ALGEBRAIC AND TRANSCENDENTAL REAL NUMBERS WRITING ASSIGNMENT 2, 18.100Q

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ABSTRACT. In abstract algebra, a common task is seeking roots of polynomials in one variable with coefficients from a particular field. Through analysis, it is then interesting to consider the set of all such roots for a particular field, $\mathbb{Q}$; this set is known as the "algebraic real numbers." The purpose of this paper is to examine the algebraic numbers: first proving this set is countable, and then showing it has a nonempty uncountable set complement (known as the "transcendental real numbers"). We will also provide examples of both algebraic and transcendental real numbers.

1. INTRODUCTION

In the study of real analysis, one of the fundamental topics investigated is the properties of various sets and subsets of the real numbers; in particular, characteristics of sets such as countability, or cardinality compared to the natural numbers, and relational structure between the points of a set provide interesting insights into the subtleties that exist between different sets.

In this paper we explore one such subset of the real numbers, yielded organically by the introduction of polynomials in one variable with rational coefficients. These polynomials evaluate to 0 at particular values known as "roots;" we seek to understand the nature of these polynomial roots, wondering if they have a unique structure as a set of their own.

Much of our work in this paper, especially § 3.2, will build on the results of Joseph Liouville, who constructed the first provable example of a real number outside this set in 1844 [1]. Without such concrete examples, we may be tempted to think of this set of polynomial roots in a manner analogous to the ancient Greeks with respect to the rationals before Hippasus proved $\sqrt{2}$ irrational [2]; in other words, we may like to incorrectly assume that all real numbers must be the root of some non-zero polynomial with rational coefficients. We shall soon see in § 2 and § 3.1, however, that applying the results of Georg Cantor and the tools of countability will lead us down the same path as Liouville [3].

To reach these conclusions, we begin by defining the term "algebraic number" before considering properties of our set of polynomial roots further.

Definition 1. A number $\alpha \in \mathbb{R}$ is called *algebraic* if there is some polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with all $a_i \in \mathbb{Q}$ such that α is a root of p.

We will use the notation A to denote the set of algebraic real numbers; in other words, let

$$A = \{\alpha \in \mathbb{R} \mid \alpha \text{ is algebraic}\}. \tag{1}$$

Example. Any rational number is algebraic, as for all $r \in \mathbb{Q}$ there exists a polynomial $p_r(x) \in \mathbb{Q}[x]$ defined as

$$p_r(x) = x - r$$

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with r as its root.

Example. $\sqrt{2}$ is an algebraic number as it is a root of the polynomial

$$p_{\sqrt{2}}(x) = x^2 - 2$$

and $p_{\sqrt{2}} \in \mathbb{Q}[x]$.

Definition 2. A real number that is not algebraic is said to be *transcendental.*

We now have a clear division of $\mathbb{R}$ into two subsets: A and A^c, the algebraic and transcendental real numbers, respectively. In the following sections of this paper, we will set out to prove the below theorems about these sets:

Theorem 1. A is countable.

Theorem 2. $A^c \neq \emptyset$; furthermore, A^c is uncountable.

Once we have shown this, we take a closer look at A^c and attempt to discover some of its members, in the footsteps of Joseph Liouville.

Definition 3 (Liouville [4])**.** A real number x is a *Liouville number* if for every positive integer n, there exist integers a and b with $b > 1$ such that

$$0 < \left| x - \frac{a}{b} \right| < \frac{1}{b^n}. \tag{2}$$

Theorem 3 (Liouville [4])**.** All Liouville numbers are transcendental.

Finally, we intend to conclude with a segue into extensions of the algebraic real numbers as we have defined in Definition 1 to other sets such as $\mathbb{C}$ or polynomial fields like $\mathbb{F}_2$.

2. Cardinality of the algebraic numbers

We begin by assessing Theorem 1. Given Definitions 1 and 2 and the observation of Example 1, we wonder about the cardinality of A; specifically, we will compare it to the benchmark cardinality of $\mathbb{N}$ to gain perspective. In other words, we wish to determine the countability of A. In stating Theorem 1, we claim that A is countable, or that there exists a bijection from A to $\mathbb{N}$.

To prove Theorem 1, we start by considering the set of polynomials in one variable with coefficients in $\mathbb{Q}$, or $\mathbb{Q}[x]$, as it holds a close relation to A by Definition 1. Note that $\mathbb{Q}$ itself is countable, as proven in Rudin[5, p. 30].

Lemma 1. The set $\mathbb{Q}[x]$ is countable.

