Amending the US Constitution via Article V : The Effect of Voting Rule Inflation

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June 24, 2008

Abstract

It is generally believed that amendment to the United States Constitution has proven to be unduly onerous. Consistent with that belief, we show that, even if the original Article V amendment requirement were deemed optimal, the *effective* supermajority required by Article V has substantially increased over time, as the size of the relevant voting bodies has increased. We demonstrate this and other comparative statics in a general model of supermajority voting rules. Calibrating the model based on the optimality of the original requirements, we show that the requirement in Article V that an amendment be supported by 2/3 of each House of Congress and 3/4 of State Legislatures would now be equivalent to a requirement of support by 53% of the House, 59% of the Senate, and 62% of State Legislatures. Without this "voting rule inflation" effect, we find that several proposed Amendments to the Constitution would likely have passed. Voting rule inflation is thus shown to be an important consideration for any constitutional designer.

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"**Article V**: *The Congress, whenever two thirds of both Houses shall deem it necessary, shall propose Amendments to this constitution, or, on the Application of the Legislatures of two thirds of the several States, shall call a Convention for proposing Amendments, which, in either Case, shall be valid to all Intents and Purposes, as Part of this Constitution, when ratified by the Legislatures of three fourths of the several States, or by Conventions in three fourths thereof, as the one or other Mode of Ratification may be proposed by Congress; Provided that no Amendment which may be made prior to the Year One thousand eight hundred and eight shall in any Manner affect the first and fourth clauses in the Ninth Section of the first Article; and that no State, without its Consent, shall be deprived of its equal Suffrage in the Senate.*"

"...for it is, in my conception, one of those rare instances in which a political truth can be brought to the test of a mathematical demonstration." Alexander Hamilton, Federalist No. 85

1 Introduction

Legal rules and institutions are fundamental determinants of economic outcomes. Moreover, the fact that *constitutions* can have important effects on economic outcomes is now well established (Persson and Tabellini (2003)). One important part of a constitution is its amendment provisions, which govern the way in which the constitution may adapt to changing and unforeseen circumstances. Amendment provisions need to strike a tricky balance. On the one hand, a permissive amendment rule may undermine the very purposes of entrenched constitutional protections; one the other, a stringent one may hinder adaptation to changing circumstances.

In adopting Article V of the United States Constitution, the Framers knew the importance of the amendment provision. It was widely understood amongst the Framers that the requirement of unanimous state consent for an amendment to the Articles of Confederation had been a serious obstacle to the capacity of the Confederation to respond to changing circumstances[1]. For instance, several proposals between 1781 and 1783 to give Congress a power of taxation had been defeated by the veto of a single state[2]. Conversely, however, the Framers were also concerned to guard against what Madison described in Federalist 43 as "the extreme facility, which would render the Constitution too mutable". The initial proposal to require ratification by only two-thirds of the states was thus abandoned in favor of an amendment proposed by Madison to Article V, requiring approval by three-quarters of the states (Lash (1994)).

The final requirements of Article V of the Constitution, in which either the states (acting through their legislatures or conventions) or 2/3 of both Houses of Congress may propose amendments, and amendments must be ratified by 3/4 of the states, were thus seen by the Framers to strike the optimal balance between concerns for flexibility and stability.

In contemporary constitutional scholarship, however, there is a widely held perception that Article V has proved far less flexible than the Framers intended or envisaged. For instance, no proposal for amendment has ever been initiated by the states (Stokes-Paulsen (1993), Levinson (1995)). The effect, in practical terms, is that Constitutional amendment will always require approval of 2/3 of both Houses and 3/4 of the states (Levinson (1995)). And it is generally suggested that this process has proven excessively difficult (Griffin (1995b), Lutz (1995)).

The result has been that many constitutional scholars have focused attention on more in-

[1] See for example., Charles Pickney, Observations on the Plan of Government Submitted to the Federal Convention, reprinted in The Records of the Federal Convention 1787 (Max Farrand ed., 1937); Alexander Hamilton, The Records of the Federal Convention 1787 (Max Farrand ed., 1937); The Federalist No. 85 (Hamilton); James Madison, The Virginia Ratification Debates, reprinted in The Debates of the Several State Conventions on the Adoption of the Federal Constitution (Jonathan Elliott, ed., 1836).

[2] Rhode Island and New York respectively: see Lash (1994).

formal mechanisms through which Constitutional change may be achieved (Ackerman (1991), Ackerman (1996), Amar (1994), Griffin (1995a), Levinson (1995), Strauss (2001)). While continuing to employ Article V processes in a highly visible way (Strauss (2001)), political actors have increasingly turned to alternative mechanisms, such as control of Supreme Court appointments, as important for effecting Constitutional change.

This paper seeks to formalize the idea that underlies this shift away from reliance on Article V as a mechanism for constitutional change. It shows that the widely held perception that Article V amendment has become more difficult has a sound logical basis. We develop a model of supermajority voting rules in which the optimal rule depends on the distribution of preferences, number of decision makers, and importance of the issues.

We identify two possible sources of "voting rule inflation". First, changes in the underlying distribution of voter preferences may make amendment more difficult. Second, changes in the size of the voting pool may affect the difficulty of amendment.

Changes in the degree of hetererogeneity of voter preferences are notoriously difficult to measure, however. In order to highlight the effect of voting rule inflation, we therefore hold the distribution of voter preferences constant, and analyze its second potential source.

We develop a general model of supermajority voting rules and show that increasing the number of voters on a Constitutional amendment, both in terms of representatives voting in Congress, and in the number of states voting, materially increases the difficulty of Article V amendment. This conforms with the conclusion of Lutz (1995), but gives a theoretical explanation for this fact, and a way to quantify the magnitude of voting rule inflation.

Calibrating the model by using parameter values which would make the 1789 Article V (2/3 of the Senate, 2/3 of the House, 3/4 of the States) rule optimal, we show that for the 2006 voting pool, the optimal Article V voting rule would in fact be 59%, 53% and 62%, respectively.

There is a simple intuition for why the voting rule inflation we identify occurs. In the context of our model, the optimal supermajority rule is determined by a trade-off between the blocking power of a small minority of voters, and the possibility of a majority taking an action which adversely impacts a minority[3]. On the one hand, a low supermajority rule is desirable because it reduces the probability of a small minority blocking a change to the social decision which could benefit a large number of others ("blocking"). On the other hand, a high rule is also desirable because it reduces the chance that an individual will be affected by the majority making a change to the social decision which hurts them a great deal, but benefits the majority, even if only by a very small amount ("expropriation"). The balance between these two effects determines the optimum.

