SOME EXERCISES

By popular demand, I’m putting up some fun problems to solve. These are meant to give intuition for messing around with spectra.

1. THE ALGEBRAIC THICK SUBCATEGORY THEOREM

In Lecture 2, we proved that the only thick subcategories of the category Sp_ω of (p-local, always) finite spectra are the subcategories C_n (defined to be the collection of those finite spectra X such that K(n − 1)_*-acyclic finite spectra).

In this problem, we will prove an algebraic analogue of this result; this will be a somewhat different version of [Rav92, Theorem 3.4.2], although the proof is the same. If you get stuck, you can either look there, or the lecture notes for Lecture 2. Let A be the category of BP_*BP-comodules which are finitely presented as BP_*-modules. This is an abelian category.

We define a thick subcategory C of A to be a subcategory satisfying the two-out-of-three property for short exact sequences: if two of A, B, and C are in C and there is a short exact sequence

$$0 \to A \to B \to C \to 0,$$


then the third is also in C.

In analogy to the subcategories C_n of Sp_ω, we define subcategories, also denoted C_n, of A by saying that M ∈ C_n iff v_{n−1}M = 0, i.e., v_{n−1} is nilpotent in M. Our goal will be to show that any thick subcategory of A is either trivial, A itself, or C_n for some n.

Our proof will rely on the following result. Recall that I_n = (p, v_1, · · ·, v_{n−1}) is an ideal of BP_*; it turns out that this is an “invariant ideal”, in the sense that BP_*/I_n is a BP_*BP-comodule.

Theorem 1 (Landweber filtration theorem). Any object M of A has a filtration

$$0 = M_k \subset \cdots \subset M_l \subset M$$

where M_j/M_{j+1} is isomorphic as a BP_*BP-comodule to (a shift of) BP_*/I_{n_j}.

• Prove the following corollary of Theorem 1.

Corollary 2. There is a strict inclusion C_{n+1} ⊆ C_n.

Concretely, this says: if v_{n}^{-1}M = 0, then v_{n−1}M = 0. In the notation of Theorem 1, show that n_j ≥ n (prove that v_{n}^{-1}BP_*/I_{n_j} = 0). Conclude that v_{n−1}M = 0.

To prove that the inclusion is strict, find an example of a BP_*BP-comodule M such that v_{n−1}M = 0 but v_{n}^{-1}M ≠ 0.
• Let $C$ denote a thick subcategory of $A$. Use Theorem 1 to prove that it suffices to show that all the quotients $BP^*/I_n$ are in $C$.

• Let $M$ be a comodule in $C \subseteq C_n$ but not in $C_{n+1}$. Using Corollary 2 prove that some $n_i = n$ for some $i$.

• Show that $C$ contains $BP^*/I_m$ for every $m \geq n$, hence it contains every quotient $BP^*/I_{n_j}$, as desired.

2. Real and complex $K$-theory

Let $KU$ denote complex $K$-theory, and let $KO$ denote real $K$-theory. Bott periodicity implies that these are “nonconnective” real spectra (in other words, they have homotopy in negative dimensions). The spectrum $KU$ naturally admits an action of the cyclic group $C_2$, coming from the complex conjugation action on complex vector bundles. A theorem of Atiyah’s shows that $KU^{hC_2} \simeq KO$. In this problem, we will study $KU^{hC_2}$, and prove this result computationally.

Our key technical tool in this problem is the homotopy fixed point spectral sequence. Recall that if $E$ is a spectrum and $G$ is a finite group acting on $E$, the homotopy fixed points $E^{hG}$ are defined via

$$E^{hG} = \text{Hom}_G(\Sigma^\infty EG, X);$$

here $\text{Hom}$ is the internal hom in spectra, also known as the function spectrum. The homotopy orbits are defined via

$$E_{hG} = (\Sigma^\infty EG \wedge X)/G.$$

One can compute the homotopy of $E^{hG}$ and $E_{hG}$: there are spectral sequences, appropriately called the homotopy fixed points spectral sequence and the homotopy orbits spectral sequence. These are of signature:

$$E_2^{s,t} = H^s(G; \pi_t E) \Rightarrow \pi_{t-s}(E^{hG});$$

$$E_2^{s,t} = H^s(G; \pi_t E) \Rightarrow \pi_{t-s}(E_{hG}).$$

We will prove Atiyah’s theorem through the following sequence of steps.