Proof. Consider a polynomial $p \in \mathbb{Q}[x]$ of degree n. We know that p has the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

for some $a_0, \ldots, a_n \in \mathbb{Q}$; there then exists a bijection mapping p to a unique n-tuple $p' = (a_0, a_1, \ldots, a_n)$ where again $a_i \in \mathbb{Q}$. Applying Theorem 2.13 from Rudin[5, p. 29] then tells us that the set of all such n-tuples is countable (as $\mathbb{Q}$ is), so we thus know the set of degree n polynomials in $\mathbb{Q}[x]$ is countable as well. As $\mathbb{Q}[x]$ is the union of all such sets over $n = 0, 1, 2, \ldots$, we can again apply Theorem 2.12 from Rudin[5, p. 29] to see that $\mathbb{Q}[x]$ is countable, as desired. ☐

We are now able to prove Theorem 1.

Proof of Theorem 1. For each $p \in \mathbb{Q}[x]$, let $R(p)$ denote the set of roots of p. By the Fundamental Theorem of Algebra[6], we know that $|R(p)| \leq \deg(p)$; that is, $R(p)$ is finite and hence certainly countable. As

$$A = \bigcup_{p \in \mathbb{Q}[x]} R(p),$$

Theorem 2.12 of Rudin[5, p. 29] again tells us that a countable union (using Lemma 1) of countable sets is itself countable, so we are done. A must be countable, as previously claimed. ☐

3. Transcendental numbers

We next look to analyzing A^c to evaluate the claims made in Theorem 2.

3.1. **Existence of the transcendental real numbers.**

Since $A \subset \mathbb{R}$ and Theorem 1 tells us $|A| < |\mathbb{R}|$, we know $R \setminus A = A^c \neq \emptyset$ must be true. This implies with certainty that elements of A^c exist, i.e. that there are real numbers which are not algebraic.

3.1.1. *Transcendental numbers are uncountable.* Furthermore, in Theorem 2 we claim A^c is uncountable. We provide a proof below.

Proof. We proceed using proof by contradiction. Assume, for the sake of contradiction, that A^c is countable. By Theorem 1, we already know that A is countable as well. Thus, Theorem 2.12 of Rudin[5, p. 29] tells us $A \cup A^c = \mathbb{R}$ must be countable as well; however, since we know $\mathbb{R}$ is uncountable, this is a contradiction. Thus our initial assumption must have been incorrect, and A^c is *not* countable, as desired. $\square$

As a result, we have successfully proven both components of Theorem 2. To reiterate, $A^c \neq \emptyset$, and is in fact uncountable. This implies that $|A^c| > |A|$, or in other words, that the number of transcendental real numbers is far greater than the number of algebraic real numbers. In fact, approaching this in a similar way to comparing $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{Q}$, we could say that almost every real number is transcendental rather than algebraic.

3.2. **Identifying transcendental real numbers.**

It is important to keep in mind, however, that it is far easier to find or construct concrete examples of algebraic numbers than transcendental numbers. We will next prove Theorem 3, or that the Liouville numbers as defined in Definition 3 are transcendental.

Lemma 2 (Filaseta [7])**.** Let α be an irrational number which is a root of $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Q}[x]$ with $f(x) \not\equiv 0$. Then there exists a constant $A = A(\alpha) > 0$ such that if a and b are integers with $b > 0$, then

$$\left|\alpha - \frac{a}{b}\right| > \frac{A}{b^n}. \tag{3}$$

Proof of Lemma 2. We proceed using proof by contradiction. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the distinct roots of $f(x)$ not equal to α. Furthermore, let M be $\max_{x \in [\alpha-1, \alpha+1]} |f'(x)|$, and $A < \min\left(1, \frac{1}{M}, \min_i \alpha_i\right)$. Assume, for the sake of contradiction, that there exist integers a, b with $b > 0$ such that Inequality 3 is not satisfied. Then we have

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{A}{b^n} \leq A < \min\left(1, \min_i \alpha_i\right).$$

As a result,

$$\alpha - 1 \leq \frac{a}{b} \leq \alpha + 1 \quad \text{and} \quad \frac{a}{b} \notin \{\alpha_1, \ldots, \alpha_m\}.$$

By the Intermediate Value Theorem, there exists an x_0 between $\frac{a}{b}$ and α such that

$$f(\alpha) - f\left(\frac{a}{b}\right) = \left(\alpha - \frac{a}{b}\right) f'(x_0),$$

where $f'(x_0)$ denotes the derivative of f at x_0 [8]. Rearranging yields

$$\left|\alpha - \frac{a}{b}\right| = \left|\frac{f(\alpha) - f(a/b)}{f'(x_0)}\right| = \left|\frac{f(a/b)}{f'(x_0)}\right|.$$

