Incentive Compatibility: Everywhere vs. Almost Everywhere*

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August 29, 2006

Abstract

A risk neutral buyer observes a private signal $s \in [a, b]$, which informs her that the mean and variance of a normally distributed risky asset are s and σ_s^2 respectively. She then sets a price at which to acquire the asset owned by risk averse "outsiders". Assume $\sigma_s^2 \in \{0, \sigma^2\}$ for some $\sigma^2 > 0$ and let $\mathbb{B} = \{s \in [a, b] \mid \sigma_s^2 = 0\}$. If $\mathbb{B} = \emptyset$, then there exists a fully revealing equilibrium in which trade occurs. If $\mathbb{B} \neq \emptyset$, no such equilibrium can exist, even if $\mathbb{B}$ is of measure zero.

1 Background

The assumption that agents possess private information is a feature of many economic models. Incentive compatibility is a basic requirement of any solution concept in such models. (See for example [1].) Often, for reasons of tractability, these models represent private information as variables that can take on a continuum of values. There are then an uncountably infinite number of incentive compatibility constraints. One naturally wonders whether to insist on all of these being satisfied, or to allow for violations on a set of measure zero. Here, we present an economic example in which the equilibria differ markedly depending on which of these one chooses.

The phenomenon we illustrate can be ruled out by imposing suitable continuity of agents' actions with respect to their information. We do not see a strong economic rationale for such an assumption. We also contend that a suggestion to ignore anomalous behavior at a certain set of states just

*We thank Roger Myerson, George Mailath and Drew Fudenberg for helpful remarks.

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because they are of zero measure goes against the spirit of all known equilibrium concepts for extensive form games. A more satisfactory resolution of this issue is important for interpreting equilibria in models with a continuum of types. Unfortunately, such an interpretation eludes us.

2 The Example

The trading environment is close to the one studied in [3], the significant difference being the fact that there are a continuum of types. There are two players. Player B(uyer) has one unit of money and Player S(eller) has a unit endowment of an asset X whose future value is the realization of a random variable $\tilde{V}$. If Player S supplies q units of X to Player B at a unit price of ρ, their respective future wealths are the random variables

$$\tilde{W}_S(\rho, q) = (1 - q\rho) + q\tilde{V} \qquad \tilde{W}_B(\rho, q) = q\rho + (1 - q)\tilde{V} \tag{1}$$

Both players only care about their future wealth. Player B is risk neutral and Player S is risk averse with $u(x) = -e^{-x}$ being her von Neumann-Morgenstern utility function over wealth x. For some of the results it suffices that $u(\cdot)$ is strictly concave.

We study the following signaling game. Player B observes a signal $s \in [a, b]$ that informs her that $\tilde{V}$ is distributed according to a normal distribution with mean s and variance $\sigma^2(s)$. She then acts as a Stackelberg leader and chooses a price at which she is willing to trade with Player S. Player S observes the price (but not the signal) and chooses how much of her endowment to sell. Thus, a strategy for Player B is a pricing function $p : [a, b] \longrightarrow \mathbb{R}_+$ that specifies the price $p(s)$ for each possible signal $s \in [a, b]$. A strategy for Player S is a mapping $Q : \mathbb{R} \longrightarrow [0, 1]$ which specifies the quantity $Q(p)$ that she is willing to supply for any $p \in \mathbb{R}$.

Assume that the signals are continuously distributed on $[a, b]$ according to a probability distribution function $F(\cdot)$.

Given Q, the payoff of a type s Player B from setting a price ρ is the expectation of $\tilde{W}_B$, i.e.

$$V(\rho, Q|s) = 1 + (s - \rho)Q(\rho). \tag{2}$$

Given a pricing strategy p, for each $\rho \in \mathbb{R}_+$, let $G_\rho(\cdot|p)$ be a probability distribution on $[a, b]$ that reflects the posterior beliefs of Player S regarding the distribution of signals. If Player B chooses the strategy p the payoff of Player S from choosing to supply q units of X having observed a price ρ is

$$U(q, p|\rho) = \int_a^b E[u(\tilde{W}_S(\rho, q))|\tilde{S} = s] dG_\rho(s|p) \tag{3}$$

Definition 1 (Perfect Bayesian Equilibrium)**.** A strategy profile $\{p, Q\}$ and a system of beliefs $\{G_\rho(\cdot|p)\}_{\rho\geq 0}$ is said to constitute a Perfect Bayesian Equilibrium if

1. For all ρ in the range of p, $G_\rho(\cdot|p)$ obeys Bayes rule.
2. $p(s) \in \text{argmax}_\rho (s-\rho)Q(\rho)$.
3. $Q(\rho) \in \text{argmax}_{q\in[0,1]} U(q, p|\rho)$ for all ρ.

