These are notes for a talk that I was coerced into giving for the Langlands support group on modular representation theory. I am certain that this talk will go horribly — there are too many indices flying around! I should’ve paid more attention to how the physicists teach general relativity.

Let $S = \text{Spec}(k)$ with $k$ a perfect field of characteristic $p$. Recall that if $X$ is an $S$-scheme, then the pullback diagram

\[
\begin{array}{ccc}
X & \overset{F_X}{\rightarrow} & X
\
\downarrow & & \downarrow
\
X^{(p)} & \overset{F_{X/S}}{\rightarrow} & X
\
\downarrow & & \downarrow
\
S & \overset{\text{Frob}}{\rightarrow} & S
\end{array}
\]

defines the absolute Frobenius $F_X$ and the relative Frobenius $F_{X/S} : X \rightarrow X^{(p)}$. In this talk, we will be concerned with the relative Frobenius, so we shall simply denote it by $F : X \rightarrow X^{(p)}$.

**Definition 1.** Let $G$ be an $S$-group, and let $F^n : G \rightarrow G^{(p^n)}$ denote the $n$th iterate of the relative Frobenius. The $n$th Frobenius kernel of $G$ is the $S$-subgroup of $G$ defined by $G_n = \ker(F^n)$.

**Remark 2.** There are inclusions $G_n \subseteq G_{n+1}$, and that $(G_n)_{n'}$ is either $G_{n'}$ (if $n' \leq n$) or $G_n$ (if $n \leq n'$).

**Example 3.** Suppose $S = \text{Spec}(k)$, and let $G = G_m = \text{Spec}(k[t^{±1}])$. Then $F : G_m \rightarrow G_m^{(p)}$ sends $t$ to $t^p$, and so $(G_m)_n = \text{Spec} k[t]/(t^{p^n} - 1) = \mu_{p^n}$. Similarly, if $G = G_a = \text{Spec}(k[t])$, then $(G_a)_n = \text{Spec} k[t]/t^{p^n} = \alpha_{p^n}$.

**Remark 4.** If $G$ is a reduced affine $S$-group scheme, then $F^n$ induces an isomorphism $G/G_{n} \rightarrow G^{(p^n)}$. Indeed, this follows from the fact that $G/G_{n}$ is the closed subgroup of $G^{(p^n)}$ given by the kernel of $k[G]^{(p^n)} \rightarrow k[G]$ sending $f$ to $f^{p^n}$.

Last time, Kevin talked about the equivalence of categories between finite $S$-group schemes and finite-dimensional cocommutative Hopf $k$-algebras.

**Example 5.** Let $\mathfrak{g}$ be a restricted Lie algebra (so there is a map $X \mapsto X^{[p]}$, referred to as a $p$-derivation). If $\mathfrak{g}$ is the Lie algebra of an algebraic group over $k$, then the $p$th power of a derivation is a derivation (because we are in characteristic $p$), and the operation of taking the $p$th power of a derivation defines a restricted Lie algebra structure on $\mathfrak{g}$. Indeed, if $\mathfrak{g} = \text{Lie}(G_a)$, then $U(\mathfrak{g}) = k[x]$, and the $p$-derivation is trivial.

Recall that $U^{[p]}(\mathfrak{g})$ denotes the quotient $U(\mathfrak{g})/(x^p - x^{[p]})$ of the universal enveloping algebra of $\mathfrak{g}$ by the relation which forces the $p$th power of any element to be equal to its $p$th derivation. This is a finite-dimensional $k$-algebra, of dimension $p^{\dim(\mathfrak{g})}$. The $k$-algebra $U^{[p]}(\mathfrak{g})$ admits the structure of a cocommutative Hopf algebra, given by sending $x \in \mathfrak{g}$ to $x \otimes 1 + 1 \otimes x$ under the diagonal, to $-x$ under the antipode, and to 0 under the counit. By the equivalence of categories discussed last time, there exists a finite algebraic $k$-group with associated Hopf algebra $U^{[p]}(\mathfrak{g})$. For instance, if $\mathfrak{g} = \text{Lie}(G_a)$, then $U^{[p]}(\mathfrak{g}) = k[x]/x^p$ — notice that this is exactly the Hopf algebra associated to $\alpha_p$. This is a special case of the following more general observation:

**Proposition 6.** Let $\mathfrak{g} = \text{Lie}(G)$. The finite $k$-group corresponding to $U^{[p]}(\mathfrak{g})$ is $G_1$. 