As $\frac{a}{b}$ is not a root of f, we can substitute

$$\left|f\left(\frac{a}{b}\right)\right| = \left|\sum_{j=0}^{n} \frac{a_j a^j}{b^j}\right| = \frac{\left|\sum_{j=0}^{n} a_j a^j b^{n-j}\right|}{b^n} \geq \frac{1}{b^n}.$$

Finally, since $|f'(x_0)| \leq M$ by definition of M, we substitute our findings to get

$$\left|\alpha - \frac{a}{b}\right| = \left|\frac{f(a/b)}{f'(x_0)}\right| \geq \frac{1}{Mb^n} > \frac{A}{b^n} \geq \left|\alpha - \frac{a}{b}\right|,$$

which is a contradiction. Hence our initial assumption must have been incorrect, and our lemma holds, as desired. □

Proof of Theorem 3 [7]. Consider α, a Liouville number as defined by Definition 3. We first show that α is irrational.

We proceed using proof by contradiction. Suppose, for the sake of contradiction, that $\alpha \in \mathbb{Q}$; then there exist integers p, q with $q \neq 0$ such that $\alpha = \frac{p}{q}$. Let k be a positive integer such that $2^{k-1} > q$. Then, for any integers a and b with $b > 1$ and $\frac{a}{b} \neq \frac{p}{q}$, we have

$$\left|\alpha - \frac{a}{b}\right| = \left|\frac{p}{q} - \frac{a}{b}\right| \geq \frac{1}{bq} > \frac{1}{2^{k-1}b} \geq \frac{1}{b^k},$$

which is a direct contradiction of Definition 3. This implies α is not a Liouville number, which means our initial assumption that α is rational must be incorrect. Hence any Liouville number must be irrational, as desired.

We next use this information and Lemma 2 to show that α is transcendental. We again proceed using proof by contradiction. Suppose, for the sake of contradiction, that α is an algebraic number. By Lemma 2, there exist constants $A, n > 0$ such that Inequality 3 is true for all integers a, b with $b > 0$. Let l be a positive integer such that $2^l \geq \frac{1}{A}$. Since α is a Liouville number, Definition 3 tells us

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{b^n} \cdot \frac{1}{b^l} \leq \frac{1}{2^l \cdot b^n} \leq \frac{A}{b^n};$$

however, this contradicts Inequality 3, so our initial assumption must have been incorrect. In other words, α cannot be an algebraic number. Therefore, we have shown that every Liouville number is transcendental, as desired. □

Thus we have proven Theorem 3; we next give an explicit example as to what a Liouville number looks like, which will be our first concrete example of a transcendental number.

Example (Filaseta [7]). We claim $\alpha = \sum_{j=0}^{\infty} \frac{1}{2^{j!}}$ is a Liouville number. For an arbitrary, but particular, integer n we consider $\frac{a}{b} = \sum_{j=0}^{n} \frac{1}{2^{j!}}$, with a and $b = 2^{n!} > 1$ integers. We have

$$0 < \left|\alpha - \frac{a}{b}\right| = \sum_{j=n+1}^{\infty} \frac{1}{2^{j!}} < \sum_{j=(n+1)!}^{\infty} \frac{1}{2^j} = \frac{1}{2^{(n+1)!-1}} \leq \frac{1}{2^{n(n!)}} = b^n,$$

as desired (using the infinite geometric sum formula and expansion $(n+1)! = (n+1)n!$). Hence, by Definition 3, α is a Liouville number, and Theorem 3 tells us α is also transcendental.

4. Extensions of the algebraic real numbers

There are two primary ways to initially extend this project beyond the limited scope of this paper:

(1) Consider the algebraic numbers as a subset of $\mathbb{C}$ rather than $\mathbb{R}$.
(2) Consider the polynomials in a different subspace, possibly $\mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{Z}$.

4.1. **Subset of the complex numbers.** Changing the superset to the complex numbers would have limited impact if the associated polynomial p must stay in $\mathbb{Q}[x]$; but, outside that intuition, some new—previously complex—points would be joining A. These new points may potentionally add interesting characteristics of their own, worth studying in further detail.

4.2. **Operations in a different field.** In a different algebraic structure, the underlying set composition of A's analogue could be interesting due to greater change from the status quo of this paper. This direction of intellectual exploration would make use of the intersection of modern algebra and real analysis, and may provide greater insight into both fields and the concept of algebraic and transcendental numbers as such.

5. Conclusion

In conclusion, we have seen that the algebraic real numbers yielded naturally by polynomials with rational coefficients have created two very interesting infinite sets with diverging properties. In the future, more work should be undertaken analyzing how modifying the definition of an "algebraic number" changes the set's properties.

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