Now, as the number of voters in a particular body increases, the balance between these two factors shifts. The probability of being in a potentially expropriated minority decreases, and the probability of being subject to blocking increases. In order to optimally balance these two factors the supermajority rule must adjust downward. Therefore any time there is a natural tendency for the number of voters in a particular body to rise over time, any static supermajority rule will not be optimal at all points in time. It can be optimal at the outset, or at some particular point in the future, but not both unless the rule itself, *in percentage terms*, changes with the number of voters.

Although it is hard to know exactly what effect lower amendment requirements would have had, because of strategic voting, if one examines actual voting records, then three amendments since 1973 would have progressed further than they actually did. The Equal Rights Amendment would have passed the House, a Balanced Budget Amendment would have passed the House and Senate, and a Flag Burning Amendment would have passed the House.

The remainder of this paper is organized as follows. Section 2 discusses the relevant literature on voting rules. Section 3 contains our model and theoretical results. Section 4 calibrates this model in the context of Article V. Section 5 consider the implications of this for proposed amendments. Section 6 concludes.

[3] We rule out the possibility of monetary transfers/side payments.

2 Related Literature

This model in this paper is related to important literatures in economics and political science. Interest in rules chosen behind the veil of ignorance can be traced to Rousseau. Early works by economists using this notion include Vickrey (1945), Harsanyi (1953) and Harsanyi (1955). Mirrlees (1971) and, of course, Rawls (1971) analyze profound questions within this framework. The formal analysis of the construction of constitutions began with Buchanan and Tullock (1962). The literature on majority voting is known to have distant origins, dating at least to Condorcet. Arrow (1951) ignited a vast literature attempting to overcome his impossibility theorem. Particularly pertinent to this paper, Arrow himself conjectured (Arrow (1951)) that a sufficient degree of social consensus could overcome his impossibility theorem[4]. This conjecture was formalized by Caplin and Nalebuff (1988) and with greater generality by Caplin and Nalebuff (1991). In fact, formal interest in voting under supermajority rules can be traced to Black (1948). Despite large literatures on related issues there is, to our knowledge, no canonical exposition of the optimal supermajority rule.

Focusing on the role of contractual incompleteness, Aghion and Bolton (1992) show that some form of majority voting dominates a unanimity requirement in a world of incomplete social contracts. They highlight the fact that if a contract could be complete then the issue of supermajority requirements is moot if rules are chosen behind the veil of ignorance. Aghion, Alesina and Trebbi (2004) utilize a related framework, also in the spirit of public good provision analyzed by Romer and Rosenthal (1983). In a similar framework, Erlenmaier and Gersbach (2001) consider "flexible" majority rules whereby the size of required supermajority depends on the proposal made by the agenda setter. Babera and Jackson (2004) consider "self-stable" majority rules, in the sense that the required supermajority does not wish to change the supermajority rule itself *ex post.* A related paper is Maggi and Morelli (2003), which finds that unanimity, in certain settings, is usually optimal if there is imperfect

[4] "The solution of the social welfare problem may lie in some generalization of the unanimity condition..." (quoted in Caplin and Nalebuff (1988))

enforcement

We undertake a more general formulation where the policy set is a continuum. This allows us to study the effect of risk and risk-aversion on the voting rule. As discussed in section 3, we consider a particularly strong form of incompleteness of the social contract. The social contract is not permitted to specify a state-contingent supermajority rule, nor are monetary transfers / side payments allowed. In the context of the model this means that the supermajority requirement cannot differ based on realized draws from the distribution of types.

3 The Model

3.1 Statement of the Problem

Let there be n voters, with n finite. The policy space is assumed to be the unit interval $[0,1]$. Voters preferences over this policy space are drawn from the distribution function $F(x)$.

Definition 1. *A Social Decision is a scalar, $\theta \in [0,1]$.*

Assumption A1. *Each voter i has a utility function of the form*

$$U_i = -u\left(|\theta - x_i|\right),$$

where $u(\cdot)$ is an increasing, convex function, amd x_i is voter i"s preferred policy.

We are thus assuming that voters are *ex ante* identical, but not (generically) *ex post.*

Definition 2. *A Supermajority Rule is a scalar $\alpha \in [\frac{1}{2}, 1]$ which determines the proportion of voters required to modify the social decision.*

There are two time periods in the model. In period 1 voters know the distribution of preferences, $F(x)$, but they do not know their draw from the distribution. In this period they determine, behind the veil of ignorance, a social choice and a supermajority rule. In period 2, after preferences are realized, the social decision can be changed if a coalition of at least αn voters prefer a new social decision. The analysis in period 1 is particularly simple, by virtue of the assumption that voters are *ex ante* identical.

As mentioned before, we restrict the (social) contracting space. State contingent supermajority rules are not permitted. An example of such a rule would be any kind of utilitarian calculus which would vary the supermajority requirement to change the status quo according to the aggregate utility to be gained *ex post*. We also rule out monetary transfers / side payments. Let $\hat{\theta}$ be the *ex ante* optimal social decision.

Definition 3. *The ex post optimal social decision is:*

$$\theta^* = \arg\max_{\theta} \sum_{i=1}^{n} u\left(|\theta - x_i^*|\right).$$

With a finite number of voters the *ex post* optimal decision may well differ from the *ex ante* optimal decision because of the realized draws from $F(x)$. It is this wedge between *ex ante* and *ex post* optimality which creates complexity in the choice of the optimal supermajority rule.

We make the following technical assumption which enables us to avail ourselves of several useful results from the theory of order-statistics.

Assumption A2. *The parent distribution of voter types $F(x)$ is absolutely continuous.*

By using order-statistics we are able to fully characterize the aggregate expected utility of a given voter for an arbitrary distribution of the population, number of voters, degree of risk-aversion and supermajority rule. We are, therefore, able to determine which rule yields the highest expected utility, and is hence optimal.

There is an obvious issue of how the *ex post* social decision is determined if a coalition has a sufficient number of members relative to the required supermajority who would be made better-off by a change to the *ex ante* social decision. In principle, any *ex post* social decision within the interval spanned by their preferences improves each of their payoffs. For simplicity we make the following assumption about how the bargaining power amongst members of such a coalition.

Assumption A3. *If a coalition has the required supermajority ex post then the social decision is that preferred by the "final" member of the coalition. That is, the member of the coalition whose preference is closest to the ex ante social decision.*

3.2 Analytic Results

Theorem 1. *Assume A1-A3. Then the optimal supermajority rule is decreasing in the number of voters, n.*

Proof. See appendix. ■

As the number of voters increases, the probability of being part of an expropriated minority decreases. The benefit gained from avoiding blocking, however, is unchanged and the probability of this increases. The risk-averse agents therefore require less insurance and hence the optimal supermajority rule decreases.