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Proof. Recall that if $H$ is any algebraic $k$-group, then $U^{|p|}(\mathfrak{h})$ injects into the algebra $\text{Dist}(H)$ of distributions. In particular, $U^{|p|}(\text{Lie}(G_1))$ injects into $\text{Dist}(G_1)$. Now, $\dim U^{|p|}(\mathfrak{g}) = p^{\dim(\mathfrak{g})}$, so since $\text{Lie}(G_n) = \mathfrak{g}$ for any $n \geq 1$, we must have $\dim U^{|p|}(\text{Lie}(G_1)) = p^{\dim(\mathfrak{g})}$. But $\dim \text{Dist}(G_1) = \dim k[G_1] \leq p^m$, and so the inclusion of $U^{|p|}(\text{Lie}(G_1))$ into $\text{Dist}(G_1)$ must be an isomorphism. Because $G_1$ is infinitesimal (the augmentation ideal is nilpotent), $\text{Dist}(G_1)$ is the Hopf algebra associated to $G_1$. We conclude that the finite $k$-group corresponding to $U^{|p|}(\mathfrak{g})$ is $G_1$. 

In particular, the representation theory of $G_1$ is the same as the representation theory of $U^{|p|}(\mathfrak{g})$, which is in turn the same as the representation theory of $\mathfrak{g}$ as a restricted Lie algebra.

In characteristic $p$, the cohomology of groups (with coefficients in some representation) is quite interesting, so it would be useful to understand how to relate the group cohomology of $G$ with the inverse limit of the group cohomology of its Frobenius kernels.

**Proposition 7.** Let $G$ be irreducible and reduced, and suppose that $H^\ast(G; k)$ is concentrated in degree 0. Suppose further that for every finite-dimensional $G$-representation $V$, the cohomology groups $H^i(G; V)$ are finite-dimensional $k$-vector spaces. Then for every finite-dimensional $G$-representation $W$, the map $H^i(G; W) \twoheadrightarrow \lim H^i(G_n; W)$ is an isomorphism.

**Proof.** For each $n \geq 1$, we have the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j}(r) = H^i(G/G_n; H^j(G_n; W)) \Rightarrow H^{i+j}(G; W).$$

Because $G$ is reduced, there is an isomorphism $G/G_r \cong G(\mathfrak{g}^n)$. Moreover, each $E_2^{i,j}(n)$ is finite-dimensional. Moreover, the maps $G_n \to G_{n+1}$ and $G/G_n \to G/G_{n+1}$ induce a map $\{E_2^{i,j}(n+1)\} \to \{E_2^{i,j}(n)\}$ of spectral sequences. Inverse limits are exact on filtered colimits of finite-dimensional vector spaces, so there is a spectral sequence $\{E_2^{i,j}(n)\} := \{ \lim E_2^{i,j}(n) \}$ converging to $H^\ast(G; W)$.

We claim that $E_2^{i,j} = 0$ for $i > j$. To see this, it suffices to show that for any $n$, there exists some $m \geq n$ (depending only on $n$ and $j$) such that the map $E_2^{i,j}(m) \to E_2^{i,j}(n)$ is zero for all $i > j$. To see this, note that the map $E_2^{i,j}(m) \to E_2^{i,j}(n)$ factors as $H^i(G/G_m; H^j(G_m; W)) \to H^i(G/G_n; H^j(G_n; W)) \to H^i(G/G_n; H^j(G_n; W)^G_m) \to H^i(G/G_n; H^j(G_n; W))$. It suffices to show that for each $n$, there is some $m$ such that $H^j(G_n; W)^G_m = H^j(G_n; W)^G$; indeed, then

$$H^i(G/G_n; H^j(G_n; W)^G_m) \cong H^i(G/G_n; H^j(G_n; W)^G) \cong H^i(G/k; H^j(G_n; W)^G),$$

which vanishes because we assumed that $H^j(G/k)$ is concentrated in degree 0.