The first condition the usual restriction that the beliefs of Player S are consistent with Bayes rule along the equilibrium path. The remaining conditions require that traders' choices are optimal given their beliefs.

Assume $E(u(\tilde{V})|s) > 0$ for all s so that Player S has no incentive to trade unless the price is positive. Player B, regardless of her type, can ensure a non-negative payoff and consequently $p(s) \leq s$ in any equilibrium.

Assume that $\sigma_s^2 \in \left\{0, \sigma^2\right\}$ and let

$$\mathbb{B} = \left\{s \in [a, b] \mid \sigma_s^2 = 0\right\} \tag{4}$$

Before proceeding further, it is useful to introduce $x(\cdot, s)$ to denote the supply function of Player S when she knows s and $\sigma_s^2 = \sigma^2$. It can be seen[1] that

$$x(\rho, s) = \begin{cases} 0 & \text{if } \rho \leq s - \sigma^2 \\ 1 - (s-\rho)/\sigma^2 & \text{if } s - \sigma^2 \leq \rho \leq s \\ 1 & \text{if } s \leq \rho. \end{cases} \tag{5}$$

If $\sigma_s^2 = 0$ and Player B knows s, the asset offers a sure return of s. As a result, the two assets are perfect substitutes if $\rho = s$. Otherwise, Player B strictly prefers the asset (bond) if $\rho < s$ ($\rho > s$). Accordingly, her supply correspondence is

$$\hat{x}(\rho, s) = \begin{cases} 1 & \text{if } s < \rho \\ [0, 1] & \text{if } \rho = s \\ 0 & \text{if } \rho < s \end{cases}$$

Our findings are summarized as two propositions below. Their proofs of the above results are in Section 4.

Proposition 1. *Suppose $a - \sigma^2 > 0$ and $\mathbb{B} = \emptyset$. There exists a fully separating equilibrium in which a positive quantity is traded in every state.*

Proposition 2. *Suppose $\mathbb{B} \neq \emptyset$. There is no trade in any separating equilibrium*[2].

[1] For instance, the algebra that leads to $x(\cdot, s)$ can be found in Chapter 4, [2].

[2] There is an open set of parameters for which a pooling equilibrium can be shown to exist however provided $\mathbb{B}$ is of measure zero.

3 Discussion

Several remarks with regard to Proposition 1 and Proposition 2 are in order.

Remark 1. Note that there are no assumptions on the cardinality of $\mathbb{B}$ (beyond it being non-empty). Therefore, one might change σ_b^2 to zero from $\sigma^2 > 0$ and destroy a revealing equilibrium in which trade occurs. This is a change on a measure zero set.

Remark 2. It is not our claim that *all* separating equilibria are eliminated with the presence of a state s such that $\sigma_s^2 \neq \emptyset$. Indeed, suppose Player B chooses the strategy p^* where $p^*(s) = s - \sigma^2$. Since p^* is strictly increasing, it will fully reveal the state. From $x(\cdot, s)$ and $\hat{x}(\cdot, s)$ above, it is immediate that a best response for $\rho \in \left[a - \sigma^2, b - \sigma^2\right]$ is to choose $Q^*(\rho) = 0$. Assume that Player S believes the state to be a or b when $\rho < a - \sigma^2$ or $\rho > b - \sigma^2$ respectively. This constitutes a fully separating equilibrium[3]. However the fact that information is revealed is moot as there is never any trade in this equilibrium.

Remark 3. A rough intuition for the phenomenon is as follows. Trade occurs when a state is common-knowledge only because Player B is risk-neutral and Player S is risk averse. However, when $s \in \mathbb{B}$ becomes common-knowledge there are no gains from trade, except at the no arbitrage price of $p(s) = s$. Now consider a s' close to s and such that $s' \notin \mathbb{B}$. Incentive compatibility forces $p(s')$ to be close to s'. The risk averse Player S must then seek full insurance and sell almost all her endowment to the Player B. It turns out that this feature implies that all types to the right of s will sell all their endowment. As the amount sold does not vary, the price paid does not either.

Remark 4. Suppose $\sigma_{s^*}^2 = 0$ but $\sigma_s^2 \neq 0$ for s in a small neighborhood of s^*. Then note that $x(s^*, s) = 1 - (s - s^* - \epsilon)/\sigma^2$ in that neighborhood for sufficiently $\epsilon > 0$. However, $\lim_{s \to s^*} x(s^* - \epsilon, s) \neq \hat{x}(s^* - \epsilon, s^*)$. The phenomenon described by Propositions 1 and 2 can be overcome if such discontinuity of the receiver's optimal action can be ruled out by assumption. (See [4]) However, we do not believe that such technical continuity restrictions can be an implication of behavioral hypotheses.