Theorem 2. *Assume A1-A3. Then the optimal supermajority rule is increasing in the coefficient of importance, $\beta \equiv -u''(\cdot)/u'(\cdot)$.*

Proof. See appendix. ■

As the coefficient of importance/risk-aversion increases voters are progressively more concerned with being expropriated. They essentially purchase insurance against this by requiring that the size of the majority required to expropriate them be large, thereby reducing the probability of that event occurring. In fact, when the coefficient of importance is sufficiently high a unanimity requirement is always optimal. If there is the prospect of a sufficiently bad payoff[5] then voters require a veto in order to insure themselves against this outcome.

[5] And as $\beta \to \infty$ expected utilty$\to -\infty$.

Before stating our final result, the following definition is useful.

Definition 4. *A distribution $\widehat{F}(\cdot)$ is Rothschild-Stiglitz Riskier than another distribution $F(\cdot)$ if either (i) $F(\cdot)$ Second Order Stochastically Dominates $\widehat{F}(\cdot)$, (ii) $\widehat{F}(\cdot)$ is a Mean Preserving Spread of $F(\cdot)$, or (iii) $\widehat{F}(\cdot)$ is an Elementary Increase in Risk from $F(\cdot)$.*

As is well known, Rothschild and Stiglitz (1970) showed that these three statements are equivalent.

Theorem 3. *Assume A1-A3. Then the optimal supermajority rule is larger for a distribution of voter types, $\widehat{F}(x)$ than for the distribution $F(x)$ if $\widehat{F}(x)$ is Rothschild-Stiglitz Riskier than $F(x)$.*

Proof. Trivial, since a Rothschild-Stiglitz increase in risk has the same effect as an increase in the coefficient of importance. ■

This result obtains for reasons closely related to those of the two previous theorems. As the spread of voter types increases more insurance is desired, which is effected by requiring the supermajority rule to be higher. This is, however, only the case if the voters' utility is more than proportionally decreasing as the social decision moves away from their ideal point (i.e. $\beta > 0$).

We now provide two examples which serve two purposes: (i) they illustrate the analytic results in a less abstract setting, and (ii) provide the basis for calibrating the model as we do in section 4.

3.2.1 Example 1

Voters' types are drawn from the uniform distribution on $[0, 1]$, $u_i = -\exp\{\beta |\theta - x_i|\}$, *and* $n = 5$.

First note that the *ex ante* optimal social decision is simply $\theta^* = \frac{1}{2}$. First we focus on the outcome under majority rule, which is simply that the *ex post* social decision is the median of the voters' draws. Consider voter i and let the other voters' draws be:

$$x_1^* \leq x_2^* \leq x_3^* \leq x_4^*$$

where x_k^* is the kth order-statistic. Now note that the density of (x_2^*, x_3^*) on $[0,1] \times [0,1]$ is[6]:

$$f(a_2, a_3) = 24a_2(1 - a_3)$$

Note that in considering the median we need only be concerned with voter i's position relative to x_2^* and x_3^*. If they are between x_2^* and x_3^* then they are the median. If $x_i^* \leq x_2^*$ then the expected loss is $\int_0^{a_2} -\exp\{\beta|t - a_2|\}\,dt$ and if $x_i^* \geq x_3^*$ it is $\int_{a_3}^1 -\exp\{\beta|t - a_3|\}\,dt$. If $x_2^* \geq x_i^* \geq x_3^*$ then the expected loss is $-\exp(0) = -1$. The expected utility of voter i is therefore:

$$\begin{aligned} E\left[u_i^M\right] &= \int_0^1 \int_0^{a_3} \begin{pmatrix} \int_0^{a_2} -\exp\{\beta(a_2 - t)\}\,dt \\ +\int_{a_2}^{a_3}(-1)dt \\ +\int_{a_3}^1 -\exp\{\beta(t - a_3)\}\,dt \end{pmatrix} 24a_2(1 - a_3)da_2da_3 \\ &= -\frac{\beta(\beta^4 - 10\beta^3 + 120\beta + 480) + 240e^\beta(\beta - 3) + 720}{5\beta^5}. \end{aligned}$$

Now consider the expected utility of voter i if we require unanimity in order to change the social decision *ex post*. Denote the *ex post* social decision as t. Let B be the event where $0 \leq x_1^* \leq x_4^* < \frac{1}{2}$ and let B' be the event where $\frac{1}{2} \geq x_1^* \geq x_4^* \geq 1$. Let $A = \Omega \backslash (B + B')$. It is clear that $\Pr(A) = \frac{7}{8}$ and that $\Pr(B) = \Pr(B') = \frac{1}{16}$. The expected utility of voter i

[6] For an absolutely continuous population the joint density of two order statistics $i < j$, from n statistics, is given by:

$$\frac{n!}{(i-1)!(j-i-1)!(n-j)!} F(x_i)^{i-1} \left(F(x_j) - F(x_i)\right)^{j-i-1} \left[(1 - F(x_j))^{n-j} f(x_i) f(x_j)\right]$$

(See Balakrishnan and Rao (1998)). For the uniform distribution this implies:

$$f(x_i, x_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} u_i^{i-1} \left(u_j - u_i\right)^{j-i-1} \left(1 - u_j\right)^{n-j}$$

conditional on event A is:

$$\begin{aligned} E\left[u_i^U|A\right] &= 2\int_0^{\frac{1}{2}} -\exp\left\{\beta\left(\frac{1}{2}-t\right)\right\}dt \\ &= \frac{2\left(1-e^{\beta/2}\right)}{\beta}. \end{aligned}$$

The density[7] of x_4^* is $f(a_4) = 4(a_4)^3$. We now need the density of x_4^* on $[0, \frac{1}{2}]$, which is found by applying the Change of Variables Theorem, yielding $g(a_4) = 2 \times 4(2a_4)^3 = 64(a_4)^3$. Therefore:

$$\begin{aligned} E\left[u_i^U|B\right] &= \int_{\frac{1}{2}}^{1} -\exp\left\{\beta\left(t-\frac{1}{2}\right)\right\}dt \\ &\quad + \int_0^{\frac{1}{2}}\left(\int_0^{a_4} -\exp\left\{\beta\left(a_4-t\right)\right\}dt - \int_{a_4}^{\frac{1}{2}} 1dt\right)64\left(a_4\right)^3 da_4 \\ &= \frac{2\left(1-e^{\beta/2}\right)}{\beta} + \frac{1}{\beta} - \frac{8(48+e^{\beta/2}(\beta^3-6\beta^2+24\beta-48)}{\beta^5} - \frac{1}{10}. \end{aligned}$$

where$\int_{\frac{1}{2}}^{1} -\exp\left\{\beta\left(t-\frac{1}{2}\right)\right\}dt$ is the term associated with $x_i \geq \frac{1}{2}$ and the term associated with $x_i \leq \frac{1}{2}$ is $\int_0^{\frac{1}{2}}\left(\int_0^{a_4} -\exp\left\{\beta\left(a_4-t\right)\right\}dt - \int_{a_4}^{\frac{1}{2}} 1dt\right)64\left(a_4\right)^3 da_4$.