• Let $\sigma$ denote a generator of $C_2$, and let $u$ denote the Bott element (represented by $1 - [L]$, where $L$ is the line bundle over $\mathbb{C}P^1$ obtained by pulling back the universal line bundle via the inclusion $\mathbb{C}P^1 \subseteq \mathbb{C}P^\infty$). Show that $\sigma(u) = -u$.

• Use this to compute that the $E_2$-page of the homotopy fixed points spectral sequence for $KU^{hC_2}$ is, as a bigraded ring, given by

$$E_2^{s,t} \simeq H^s(C_2; \pi_* KU) \simeq \mathbb{Z}[u^{\pm 2}, \alpha]/2\alpha,$$

where the bidegree of $u$ and $\alpha$ as $(s, t)$ is given by $|u| = (0, 4)$, and $|\alpha| = (1, 2)$.

• Suppose $C_2$ acts trivially on the sphere spectrum $S$. Show that $S^{hC_2}$ is $\Sigma^\infty_+ \mathbb{R}P^\infty$. Conclude that the homotopy fixed point spectral sequence for $S^{hC_2}$ is the same as the Atiyah-Hirzebruch spectral sequence for $\pi_* \Sigma^\infty_+ \mathbb{R}P^\infty$. 

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(the $E_2$-page of this spectral sequence is $H^*(\Sigma^\infty_+ \mathbb{RP}^\infty; \pi_\ast S)$). In the Atiyah-Hirzebruch spectral sequence, we have an element $\eta$ in bidegree $(1,2)$.

- The unit map $S \to KU$ is equivariant for this action, so we get a map $S^{hC_2} = \Sigma^\infty_+ \mathbb{RP}^\infty \to KU^{hC_2}$. Show that under the map of homotopy fixed point spectral sequences, the element $\eta$ maps to $\alpha$.

- Use the attaching maps in the Atiyah-Hirzebruch spectral sequence for $\Sigma^\infty_+ \mathbb{RP}^\infty$ to write down some differentials in this spectral sequence. Use naturality to transport these differentials to the homotopy fixed point spectral sequence for $KU^{hC_2}$; in particular, prove that in the homotopy fixed point spectral sequence for $KU^{hC_2}$, we have a $d_3$-differential

$$d_3(u^2) = \alpha^3.$$ 

See the first section of https://sanathdevalapurkar.github.io/chromotopy/2017/09/06/hfpss.html if you get stuck.

- Compute the $E_3$-page for the homotopy fixed point spectral sequence for $KU^{hC_2}$, and show that there can be no more differentials (for degree reasons).

- Conclude that

$$\pi_\ast(KU^{hC_2}) \simeq \mathbb{Z}[\alpha, 2u^2, u^{\pm 4}] / (2\alpha).$$

with $|\alpha| = 1$, and $|u^2| = 4$.

- Recall that Bott periodicity gives that

$$\pi_\ast(KO) \simeq \mathbb{Z}[x, y, z^{\pm 1}] / (2x, x^3, xy, y^2 - 4z),$$

with $|x| = 1$, $|y| = 4$, and $|z| = 8$. Construct a map $KO \to KU^{hC_2}$, and show that it induces an isomorphism on homotopy groups. Conclude that $KO \simeq KU^{hC_2}$.

- Let $bu$ denote connective complex $K$-theory, and $bo$ connective real $K$-theory. Is it true that

$$bo \simeq bu^{hC_2}?$$

Hint: nope.

However, $bo$ and $bu$ still have a very close relationship. In Problem 3 we will compute the cohomology of $bo$ using the cofiber sequence $\Sigma bo \to bo \to bu$, which arises from the cofiber sequence $O \to U \to U/O$.

- Show that the map $\Sigma bo \to bo$ can be identified with $bo \wedge \eta$, where $\eta \in \pi_1(S)$ is the first Hopf element, which is detected in the Adams spectral sequence by $h_1 = [\xi^2]$.

- Conclude that $bu \simeq \Sigma^{-2} bo \wedge \Sigma^\infty \mathbb{CP}^2$. Hint: show that $C(\eta)$, the cofiber of $\eta$, can be identified with $\Sigma^{-2} \Sigma^\infty \mathbb{CP}^2$.

- Likewise, show that $KU \simeq \Sigma^{-2} KO \wedge \Sigma^\infty \mathbb{CP}^2$.