If $M$ is a $G$-module and $G$ is irreducible, then $M^G = \bigcap_{n>0} M^G_n$. There is a descending chain $M^G \supseteq M^G_2 \supseteq M^G_1$, so if $M$ is moreover finite-dimensional, then there exists some $N \gg 0$ such that $M^G = M^G_N$. In particular, it follows that for each $n$ and $j$, there is some $m$ such that $H^j(G_n; W)^G_m = H^j(G_n; W)^G$, as desired.

Since $E_2^{i,j} = 0$ for $i > j$, we have $H^j(G; W) \cong \lim H^j(G_n; W)^G$. Since there exists some $m \geq n$ such that the map $H^j(G_m; W) \to H^j(G_n; W)$ takes values in $H^j(G_n; W)^G$, we find that $\lim H^j(G_n; W)^G \cong \lim H^j(G_n; W)^G$. This proves the desired result. 

We now turn to the May spectral sequence to calculate the cohomology of these Frobenius kernels. Recall how this goes:

**Definition 8.** Let $A$ be a commutative Hopf algebra over $k$, and let $I$ denote the kernel of the augmentation $A \to k$. If $M$ is an $A$-comodule, then the Hochschild complex $A^{\otimes \bullet} \otimes M$ computes the cohomology of $M$ as an $A$-module. The powers of the augmentation ideal give a filtration of the Hochschild complex, where the $n$th filtered piece in degree $j$ is $\sum I^{a_i} \otimes \cdots \otimes I^{a_1} \otimes M$ with $\sum a_i = n$. The associated graded of this filtration in degree $j$ is therefore $\bigoplus I^{a_1}/I^{a_1+1} \otimes \cdots \otimes I^{a_j}/I^{a_j+1} \otimes M$, where the sum is taken over all $a_i$ such that $\sum a_i = n$.

It’s an easy calculation that, in fact, the differentials respect the filtration, and hence descend to the associated graded. Moreover, if $\text{gr}(A)$ is defined to be $\bigoplus_{n \geq 0} I^n/I^{n+1}$, then $\text{gr}(A)$ is a Hopf algebra over $k$, and $M$ can be given the structure of a trivial $\text{gr}(A)$-module. The associated graded of the
Hochschild complex of $M$ as a $G$-module is then isomorphic (as a complex) to the Hochschild complex of $M$ as a $\text{gr}(A)$-module. In particular, if $\bigcap I^n = 0$, then we obtain we obtain a convergent spectral sequence

$$E_1^{i,j} = H^{i+j}(\text{gr}(A); k)_j \otimes M \Rightarrow H^{i+j}(A; M).$$

This is known as the May spectral sequence.

**Example 9.** If $G$ is an algebraic $k$-group, setting $A = k[G]$ produces a commutative Hopf algebra $\text{gr}(A)$, and hence a $k$-group scheme $\text{gr}(G)$ such that $k[\text{gr}(G)] = k(G)$. It follows that if $G$ is irreducible, then there is a May spectral sequence

$$E_1^{i,j} = H^{i+j}(\text{gr}(G); k)_j \otimes M \Rightarrow H^{i+j}(G; M).$$

This spectral sequence is multiplicative.

**Proposition 10.** Let $G$ be reduced and irreducible. Then there is an isomorphism

$$E_1^{i,j} \cong \bigoplus M \otimes (\text{Sym}^{a_1} g^*)(p) \otimes (\text{Sym}^{a_2} g^*)(p^2) \otimes \cdots \otimes \Lambda^{b_2} g^* \otimes (\Lambda^{b_2} g^*)(p) \otimes \cdots$$

for $p$ odd, where the sum is over all finite sequences $\{a_n\}$ and $\{b_n\}$ with $i + j = \sum 2a_n + b_n$ and $i = \sum a_n p^n + b_n p^{n-1}$.

If $p = 2$, then

$$E_1^{i,j} \cong \bigoplus M \otimes \text{Sym}^{a_1} g^* \otimes (\text{Sym}^{a_2} g^*)(p) \otimes \cdots,$$

where the sum is over all finite sequences $\{a_n\}$ such that $i + j = \sum a_n$ and $i = \sum a_n 2^n - 1$.

