Computer Algebra and Elliptic Functions

Steven Finch

July 29, 2013

Let

$$F[\phi,m]=\int\limits_0^{\sin(\phi)}\frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-m\,\tau^2}}$$

denote the incomplete elliptic integral of the first kind and $K[m]=F[\pi/2,m]$; we purposefully choose formulas to be consistent with the computer algebra package MATHEMATICA. Consider the domain Ω in the complex plane with points $u+iv$ satisfying one of the four conditions:

$$\begin{array}{llll} u\geq 0, & v\geq 0, & (u+1)^2+v^2\leq 2, & u^2+(v+1)^2\leq 2; \\ u\leq 0, & v\geq 0, & (u-1)^2+v^2\leq 2, & u^2+(v+1)^2\leq 2; \\ u\leq 0, & v\leq 0, & (u-1)^2+v^2\leq 2, & u^2+(v-1)^2\leq 2; \\ u\geq 0, & v\leq 0, & (u+1)^2+v^2\leq 2, & u^2+(v-1)^2\leq 2. \end{array}$$

(Figure 1). Define three complex-valued functions on Ω:

$$f(u,v)=\frac{1}{4}\left(-i\,F\left[\theta(u,v),\frac{1}{4}\right]+F\left[\theta(u,v),\frac{3}{4}\right]\right),$$

$$g(u,v)=\frac{1}{4}\left(i\,F\left[\theta(u,v),\frac{1}{4}\right]+F\left[\theta(u,v),\frac{3}{4}\right]\right),$$

$$h(u,v)=\left(2-\sqrt{3}\right)F\left[\arcsin\left(i\left(2+\sqrt{3}\right)(u+i\,v)^2\right),\left(2-\sqrt{3}\right)^4\right]$$

where

$$\theta(u,v)=\arcsin\left(\frac{2(1+i)(u+i\,v)}{\sqrt{1+4i(u+i\,v)^2-(u+i\,v)^4}}\right).$$

Two basic Jacobi elliptic functions, characterized via

$$u=\int\limits_0^{\operatorname{sn}(u,m)}\frac{d\tau}{\sqrt{1-\tau^2}\sqrt{1-m\,\tau^2}}=\int\limits_{\operatorname{cn}(u,m)}^{1}\frac{d\tau}{\sqrt{1-\tau^2}\sqrt{m\,\tau^2+(1-m)}},$$

are also needed. Actually, only one is required because $\operatorname{sn}(u,m)^2+\operatorname{cn}(u,m)^2=1$.

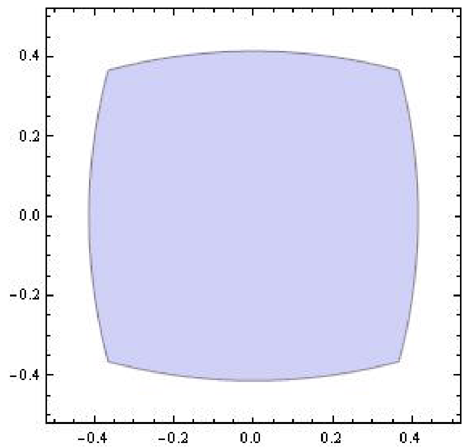

Figure 1: Domain Ω

0.1. Schwarz D surface. Examine the simultaneous system of 3 equations in 5 unknowns:

$$x = \frac{1}{2} + \kappa \operatorname{Im}(f(u,v)),$$
$$y = \frac{1}{2} + \kappa \operatorname{Im}(g(u,v)),$$
$$z = -\frac{1}{2} + \kappa \operatorname{Re}(h(u,v))$$

where

$$\kappa = \frac{2}{K[1/4]} = 1.1864152923...$$

is a normalization constant. The map $\Omega \ni (u,v) \mapsto (x,y,z)$ defines a parametric surface in three-dimensional real space (Figure 2). This surface can also be given implicitly by the equation

$$\Phi(x)\Phi(y) + 1 = \Phi(y)\Phi(z) + \Phi(z)\Phi(x)$$

where

$$\Phi(\xi) = \sqrt{\frac{1 - \operatorname{cn}\left(\sqrt{3}\rho\,\xi, -1/3\right)}{1 + \operatorname{cn}\left(\sqrt{3}\rho\,\xi, -1/3\right)}}$$

and

$$\rho = 2/\kappa = 1.6857503548....$$

Such an assertion can be verified numerically. What is missing is an algebraic proof that u, v can indeed be eliminated from the 3 equations to yield the implicit representation.