Remark 5. There is a discussion in Section 4.8 of [5] on the sensitivity of sequential equilibria to the addition of chance moves of small probability. The example presented here seems to be of a different flavor although similar in spirit. Myerson's example consists of two states, say 1.1 and 2.2 to use

[3] That deviating to a price $\rho \in [a - \sigma^2, b - \sigma^2]$ does not benefit is clear. To check that deviation to a price outside this range also does not pay can be verified along the lines in the proof of Proposition 1.

his notation. Whether one considers 2.2 to be a zero-probability event or an impossible event alters the behavior at 1.1, which of course occurs with probability one. It is harder to make such a distinction here. Here, *every* realization of the signal in $[a, b]$ has zero probability. Therefore no single state is infinitely more likely than another.

4 Proofs

Given an equilibrium strategy profile $\{p, Q\}$, let $q(s) = Q(p(s))$ denote the equilibrium quantity that is traded in the event $\tilde{S} = s$. Also let $1 + V(s', s)$ denote the payoff of a Player B of type s if she mimics the action of type s' and $v(s) = V(s, s)$. That is,

$$V(s', s) = q(s')[s - p(s)] \text{ and } v(s) = q(s)[s - p(s)]$$

The incentive compatibility conditions for Player B can now be written as

$$v(s) \geq V(s', s) \qquad \forall s, s' \in [a, b]. \tag{6}$$

Proof of Proposition 1. The proof is by construction. Consider the function $g : (0, 0.5] \longrightarrow \mathbb{R}$ where

$$g(q) \quad = \quad b + (\log(2q) + 1 - 2q)\sigma^2$$

Note that $g'(q) = \sigma^2(1-2q)/q > 0$ and hence g^{-1} is well defined. Moreover, $g(0.5) = b$ and $\lim_{q \to 0} g(q) = -\infty$. Therefore, for any $a < b$, there exists $q_a > 0$ such that $g(a) = q_a$.

In the equilibrium we construct, a quantity $q(s) = g^{-1}(s)$ will be traded in state s. Since Player S knows the state in equilibrium, the price $p(s)$ that must be set to sustain a supply of $q(s)$ must satisfy $q(s) = x(p(s), s)$ which, by (5), gives

$$p(s) \quad = \quad s - \sigma^2(1 - g^{-1}(s)).$$

Our assumption that $a - \sigma^2 > 0$ ensures that $p(s) > 0$ for all s.

To construct the equilibrium then, take the beliefs $\{G_\rho\}_{\rho \geq 0}$ as follows: For any $\rho \in [p(a), p(b)]$ the state is fully revealed and G_ρ is the belief that $s = p^{-1}(\rho)$ is the true state. For $\rho \geq p(b)$, assume that G_ρ is the belief that the true state is b and for $\rho \leq p(a)$ that a is the true state. Given these beliefs, the equilibrium strategy of Player S is

$$Q(\rho) \quad = \quad \begin{cases} x(\rho, a) & \text{if } \rho \leq p(a), \\ q(s) & \text{if } \rho \in [p(a), p(b)] \text{ where } s = p^{-1}(\rho), \\ x(\rho, b) & \text{if } p(b) \leq \rho. \end{cases}$$

We now claim that the strategy profile $\{p(\cdot), Q(\cdot)\}$ together with the beliefs $\{G_\rho\}_{\rho\geq 0}$ constitute a Perfect Bayesian Equilibrium. It is evident from construction that $Q(\cdot)$ is optimal given the beliefs and that $\{G_\rho\}_{\rho\geq 0}$ is consistent with Bayes rule whenever applicable. The proof is complete upon checking that $p(\cdot)$ is optimal for Player B.

By choosing $p(s)$, Player B of type s receives in equilibrium the payoff

$$1 + (s - p(s))q(s)$$

Choosing $\rho \leq p(a)$ results in a payoff of

$$1 + (s - \rho)x(\rho, a)$$

The above achieves a global maximum at $\rho^* = (s + a)/2 - \sigma^2/2$ which lies to the right[4] of $p(a)$. Therefore, the above is bounded above by $1 + (s - p(a))x(p(a), a) \equiv 1 + (s - p(a))q(a)$, which is precisely the payoff of Player B of type s should she pretend to be of type a and set the price $p(a)$.