Under event B' the expected utility is given by:

$$\begin{aligned} E\left[u_i^U|B'\right] &= \int_0^{\frac{1}{2}} -\exp\left\{\beta\left(\frac{1}{2}-t\right)\right\}dt \\ &\quad + \int_{\frac{1}{2}}^{1}\left(\int_{a_1}^{1} -\exp\left\{\beta\left(t-a_1\right)\right\}dt - \int_{\frac{1}{2}}^{a_1} 1dt\right)64\left(1-a_1\right)^3 da_1 \\ &= \frac{2\left(1-e^{\beta/2}\right)}{\beta} + \frac{1}{\beta} - \frac{8(48+e^{\beta/2}(\beta^3-6\beta^2+24\beta-48)}{\beta^5} - \frac{1}{10}. \end{aligned}$$

[7] For the uniform distribution the density of the *ith* order statstic is:

$$f_i(u) = \frac{n!}{(i-1)!(n-i)!}u^{i-1}(1-u)^{n-i}$$

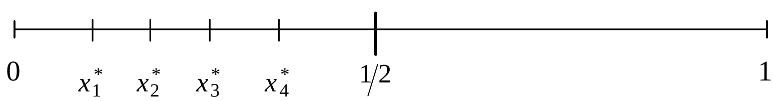
Therefore the total expected utility under unanimity is:

$$
\begin{aligned}
E\left[u_i^U\right] &= \frac{7}{8}\mathbb{E}\left[u_i^U|A\right] + \frac{1}{16}\mathbb{E}\left[u_i^U|B\right] + \frac{1}{16}\mathbb{E}\left[u_i^U|B'\right] \\
&= \frac{-3840 + \beta^4(\beta - 160) + 10e^{\beta/2}(-384 + \beta(192 + \beta(\beta(15\beta + 8)48)))}{80\beta^5}.
\end{aligned}
$$

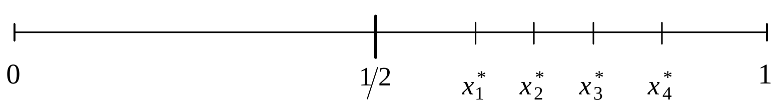
For majority rule to be preferable to unanimity therefore requires $E\left[u_i^M\right] > E\left[u_i^U\right]$. Solving numerically shows that this is the case if and only if $0 \leq \beta \lessapprox 3.9$. Therefore when the decision is relatively unimportant majority rule dominates, but with a sufficiently high enough degree of importance unanimity is preferred.

Now consider the case where the social decision can be altered *ex post* if four voters agree. In this example with five voters this reflects the only supermajority which is greater than simple majority but less than unanimity.

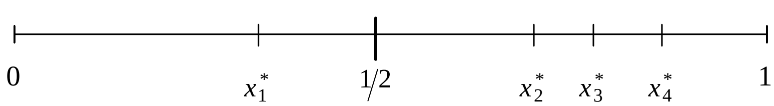
Now define events B, B', C and C' as follows. B is the event where $0 \leq x_1^* \leq x_2^* \leq x_3^* \leq x_4^* \leq \frac{1}{2}$. B' is the event where $\frac{1}{2} \leq x_1^* \leq x_2^* \leq x_3^* \leq x_4^* \leq 1$. C is the event where $0 \leq x_1^* \leq \frac{1}{2} \leq x_2^* \leq x_3^* \leq x_4^* \leq 1$. C' is the event where $0 \leq x_1^* \leq x_2^* \leq x_3^* \leq \frac{1}{2} \leq x_4^*$. Also, let $A = \Omega \backslash \left(B + B' + C + C'\right)$.



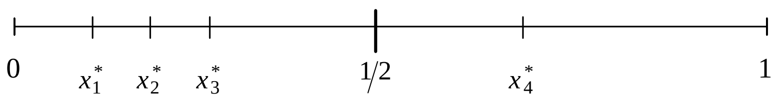
Event B



Event B'



Event C



Event C'

Note that $\Pr(B) = \Pr\left(x_4^* \leq \frac{1}{2}\right) = \frac{1}{16} = \Pr(B')$. $\Pr(C') = \Pr\left(x_3^* \leq \frac{1}{2} \wedge x_4^* \geq \frac{1}{2}\right) = \frac{1}{4} = \Pr(C')$. Also note that $\Pr(A) = \frac{3}{8}$.

As before, if event A occurs then there is no change to the *ex ante* social decision and hence the expected utility of voter i is:

$$
\begin{aligned}
E\left[u_i^S|A\right] &= 2\int_0^{\frac{1}{2}} -\exp\left\{\beta\left(\frac{1}{2}-t\right)\right\} dt \\
&= \frac{2(1-e^{\beta/2})}{\beta}.
\end{aligned}
$$

Note[8] that the density of x_2^* conditional on event C is simply the density of the first order-statistic of three on $[\frac{1}{2}, 1]$. In fact, order statistics from a continuous parent form a Markov Chain. It follows that the density of the first-order statistic of three on $U[0,1]$ is $3\left(1-a_2\right)^2$. By a change of variables, the density on $[\frac{1}{2}, 1]$ is therefore $24(1-a_2)^2$. Hence the expected utility conditional on event C is:

$$
\begin{aligned}
E\left[u_i^S|C\right] &= \int_{\frac{1}{2}}^1 \left(-1\int_{1/2}^{a_2} dt + \int_{a_2}^1 -\exp\left\{\beta\left(t-a_2\right)\right\} dt\right) 24(1-a_2)^2 da_2 \\
&\quad + \int_0^{1/2} -\exp\left\{\beta\left(\frac{1}{2}-t\right)\right\} dt \\
&= -\frac{1}{8} + \frac{1}{\beta} + \frac{1-e^{\beta/2}}{\beta} - \frac{6(-8+e^{\beta/2}(8-4+\beta^2)}{\beta^4}.
\end{aligned}
$$

[8] This fact is quite general. The conditional pdf of an order-statistic is given by:

$$
f_{X_r|X_{s=v}}(x) = \frac{(s-1)!}{(r-1)!(s-r-1)!}\frac{f(x)F(x)^{r-1}(F(v)-F(x))^{s-r-1}}{F(v)^{s-1}}
$$

The density of x_3^* conditional on event C' is the third of three uniformly distributed order-statistics on $[0, \frac{1}{2}]$, which is $g(a_2|C') = 24\,(a_3)^2$. Hence the expected utility conditional on event C' is:

$$\begin{aligned}
E\left[u_i^S|C'\right] &= \int_0^{\frac{1}{2}} \left(-1\int_{a_3}^{\frac{1}{2}} dt + \int_0^{a_3} -\exp\left\{\beta\,(a_3 - t)\right\} dt\right) 24\,(a_3)^2\, da_3 \\
&\quad + \int_{1/2}^{1} -\exp\left\{\beta\left(t - \frac{1}{2}\right)\right\} dt \\
&= -\frac{1}{8} + \frac{1}{\beta} + \frac{1 - e^{\beta/2}}{\beta} - \frac{6(-8 + e^{\beta/2}(8 - 4 + \beta^2)}{\beta^4}.
\end{aligned}$$