**Proof sketch.** Obviously, we only need to understand $H^*(\text{gr}(G); k)$. We will work at an odd prime; the argument is the same for the even prime. There is a surjection $\text{Sym}(I/I^2) \rightarrow k[\text{gr}(G)]$, and this is an isomorphism if $G$ is reduced. In particular, the cohomology of $\text{gr}(G)$ is precisely the cohomology of the Hopf algebra $\text{Sym}(I/I^2) = \text{Sym}(g^*)$ (with trivial diagonal). The claimed result now follows from the general claim that if $V$ is a finite-dimensional $k$-vector space, then

$$H^*(\text{Sym}(V^*); k) \cong \text{Sym} \left( \bigoplus_{n \geq 1} (V^*)(p^n) \right) \otimes \Lambda \left( \bigoplus_{n \geq 0} (V^*)(p^n) \right),$$

if I haven’t messed up the indexing. One can check this by writing down a Koszul resolution for $k$ as a $\text{Sym}(V^*)$-comodule. 

**Remark 11.** Recall that in characteristic 0, we have $H^*(\text{Sym}(V^*); k) \cong \Lambda V$.

The spectral sequence of Proposition 10 also works for the cohomology of the Frobenius kernels; the idea here is that one truncates the contributions coming from a large enough $I$-adic filtration.

**Proposition 12.** Let $G$ be reduced and irreducible. There is a May spectral sequence

$$E_1^{i,j} = H^{i+j}(\text{gr}(G); k)_j \otimes M \Rightarrow H^{i+j}(G; M).$$

There is an isomorphism

$$E_1^{i,j} \cong \bigoplus M \otimes (\text{Sym}^{a_1} g^*)(p) \otimes (\text{Sym}^{a_2} g^*)(p^2) \otimes \cdots \otimes \Lambda^{b_2} g^* \otimes (\Lambda^{b_2} g^*)(p) \otimes \cdots$$

for $p$ odd, where the sum is over all finite sequences $\{a_n\}_{1 \leq n \leq N}$ and $\{b_n\}_{1 \leq n \leq N}$ with $i + j = \sum 2a_n + b_n$ and $i = \sum a_n p^n + b_n p^{n-1}$.

If $p = 2$, then

$$E_1^{i,j} \cong \bigoplus M \otimes \text{Sym}^{a_1} g^* \otimes (\text{Sym}^{a_2} g^*)(p) \otimes \cdots,$$

where the sum is over all finite sequences $\{a_n\}_{1 \leq n \leq N}$ such that $i + j = \sum a_n$ and $i = \sum a_n 2^{n-1}$. 

Example 13. Let $G = \mathbb{G}_a$. Then

$$H^*(\mathbb{G}_a; k) \cong \begin{cases} k[x_1, x_2, \cdots] & \text{char}(k) = 2 \\ k[y_1, y_2, \cdots] \otimes \Lambda(x_1, x_2, \cdots) & \text{char}(k) > 2, \end{cases}$$

where $|x_i| = 1$ and $|y_i| = 2$. Similarly,

$$H^*(\alpha_p; k) \cong \begin{cases} k[x_1, \cdots, x_n] & \text{char}(k) = 2 \\ k[y_1, \cdots, y_n] \otimes \Lambda(x_1, \cdots, x_n) & \text{char}(k) > 2. \end{cases}$$

(This is the cohomology of $B(\mathbb{Z}/p)^n$.)

Example 14. Suppose $N = 1$, and work at an odd prime. Then one can check that

$$E_1^{i,j} = \begin{cases} M \otimes (\text{Sym}^s g^*)^{(p)} \otimes \Lambda^{t-s} g^* & \text{if } i = s(p-1) + t, \ j = -(p-2)s \\ 0 & \text{else.} \end{cases}$$

This spectral sequence is rather sparse: because $E_1^{i,j}$ vanishes for $(p-2) \nmid j$, we find that the $d_r$-differential vanishes for $r \neq 1 \pmod{p-2}$. Let us define a reindexed spectral sequence $\{\tilde{E}_r^{i,j}\}$ by setting $\tilde{E}_r^{i,j} = E_{(p-2)r+1}^{i+j-(p-2)i}$. One then ends up with a spectral sequence

$$\tilde{E}_1^{i,j} = M \otimes (\text{Sym}^1 g^*)^{(p)} \otimes \Lambda^{j-i} g^* \Rightarrow H^{i+j}(G_1; M).$$

This is jarring to write down; I've never written down the $E_{0}$-page of a spectral sequence.