Choosing $\rho \geq p(b)$ results in a payoff of

$$1 + (s - \rho)x(\rho, b)$$

The above achieves a global maximum at $\rho^{**} = (s + b)/2 - \sigma^2/2$, which lies to the left of $p(b) = b - \sigma^2/2$. Therefore, for $\rho \geq p(b)$, the above is bounded above by $1 + (s - p(b))x(p(b), b) \equiv 1 + (s - p(b))q(b)$, which is precisely the payoff of type s if she pretends to be type b and sets a price of $p(b)$.

The proof is complete if no type in $[a, b]$ gains by mimicking the action of a another type. Given a $q, \tilde{q} \in [q_a, 0.5]$, let $s = g^{-1}(x)$ and $t = g^{-1}(x)$. The payoff of type s if she mimics the action of type t can be written as

$$\begin{aligned} U(t, s) &= 1 + (s - p(t))q(t) \\ &= 1 + (g(q) - g(x) + \sigma^2(1 - x))x \\ &\equiv f(x, q) \end{aligned}$$

The proof is complete if $f(\cdot, q)$ can be shown to have a global maximum at q, for each $q \in [q_a, 0.5]$. This follows immediately since

$$\begin{aligned} \frac{\partial f(x, q)}{\partial x} &= (g(q) - g(x)) - xg'(x) + \sigma^2(1 - 2x) \\ &= g(q) - g(x) \\ \frac{\partial^2 f(x, q)}{\partial x^2} &= -g'(x) \ < \ 0 \end{aligned}$$

□

[4]Simple algebra immediately yields $\rho^* - p(a) = \frac{(s-a)}{2} + \sigma^2(0.5 - q_a) > 0$.

Lemmas 1-3 below are used in the proof of Proposition 2. The first of these lists rather standard implications. We include its proof here primarily for completeness.

Lemma 1. *In any equilibrium, $v(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are weakly increasing.*

Proof. Let $s > s'$. Using (6),

$$\begin{aligned} v(s) - v(s') &\geq V(s', s) - v(s') \\ &= q(s')[s - p(s')] - q(s')[s' - p(s')] \\ &= q(s')[s - s'] \end{aligned}$$

from which it follows that $v(s) \geq v(s')$.

(6) also implies

$$v(s) + v(s') \geq V(s', s) + V(s, s').$$

Upon simplification, the above yields $(s - s')[q(s) - q(s')] \geq 0$, which in turn implies that $q(\cdot)$ is non-decreasing. Having argued that $v(\cdot)$ and $q(\cdot)$ are both non-decreasing, the asserted monotonicity of $p(\cdot)$ follows readily. □

It is worth noting in the above proof that when $s > s'$ and $q(s') > 0$ implies $p(s) > p(s')$ and $v(s) > v(s')$. This feature of the equilibrium is used elsewhere in the proof of Proposition 2.

Lemma 2. *Let $s^* \in \mathbb{B}$ and suppose s^* is fully revealed in some equilibrium. Then $q(s) = 0$ for all $s \in [a, s^*]$.*

Proof. Suppose $q(s) = 0$. Then by Lemma 1, $q(t) = 0$ for all $t \leq s$.

Now suppose that $q(s) > 0$ and $q(t) > 0$ for some $t < s$. Assume, by way of contradiction, that s is fully revealed. Therefore, upon observing $p(s)$, it becomes common-knowledge that s is the true signal and, since $\sigma^2(s) = 0$, that the future value of X is s with certainty. As there is no further risk, Player S would not trade unless $p(s) = s$. Therefore $v(s) = 0$. On the other hand, $V(t, s) = (s - p(t))q(t) \geq (s - t)q(t) > 0$ in violation of (6). □

Lemma 3. *Consider a fully separating equilibrium in which $q(s) > 0$ for some s. Then $q(s) > 0$ for all $s \in [a, b]$.*

Proof. Assume to the contrary. Then $s^* = \sup \{s \in [a, b] \mid q(s) > 0\}$ is well defined and $s^* \in (a, b)$. Let p^+, q^+ denote the right hand limit of $q(\cdot)$ and $p(\cdot)$ respectively at s^*. If $p^+ < s^*$, then some type between p^+ and s^*, say t, can pretend to be a type s sufficiently close to the right of s^*, say s such that $(t - p(s))q(s) > 0$. Therefore, $p^+ = s^*$.

From the previous Lemma, $\sigma_s^2 > 0$ for all $s > s^*$. Therefore, everywhere to the right of s^*, one must have

$$q(s) = 1 - (1 - p(s))/\sigma^2 \tag{7}$$

Taking liminf on both sides, we note that $q^+ = 1$. By the monotonicity of $q(\cdot)$, we must have $q(s) = 1$ for all s.

□

Proof of Proposition 2. Lemmas 2 and 3 readily imply that any fully separating equilibrium must necessarily involves no trade at all states. □

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