- Show, using the nilpotency of $\eta$, that if $\mathcal{C}$ is a thick subcategory of the category $\text{Sp}^\omega$ of finite spectra which contains $\Sigma^\infty \mathbb{CP}^2$, then $\mathcal{C} = \text{Sp}^\omega$. 


3. Computations with the Adams spectral sequence

The goal of this problem is to lead you through the computation of the homotopy of $bu$, and then the homotopy of $bo$. For the purposes of this problem, we will always work at the prime 2. Feel free to refer to [Rog12 §6.3, 6.4].

Recall that the Adams spectral sequence runs

$$\text{Ext}^*_A(F_2, H^*(X)) \Rightarrow \pi_\bullet X^\wedge.$$  

Our goal will be to use this spectral sequence for $X = bu, bo$.

Let us recall some notation: $A(n)$ is the subalgebra of $A$ generated by $Sq^1, \cdots, Sq^{2^n}$, and $E(n)$ is the subalgebra generated by the Milnor primitives $Q_0, \cdots, Q_n$ (where $Q_0 = Sq^1$ and $[Q_0^{2^k}, Q_{k-1}] = Q_k$). It’s well-known that the duals (which sit inside $A^*$) of these subalgebras are given by

$$A(n)_* = A_*/I(n), \quad E(n)_* = A_*/J(n),$$

where

$$I(n) = \langle \xi_1^{2n+1}, \xi_2^{2n}, \cdots, \xi_n^{2n}, \xi_{n+1}^2, \xi_k|k \geq n+2 \rangle, \quad J(n) = \langle \xi_1^2, \cdots, \xi_{n+1}^2, \xi_k|k \geq n+2 \rangle.$$  

If $A$ is a $k$-algebra and $B \subseteq A$ a subalgebra with an augmentation $B \to k$, we define

$$A/\!\!/B = A \otimes_B k.$$  

The change-of-rings isomorphism (which you don’t need to prove; see [Rog12 Lemma 6.16]) states:

**Theorem 3.** Let $A$ be an algebra. Let $B$ be a subalgebra of $A$ such that $A$ is flat as a right $B$-module. Let $N$ be a left $A$-module, and let $M$ be a left $B$-module. Then:

$$\text{Ext}^s_t(A \otimes_B M, N) \simeq \text{Ext}^s_t(A, N).$$

To show this, one uses the usual isomorphism $\text{Hom}_A(A \otimes_B M, N) \simeq \text{Hom}_B(M, N)$, and the fact that $A$ is flat over $B$ to note that if $P_*$ is any projective resolution of $M$ as a $B$-module, then $P_* \otimes_A B$ is a projective resolution of $A \otimes_B M$ as an $A$-module.

As a warmup, consider the case $X = HZ_2$.

- Use the change-of-rings isomorphism to prove that the Adams $E_2$-page for $HZ_2$ is $\text{Ext}^*_A(F_2, F_2)$. (Hint: show that $H^*(HZ_2) = A/\!/A(0)$.)

- Prove that this $E_2$-page is given by $P(h_0)$, where $|h_0| = (0, 1)$, where the bidegree is written in the form $(t-s, s)$. This is the associated graded for $Z_2$.

To prove more general results, we will need the following lemma.

**Lemma 4.** Let $k$ be a field. Then

$$\text{Ext}^*_{F(k)}(k, k) = P(y);$$
here, $E(x)$ is an exterior algebra on $x$ (of possibly positive degree), and $P(y)$ is a polynomial algebra on $y$ (of degree $1 + |x|$).

To prove this lemma, use the injective resolution of $k$ as a $E(x)$-module given by

$$0 \to k \to E(x) \xrightarrow{d} E(x) \xrightarrow{d} E(x) \to \cdots,$$

where $d(x) = 1$ and $d(1) = 0$. Show that $\text{Hom}_{E(x)}(k, E(x)) \simeq k$, and conclude the lemma.

Let’s get on with our computation of $\pi_*bu_2^\wedge$.

- Prove that $H^*(bu) \simeq A/E(1)$. To do this, use Bott periodicity to show that there is a short exact sequence of $A$-modules $0 \to \Sigma^3H^*(bu) \to H^*(HZ_2) \to H^*(bu) \to 0$. Compare this with the short exact sequence $0 \to \Sigma^3A/E(1) \to A/E(0) \to A/E(1) \to 0$, and use induction on the internal degree to prove the desired result.