Example 15. One can explicitly write down the 0-line of the $E_{1}$-page of the spectral sequence in Example 14. Since $d_0^{0,j} : E_0^{0,j} \to E_0^{0,j+1}$ and $E_0^{0,j} = M \otimes \Lambda^j g^*$, we find that

$$d_0^{0,j} : M \otimes \Lambda^j g^* \to M \otimes \Lambda^{j+1} g^*.$$ 

A rather tedious calculation shows that the complex $(E_0^{0,*}, d_0^{0,*})$ is isomorphic to the Chevalley-Eilenberg complex $M \otimes \Lambda^* g$ computing the cohomology of $g$. I haven't had time to go through this in detail myself. In any case, one ends up with an isomorphism $E_1^{i,j} \cong H^j(g; M)$.

In fact, one can calculate the entire $E_1$-page of this spectral sequence.

Theorem 16 (Friedlander-Parshall, Jantzen). The $E_1$-page of the spectral sequence in Example 14 can be described as follows:

$$E_1^{i,j} = H^{j-i}(g; M) \otimes (\text{Sym}^s g^*)^{(p)}.$$ 

Proof sketch. The $d_0$-differential (still jarring) goes $d_0^{0,j} : \tilde{E}_0^{i,j} \to \tilde{E}_{0}^{i,j+1}$, i.e.,

$$d_0^{0,j} : M \otimes \Lambda^{j-i} g^* \otimes (\text{Sym}^s g^*)^{(p)} \to M \otimes \Lambda^{j-i+1} g^* \otimes (\text{Sym}^s g^*)^{(p)}.$$ 

This differential is a derivation, so

$$d_0^{0,j}(m \otimes x \otimes y) = d_0^{0,j-i}(m \otimes x) \otimes y + m \otimes x \otimes d_0^{0,j}(y).$$ 

Here, $d_0^{0,j}$ denotes the differential appearing in the $E_0$-page of the spectral sequence for $M = k$. Example 15 implies that in order to get the statement of the theorem, it suffices to show that $d_0^{0,j}$ vanishes for all $i$. In turn, it suffices to show that $a_0^{1,1} = 0$.

It is rather easy to observe that if $H$ is a closed subgroup of $G$, then the vanishing of $d_0^{0,j}$ for $H$ implies the vanishing of $d_0^{0,j}$ for $G$. Since we can always choose some large enough $n$ such that $G$ is a closed subgroup of $\text{SL}_n$, it suffices to show that $d_0^{1,1}$ vanishes for $\text{SL}_n$ (at least, for certain $n$).

Recall that

$$d_0^{1,1} : (g^*)^{(p)} \to g^* \otimes (g^*)^{(p)}.$$ 

There is a natural $G$-action on the spectral sequence of Proposition 10, and so this is a $G$-equivariant, and hence $g$-equivariant, homomorphism. One now argues that if $G$ is defined over $F_p$, then the $g$-action on $(g^*)^{(p)}$ is trivial. Therefore, $d_0^{1,1}$ factors through $(g^*)^{(p)} \otimes (g^*)^{(p)} = (g/\mathfrak{g}, \mathfrak{g})^* \otimes (g^*)^{(p)}$. One now concludes using the observation that if $p \mid n$, then $\mathfrak{sl}_n = [\mathfrak{sl}_n, \mathfrak{sl}_n]$. □
Remark 17. The Friedlander-Parshall spectral sequence is usually written with another indexing: suppose we reindex again, and write
\[ 'E^{i,j}_{2r} = 'E^{i,j}_{2r-1} = \begin{cases} E^{m,m+j}_r & i = 2m \\ 0 & i = m' + 1 \end{cases} \]
so that \( d^{2m,j}_{2r} = d^{m,m+j}_r \). Then Theorem 16 implies that
\[ 'E^2 \cong H^j(g; M) \otimes (\text{Sym}^i g^*)(p) \Rightarrow H^{i+j}(G_1; M). \]

Remark 18. These spectral sequences imply that \( H^*(G_1; k) \) is a finitely generated \( k \)-algebra. This was generalized by Friedlander and Suslin, who showed that if \( G \) is a finite group scheme, then \( H^*(G; k) \) is a finitely generated \( k \)-algebra. Moreover, they showed that if \( M \) is a finite-dimensional \( G \)-module, then \( H^*(G; M) \) is a finitely generated \( H^*(G; k) \)-module.

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