Now note that the joint density of (x_3^*, x_4^*) on $[0, 1]$ is $f(x_3, x_4) = 12\,(a_3)^2$ and so on $[0, \frac{1}{2}]$ it is $192\,(a_3)^2$. The expected utility conditional on event B is therefore:

$$\begin{aligned}
E\left[u_i^S|B\right] &= \int_0^{\frac{1}{2}}\int_0^{a_4} \begin{pmatrix} -\int_{a_3}^{a_4} 1dt \\ +\int_0^{a_3} -\exp\left\{\beta\,(a_3 - t)\right\} dt \\ +\int_{a_4}^{1} -\exp\left\{\beta\,(t - a_4)\right\} dt \end{pmatrix} 192\,(a_3)^2\, da_3 da_4 \\
&= \frac{-3840e^{\beta/2} - \beta^4(\beta - 20) + 1920(\beta + 6) + 80e^{\beta/2}(\beta^4 + 72\beta - 96)}{10\beta^5}.
\end{aligned}$$

The joint density of (x_1^*, x_2^*) on $[0, 1]$ is $f(x_1, x_2) = f(x_1, x_2) = 12\,(1 - a_2)^2$ and so on $[\frac{1}{2}, 1]$ it is $192\,(1 - a_2)^2$. The expected utility conditional on event B' is:

$$\begin{aligned}
E\left[u_i^S|B'\right] &= \int_{\frac{1}{2}}^{1}\int_1^{a_1} \begin{pmatrix} -\int_{a_1}^{a_2} 1dt \\ +\int_0^{a_1} -\exp\left\{\beta\,(a_1 - t)\right\} dt \\ +\int_{a_2}^{1} -\exp\left\{\beta\,(t - a_2)\right\} dt \end{pmatrix} \left(-192\,(1 - a_2)^2\right) da_2 da_1 \\
&= \frac{-3840e^{\beta/2} - \beta^4(\beta - 20) + 1920(\beta + 6) + 80e^{\beta/2}(\beta^4 + 72\beta - 96)}{10\beta^5}.
\end{aligned}$$

Therefore the total expected utility under a supermajority of four voters (ie. 80% supermajority) is:

$$E\left[u_i^S\right] = \frac{3}{8}E\left[u_i^S|A\right] + \frac{1}{16}E\left[u_i^S|B\right] + \frac{1}{16}E\left[u_i^S|B'\right] + \frac{1}{4}E\left[u_i^S|C\right] + \frac{1}{4}E\left[u_i^S|C'\right].$$

Which, upon simplification, is:

$$E\left[u_i^S\right] = \frac{\begin{pmatrix} -1920e^{\beta} + \beta^4(80-3\beta) + 1920(\beta+3) \\ -10e^{\beta/2}(284 + \beta(-192 + \beta(\beta+4)(5\beta-12))) \end{pmatrix}}{40\beta^5}$$

For an 80% supermajority to be preferable to majority rule therefore requires $E\left[u_i^S\right] > E\left[u_i^M\right]$. Solving numerically shows that this is the case if and only if $\beta \gtrapprox 2.69$. For unanimity to be superior to an 80% supermajority rule requires $E\left[u_i^U\right] > E\left[u_i^S\right]$. Solving numerically reveals that this the case for $\beta \gtrapprox 9.02$. That is, the 80% supermajority rule dominates unanimity until the degree of importance becomes sufficiently large. For sufficiently large degrees of importance unanimity dominates because the fear of expropriation dominates and a veto provides them with insurance against this possibility. Therefore, in this example, for $0 \gtrapprox \beta \gtrapprox 2.69$ majority rule is optimal, for $2.69 \gtrapprox \beta \gtrapprox 9.02$ an 80% supermajority requirement is optimal, and for $\beta \gtrapprox 9.02$ a unanimity requirement is optimal. This is reflected in the following figure.

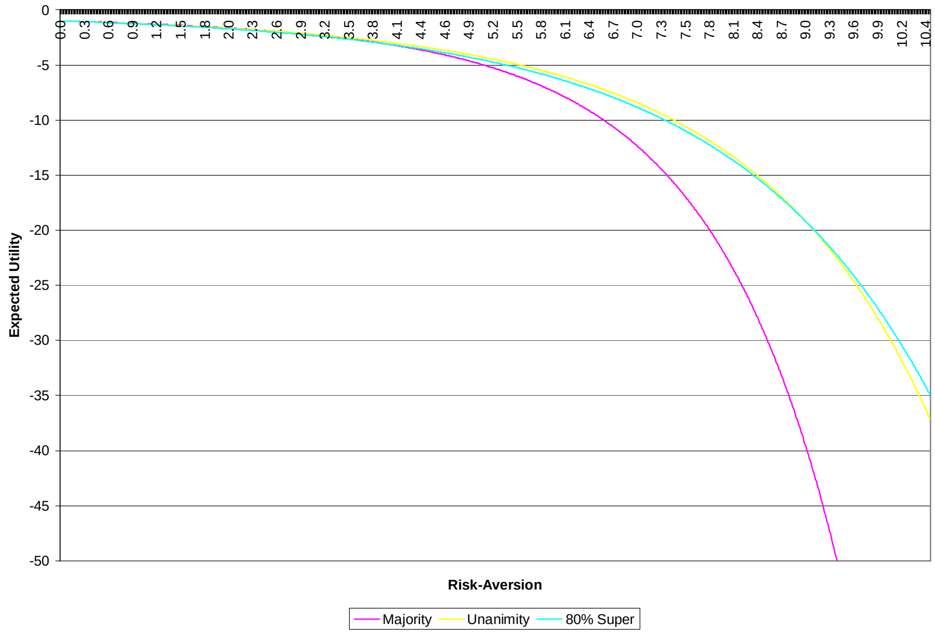


Figure 1: Example of Supermajority Rules and Risk-Aversion

3.2.2 Example 2

Voters' types are drawn from the uniform distribution on $[0,1]$, $u_i = -\exp\{\beta\,|\theta - x_i|\}$, *and* $n = 3$.

This example illustrates that as the number of voters increases the optimal supermajority rule decreases. We again use the uniform distribution, but with 3 voters rather than 5.