0.2. Schwarz P surface. Examine the simultaneous system of 3 equations in 5 unknowns:

$$x = \kappa \operatorname{Re}(f(u,v)),$$

$$y = \kappa \operatorname{Re}(g(u,v)),$$

$$z = \frac{1}{2} + \kappa \operatorname{Im}(h(u,v))$$

where

$$\kappa = \frac{3}{2K[1/9]} = 0.9274219745....$$

is a normalization constant. The map $\Omega \ni (u,v) \mapsto (x,y,z)$ defines a parametric surface in three-dimensional real space (Figure 3). Except for translation and scaling, this surface is conjugate to the preceding. What is remarkable is that an implicit representation here remains open [1]. A proof for the preceding case (Schwarz D) might carry over in some manner to here (Schwarz P).

Just as the D surface is the solution of Plateau's problem for 4 edges of a regular tetrahedron, the P surface can be shown to be the solution of Plateau's problem for 4 edges of a regular octahedron [2].

Other choices of the region Ω are possible [3] – we have not explored this avenue – and may simplify the algebra.

References

[1] J. C. C. Nitsche, Periodic surfaces that are extremal for energy functionals containing curvature functions, *Statistical Thermodynamics and Differential Geometry of Microstructured Materials*, Proc. 1991 Minneapolis conf., ed. H. T. Davis and J. C. C. Nitsche, Springer-Verlag, 1993, pp. 69–98; MR1226921 (94m:58052).

[2] S. R. Finch, Partitioning problem, unpublished note (2013).

[3] H. Terrones and A. L. Mackay, Negatively curved graphite and triply periodic minimal surfaces, *J. Math. Chemistry* 15 (1994) 183–195; MR1279755 (95b:53012).

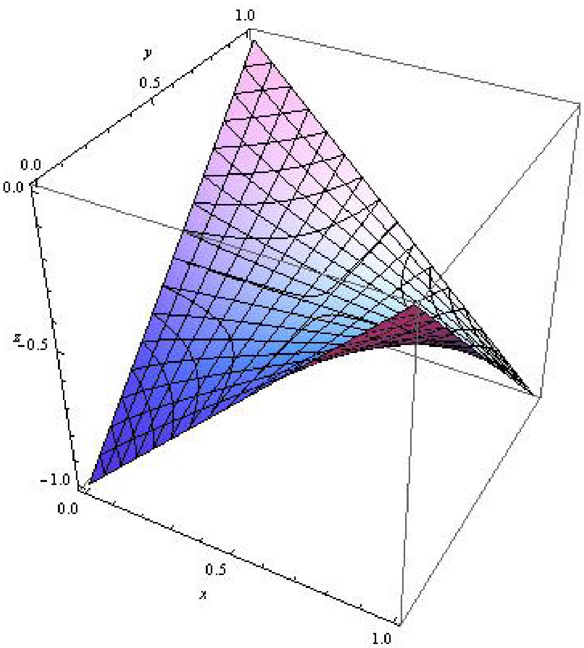

Figure 2: D tetrahedral saddle

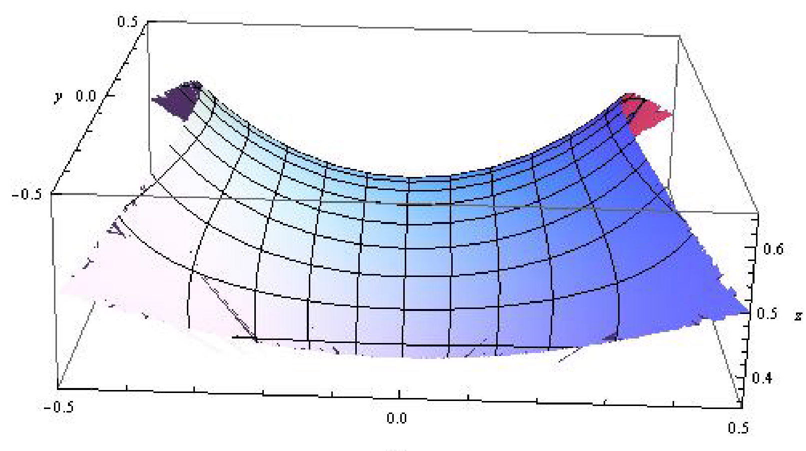

Figure 3: P saddle