- Use the change-of-rings isomorphism to prove that the Adams $E_2$-page for $bu$ is given by $F_2[h_{1,0}, h_{2,0}]$, where $|h_{1,0}| = (0, 1)$ and $|h_{2,0}| = (2, 1)$. Again, the bidegree is written in the form $(t - s, s)$.

- Show that the spectral sequence collapses at the $E_2$-page, and that $\pi_*bu_2^\wedge \simeq \mathbb{Z}_2[\beta]$, where the class of $2 \in \pi_*bu_2^\wedge$ is $h_{1,0}$ (hint: use the map $bu_2^\wedge \to H\mathbb{Z}_2$, and then use the computation for $HZ_2$), and the class of $\beta$ is $h_{2,0}$.

We can now move on to the harder case of $\pi_*bo_2^\wedge$.

- Show that there is a cofiber sequence $\Sigma bo \to bo \to bu$. (Hint: use the cofiber sequence of spaces $O \to U \to U/O$ and real Bott periodicity.) This is called the Wood cofiber sequence.

- Prove that $H^*(bo) \simeq A/A(1)$. To do this, use the above result to show that there is a short exact sequence of $A$-modules $0 \to \Sigma^2H^*(bo) \to H^*(bu) \to H^*(bo) \to 0$. Compare this with the short exact sequence $0 \to \Sigma^2A/A(1) \to A/A(1) \to A/A(1) \to 0$, and use induction on the internal degree to prove the desired result. This is hard to write down rigorously — for details, see section 7 of Catherine Ray’s masters thesis.

- Use the change-of-rings isomorphism to compute the Adams $E_2$-page. In particular, show that

$$E_2^{s,*} \simeq F_2[h_0, h_1, v, w_1]/(h_0h_1, h_1^3, h_1v, v^2 = h_0^2w_1),$$

where $|h_0| = (0, 1)$, $|h_1| = (1, 1)$, $|v| = (4, 3)$, and $|w_1| = (8, 4)$. Again, the bidegree is written in the form $(t - s, s)$. This spectral sequence collapses for degree reasons, so this gives the computation of $\pi_*bo_2^\wedge$.

What about $\pi_*KO$ and $\pi_*KU$? Well, the Adams spectral sequence isn’t of much help here, because:

- Show that $KO \wedge H\mathbb{F}_p$ and $KU \wedge H\mathbb{F}_p$ are contractible for any prime $p$. 

4. The Chern character

Let $L_1, \ldots, L_n$ be line bundles over a space $X$. Suppose $\xi$ is a rank $n$ vector bundle such that $\xi = L_1 \oplus \cdots \oplus L_n$. Define $c(\xi) = \sum_{i=1}^{n} \exp(c_1(L_i))$.

- Show that the $i$th Chern class $c_i(\xi)$ of $\xi$ can be written as $e_i(c_1(L_1), \ldots, c_1(L_n))$.
- Use the above result and Newton’s identities to write $c(\xi)$, with $\xi$ still as above, in terms of the Chern classes of $\xi$. Namely, prove that

$$c(\xi) = \text{rank}(\xi) + \sum_{i=1}^{n} \frac{1}{i!} p_i,$$

where the $p_i$ are the power sums.

- Show that $c(\xi \otimes \eta) = c(\xi)c(\eta)$, and that $c(\xi \oplus \eta) = c(\xi) + c(\eta)$.

This allows us to extend the definition of the Chern character to all vector bundles; this begets a ring homomorphism $\text{ch} : KU(X) \rightarrow \bigoplus_{i=0} H^{2i}(X; \mathbb{Q})$.

It turns out that we can define the Chern character using the theory of formal group laws:

- Show that $KU \wedge H\mathbb{Q}$ is homotopy equivalent to $\bigvee_{i \in \mathbb{Z}} \Sigma^{2i} H\mathbb{Q}$.
- Prove that $\pi_0(KU \wedge H\mathbb{Q})$ is the ring which is “universal” for maps from $KU^0$ to rings over which the multiplicative formal group is isomorphic to the additive formal group.
- Conclude that the Chern character can be viewed as the map of cohomology theories coming from the map of spectra $KU \rightarrow KU \wedge H\mathbb{Q}$, induced by the unit map $S \rightarrow H\mathbb{Q}$.

References
