotic derivatives are often extensions of vanilla options with one property or another taken to an extreme in some form. As such, limits and infinite series are well-suited to model and price these instruments, as well as stress-test their risk metrics on the sell-side before making offerings to clients. Real analysis as a whole is an invaluable skill set for these concerns, as well as of course dealing with abnormal or overly-engineered derivative payoff structures.

5. Conclusion

An ever-growing instrument in today's financial markets, exotic derivatives will always require deeper mathematical understanding to be accurately priced even with powerful tools such as Black-Scholes readily at our disposal. A finer understanding of topics such as approximation using converging sequences and infinite sums of vanilla portfolios is an invaluable method for pricing the next generation of exotics, and are worth understanding as a future trader or quantitative analyst.

[2]A bullish or long "call spread" is an options strategy consisting of buying a call at strike K_1 and selling a call on the same underlying and expiry at strike $K_2 > K_1$. Its payoff diagram can be visualized by combining the long and short call payoff graphs from Figure 1. A bearish or short call spread is the negation of this strategy.

[3]Explicitly, this implies our replicating portfolio has the same price as our digital option.

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