The expected utility under majority rule (here 2 out of three voters) is[9]:

$$E[U_M] = \int_0^1 \int_0^{a_2} \left[\begin{array}{c} \int_0^{a_1} -\exp\{\beta(a_1 - t)\}\,dt \\ + \int_{a_2}^1 -\exp\{\beta(t - a_2)\}\,dt \\ -1 \int_{a_1}^{a_2} dt \end{array} \right] 2da_1 da_2$$

$$= \frac{12 - 12e^{\beta}\beta(12 - \beta(\beta - 6))}{3\beta^3}.$$

[9]Note that the joint density of (x_1, x_2) where there are just two order statistics is simply 2.

The expected utility under a unanimity requirement is:

$$\begin{aligned} E\left[U_U\right] &= \begin{array}{c} \int_0^{1/2} -\exp\left\{\beta(1/2-t)\right\} dt \\ +\left(\int_0^{1/2}\left(\int_0^{a_2} -\exp\left\{\beta(a_2-t)\right\} dt - 1\int_{a_2}^{1/2} dt\right) 8a_2 da_2\right) \end{array} \\ &= \frac{-48+\beta^2(\beta-12)+6e^{\beta/2}(-8+\beta(\beta+4))}{6\beta^3}. \end{aligned}$$

Now consider $\beta = 5$. In this case, where $n = 3$, a unanimity requirement is optimal and yields expected utility of approximately -3.44. Where $n = 5$ (ie. example 1) and $\beta = 5$ an 80% supermajority is optimal and the expected utility is approximately -4.12. For majority rule under $n = 3$ expected utility is -4.50. This illustrates the general point made in Theorem 1, that the optimal supermajority requirement decreases as the number of voters gets larger. In this example it falls from 100% to 80%. This is voting rule inflation.

4 Calibration

We now apply this technique to the problem of comparing the supermajority rules of 1789 and the present.

4.1 1789 versus Today

A key difference between 1789 and the present, is that there has been an increase in the number of Senators, House members and States. Table 1 shows the changes in the number of relevant voters between 1789 and the present.

Table 1: Numbers of Voting Parties in 1789 and 2006

	1789	**2006**
Senators	22	100
House Members	65	435
States	13	50

It should be noted, however, that has been little change to this picture over the last 50 years since Alaska entered the Union in 1953.

We assume that the distribution of preferences within each of the three bodies has remained unchanged. This allows us to isolate the effect of a pure increase in numbers on the voting rule, rather than conflating that effect with a change in the distribution of preferences. We show below, however, that a less "spread-out" distribution of preferences implies that a lower rule is optimal. To the extent that voter preferences in the Congress and the States were more spread-out in 1789 than they are today, our results here will be a *lower bound* on the impact of changes on the optimal voting rule. Conversely, of course, if voter preferences in Congress and State legislatures (or, more hypothetically, State conventions), are more spread-out now than in 1789 then our results may, to some extent, overstate the degree of voting rule inflation.

In the calculations which are presented below we assume, for tractability, that voter preferences in each of the three bodies are uniformly distributed. This is clearly not an accurate description of those preferences. It does, however, lower the computational burden. Furthermore, investigations of other distributions which may more closely proxy actual preferences (such as the double gamma[10]), indicate that our results are fairly insensitive to the choice of distribution. The other free parameter is β (which can be thought of as equivalent to the coefficient of absolute risk aversion given the utility function). We solve for the value of β which made the 1789 rules optimal. It turns out that the requirement of 3/4 of the 13 states and 2/3 of the 22 senators were both optimal voting rule assuming $\beta = 10$ and a uniform distribution. Furthermore, the requirement of 2/3 of the 65 members of the house was very close to optimal – the optimum being 62%. Therefore we use that parameter value throughout our exercise. Our approach is to explicitly calculate the expected utility under each possible voting rule and determine which is optimal in the sense of maximizing the sum of expected utilities.

[10] Poole and Rosenthal (2000) argue that the preferences of members of the Senate and House are approximately described by the double gamma distribution.

Note that the number of events to be considered expands with the number of voters. Let event 1 be the even $x_1^* \leq ... \leq x_n^* \leq \hat{\theta} = 1/2$, where x_i^* is the ith order statistic. Let event 2 be the event that $x_1^* \leq ... \leq x_{n-1}^* \leq 1/2 \leq x_n^*$, and so on up to event $n+1$. The probabilities of these events follow straightforwardly from the Binomial Theorem. If s is the required number of voters in a supermajority, the expected utility of a particular voting rule is then simply

$$E[U^s] = \sum_{i=1}^{l} \Pr(Ev.l) \cdot E[U^s] \, |Ev.l.$$

Table 2 reports the voting rule equivalents for the three sets of voting bodies. The 1789 rule is simply what is specified in the Constitution. The current rule adjust the 1789 so that the difficulty of making an amendment under Article V is the same as if there had been the same number of Senators, House members and States as there are today.

Table 2: Voting Rule Equivalents

	1789	**2006**
Senate	67%	59%
House	67%	53%
States	75%	62%

These calculations reveal that is has was indeed much easier to amend the Constitution in 1789. In equivalent terms, the voting rule was only 62% of the states having to agree, rather than 75% today. Similarly, the equivalent requirements in the Senate and the House were much lower. This confirms the conjecture that it has become much harder to amend the US Constitution. Moreover, our analysis is able to provide some insight into why this is the case.

4.2 Application: Failed Amendments

We now consider what impact, if any, voting rule inflation can be observed to have had on the defeat of significant proposed amendments. A natural question to ask is whether, absent the voting rule inflation, these amendments would have passed.

It is, of course, impossible to provide a definitive answer to this, because members of the House and Senate obviously vote strategically. That is, some Members' votes are determined by the probability of the amendment passing. For instance, a Democrat from a conservative district may vote with Republicans on a bill that she would like to see passed if she is confident that the bill will be defeated anyway. That is, conditional on her vote not being pivotal, she prefers to be on the record as being against the amendment. Behaviors such as this mean that one cannot simply look at the *equilibrium* voting pattern since the voting rule may cause endogenous changes in voting behavior.

However, it is still possible to gain some sense of the our effect on observed voting patterns. This is because examining equilibrium voting behavior provides a useful benchmark against which the impact of strategic considerations can be assessed.

Table 3 documents each amendment which was voted on by Congress since the 93rd Congress (1973).

[Table 3 Here]

Of the 22 proposed amendments which actually went to a vote of either the House or Senate, but which ultimately failed, we argue that 10 of these might well have gained it under the adjusted requirement.

When one restricts attention to *distinct* amendments, just 12 were voted on. None passed[11] - but three "would" have passed under the adjusted requirement (the Equal Rights Amendment would have passed the House, a Balanced Budget Amendment would have passed the House and Senate, and a Flag Burning Amendment would have passed the House).

[11] The 26th amendment occured prior to the sample period. It was ratified by July 1, 1971 and a certificate of validity was granted on July 7.

4.3 Amendments That Didn't Bark

Strategic considerations also suggest than amendments may not have been proposed as a result of the voting rule. If the probability of an amendment being passed under the current rule is relatively small then legislators may not risk the "political capital" involved in proposing an amendment which has only a small chance of passing. This implies that the impact of an adjusted amendment provision would likely be larger than simply the outcome of votes on observed amendments. That is, it is likely that under a less stringent voting rule that other amendments would have been proposed - and some may well have been passed. Consequently, there is an additional channel through which a less stringent supermajority requirement could have an observed impact on Constitutional amendments.

5 Discussion and Conclusion

The mechanism by which a Constitution may be amended has crucial importance for both the success and the legitimacy of the system of government it establishes.

First and foremost, as the experience prior to 1789 under the Articles of Confederation demonstrated, if a constitution cannot be amended to respond to changing understandings or circumstances, the polity it establishes may fail.

Of course, where formal mechanisms prove too onerous, informal modes of constitutional change may emerge to prevent a constitutional system from collapsing, as arguably occurred during the New Deal (Ackerman (1996)). However, the move to more informal mechanisms of amendment should not be thought of as costless.

The shift to more informal modes of amendment carries with it a real potential cost in terms of constitutional legitimacy. There is the danger that "amendment" by the United States Supreme Court will be strongly counter-majoritarian, and thus raise substantial questions of democratic legitimacy (c.f. Bickel (1962)). Further, even if, as is perhaps more likely (Dahl (1989)), the Supreme Court acts in a way which is pro-majoritarian over time, this mode of "amendment" represents a much less democratic form of politics than that envisaged by the Framers.

Informal modes of constitutional change therefore do not displace the central role of formal modes of amendment (Levinson (1995)). Constitutional scholars continue to question whether the current voting requirements in Article V should not be lowered in some way (Griffin (1995a), Levinson (1995), Lutz (1995)). While this paper does not directly address this question, it provides theoretical support for the intuition that the current mechanism is much more onerous than that initially seen as optimal.

The finding of "voting rule inflation" also has broader implications for constitutional design, in national as well as commercial, contractual and international settings. Wherever the voting population under a particular "constitution" may be expected to increase, attention should be given to the effect this will have on the difficulty of amending the relevant constitutive document. If initial judgments about optimality are to be given long-term effect, a more flexible approach than has traditionally been employed may be required. This may even require adoption of "floating" super-majority requirements, which are linked to the size of the voting population.

Constitutions "meant to endure for ages to come[12]" should pay attention to the possibility of voting rule inflation.

[12] Marshall, C.J. in *McCulloch v. Maryland* (1819).

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Table 1: Amendments Voted On Since 93rd-108th Congress

Congress	Resolution	Introduced	Sponsor	Official Title	House Vote	Senate Vote	% Yea
93rd	None						
94th	None						
95th	H.J.RES.554	7/25/1977	Rep Edwards	Joint resolution to amend the Constitution to provide representation of the District of Columbia in the Congress			
96th	H.J.RES.74	1/15/1979	Rep Mottl	Joint resolution to amend the Constitution of the United States to prohibit compelling the attendance of a student in a public school other than the public school nearest the residence of such student	209-216		48.0%
	S.J.RES.28	1/25/1979	Sen Bayh	Joint resolution to amend the Constitution to provide for the direct popular election of the President and Vice President of the United States		51-48	51.0%
97th							
98th	**H.J.RES.1**	**1/3/1983**	**Rep Rodino**	**Joint resolution proposing to amend the Constitution of the United States relative to equal rights for men and women**	**278-147**		**63.9%**
	S.J.RES.73	3/24/1983	Sen Thurmond	A joint resolution proposing an amendment to the Constitution of the United States authorizing the Congress and the States relating to voluntary school prayer		56-44	56.0%
99th	None						
100th	None						
101st	**H.RES.417**	**5/11/1989**	**Rep Stenholm**	**Proposing an amendment to the Constitution to provide for a balanced budget for the United States Government and for greater accountability in the enactment of tax legislation**	**279-150**		**64.1%**
	H.J.RES.350	**6/29/1989**	**Rep Michel**	**Proposing an amendment to the Constitution of the United States authorizing the Congress and the States to prohibit the physical desecration of the flag of the United States**	**254-177**		**58.4%**
	S.J.RES.180	7/18/1989	Sen Dole	A joint resolution proposing an amendment to the Constitution of the United States authorizing the Congress and the States to prohibit the physical desecration of the flag of the United States		51-48	51.0%
	S.J.RES.332	5/25/1990	Sen Dole	A joint resolution proposing an amendment to the Constitution of the United States authorizing the Congress and the States to prohibit the physical desecration of the flag of the United States		58-42	58.0%

Congress	Resolution	Introduced	Sponsor	Official Title	House Vote	Senate Vote	% Yea
102nd	**H.J.RES.290**	**6/26/1991**	**Rep Sandholm**	**Proposing an amendment to the Constitution to provide for a balanced budget for the United States Government and for greater accountability in the enactment of tax legislation**	**280-153**		**64.4%**
	S.RES.298	5/19/1992	Sem Byrd	A resolution declaring an article of amendment to be the Twenty-seventh Amendment to the Constitution of the United States		99-0	99.0%
103rd	**H.J.RES.103**	**2/4/1993**	**Rep Sandholm**	**Proposing an amendment to the Constitution to provide for a balanced budget for the United States Government and for greater accountability in the enactment of tax legislation**	**271-153**		**62.3%**
	S.J.RES.41	**2/4/1993**	**Sen Simon**	**Proposing an amendment to the Constitution to provide for a balanced budget**		**63-37**	**63.0%**
104th	**H.J.RES.1**	**1/4/1995**	**Rep Barton**	**Proposing a balanced budget amendment to the Constitution of the United States**		**64-35**	**64.0%**
	H.J.RES.73	3/12/1995	Rep McCollum	Proposing an amendment to the Constitution of the United States with respect to the number of terms of office of Members of the Senate and the House of Representatives	227-204		52.2%
105th	H.J.RES.2	1/7/1997	Rep McCollum	Proposing an amendment to the Constitution of the United States with respect to the number of terms of office of Members of the Senate and the House of Representatives	217-211		49.9%
	H.J.RES.78	5/18/1997	Rep Istook	Proposing an amendment to the Constitution of the United States restoring religious freedom	224-203		51.5%
	H.J.RES.119	5/14/1998	Rep DeLay	Proposing an amendment to the Constitution of the United States to limit campaign spending	29-345		6.7%
	S.J.RES.1	**1/21/1997**	**Sen Hatch**	**Proposing an amendment to the Constitution of the United States to require a balanced budget**		**66-34**	**66.0%**
106th	**H.J.RES.94**	**4/6/2000**	**Sen Sessions**	**Proposing an amendment to the Constitution of the United States with respect to tax limitations**	**234-192**		**53.8%**
107th	**H.J.RES.41**	**3/22/2001**	**Sen Sessions**	**Proposing an amendment to the Constitution of the United States with respect to tax limitations**	**232-189**		**53.3%**
	S.J.RES.4	2/7/2001	Sen Hollings	A joint resolution proposing an amendment to the Constitution of the United States relating to contributions and expenditures intended to affect elections		40-56	40.0%
108th	None						

6 Appendix

Proof of Theorem 1. Let event 1 be the event that $x_1^* \leq ... \leq x_n^* \leq \hat{\theta}$, where x_i^* is the ith order statistic. Let event 2 be the event that $x_1^* \leq ... \leq x_{n-1}^* \leq \hat{\theta} \leq x_n^*$, and so on up to event $n+1$. Note then that the probability of event j occurring is given by

$$\begin{aligned}\pi_j &= \Pr(\text{Event } j) \\ &= F^{(j-1)}\left(\hat{\theta}\right)\left[1 - F\left(\hat{\theta}\right)\right]^{(n-j+1)}.\end{aligned}$$

When the supermajority rule is $\lambda = \psi(\alpha n)$, where ψ is the ceiling function which rounds its argument up to the nearest integer, utility conditional on draws $x_1^*, ..., x_n^*$ is

$$\bar{V} = \sum_{i=1}^{n} \sum_{j=1}^{n+1} \pi_j u\left(\left|\theta^* - x_i^*\right|\right).$$

By A3, the ex post social choice under supermajority rule λ is x_λ^* for $j \leq (n+1)/2$ and $x_{n+1-\lambda}^*$ for $j > (n+1)/2$. We can thus write $\bar{V}$ as

$$\bar{V} = -\sum_{i=1}^{n}\left(\sum_{j=1}^{(n+1)/2} \pi_j u\left(\left|x_{\psi(\alpha n)}^* - x_i^*\right|\right) + \sum_{j=((n+1)/2)+1}^{n+1} \pi_j u\left(\left|x_{n+1-\psi(\alpha n)}^* - x_i^*\right|\right)\right).$$

Expected utility involves integrating over all possible realizations of the order statistics–that is, over their joint pdf. Thus, expected utility is

$$E\bar{V} = -\int \cdots \int \sum_{i=1}^{n}\left(\begin{array}{c}\sum_{j=1}^{(n+1)/2} \pi_j u\left(\left|x_{\psi(\alpha n)}^* - x_i^*\right|\right) + \\ \sum_{j=((n+1)/2)+1}^{n+1} \pi_j u\left(\left|x_{n+1-\psi(\alpha n)}^* - x_i^*\right|\right)\end{array}\right) f(a_1, ..., a_n)\, da_1 ... da_n.$$

This can be simplified by noting that the joint pdf of all n order statistics is $n!$, since the unordered sample has density equal to 1 and there are $n!$ different permutations of the sample corresponding to the same sequence of order statistics. Thus we have

$$E\bar{V} = -\int \cdots \int \sum_{i=1}^{n} \left(\begin{array}{c} \sum_{j=1}^{(n+1)/2} \pi_j u \left(\left| x^*_{\psi(\alpha n)} - x^*_i \right| \right) + \\ \sum_{j=((n+1)/2)+1}^{n+1} \pi_j u \left(\left| x^*_{n+1-\psi(\alpha n)} - x^*_i \right| \right) \end{array} \right) n! da_1 ... da_n.$$

Denote the optimal supermajority rule as $\alpha^* = \arg\max_\alpha \left\{ E\bar{V} \right\}$. By the Monotonicity Theorem of Milgrom and Shannon (1994), a necessary and sufficient condition for α^* to be nonincreasing in n is that $E\bar{V}$ have decreasing differences in (α, n). This requires that for all $n' \geq n$, $E\bar{V}(n', \alpha) - E\bar{V}(n, \alpha)$ is nonincreasing in α. Assuming for simplicity that n and n' are odd (the generalization to even integers is simply a matter of notation) this entails

$$\begin{array}{l} \int \cdots \int \sum_{i=1}^{n} \left(\begin{array}{c} \sum_{j=1}^{(n+1)/2} \pi_j(n) u \left(\left| x^*_{\psi(\alpha n)} - x^*_i \right| \right) + \\ \sum_{j=((n+1)/2)+1}^{n+1} \pi_j(n) u \left(\left| x^*_{n+1-\psi(\alpha n)} - x^*_i \right| \right) \end{array} \right) n! da_1 ... da_n, \\ -\int \cdots \int \sum_{i=1}^{n'} \left(\begin{array}{c} \sum_{j=1}^{(n'+1)/2} \pi_j(n') u \left(\left| x^*_{\psi(\alpha n')} - x^*_i \right| \right) + \\ \sum_{j=((n'+1)/2)+1}^{n'+1} \pi_j(n') u \left(\left| x^*_{n'+1-\psi(\alpha n')} - x^*_i \right| \right) \end{array} \right) (n')! da_1 ... da_{n'} \end{array} \tag{1}$$

to be nonincreasing in α for all $n' \geq n$ and all α. An increase in α makes the term $\sum_{j=1}^{(n+1)/2} \pi_j(n) u \left(\left| x^*_{\psi(\alpha n)} - x^*_i \right| \right)$ larger, since $\left| x^*_{\psi(\alpha n)} - x^*_i \right|$ increases and the probabilities $\pi_j = F^{(j-1)}\left(\hat{\theta}\right) \left[1 - F\left(\hat{\theta}\right)\right]^{(n-j+1)}$ are unchanged. Similarly

$$\sum_{j=((n+1)/2)+1}^{n+1} \pi_j(n) u \left(\left| x^*_{n+1-\psi(\alpha n)} - x^*_i \right| \right)$$

is larger and so the first line of (1) is overall larger. Note, however, that the second line of (1) increases by more than the first line for a given change in α. The first term inside the parentheses in the second line increases by more than its corresponding term in the first line since each term $u(|\cdot|)$ is weakly larger in the second line by construction of the ordering of the order statistics, probabilities sum to 1, and $n'! > n!$ This argument is true for all α and $n' > n$, and hence the proof is complete. ■

Proof of Theorem 2. By a similar argument to the proof of the above theorem we require $E\bar{V}$ to have increasing differences in (β, α). This requires that for all $\beta' \geq \beta$, $E\bar{V}(\beta', \alpha) - E\bar{V}(\beta, \alpha)$ is nondecreasing in α. That is

$$\int \cdots \int \sum_{i=1}^{n} \left(\begin{array}{c} \sum_{j=1}^{(n+1)/2} \pi_j(n)\, u_\beta \left(\left| x^*_{\psi(\alpha n)} - x^*_i \right| \right) + \\ \sum_{j=((n+1)/2)+1}^{n+1} \pi_j(n)\, u_\beta \left(\left| x^*_{n+1-\psi(\alpha n)} - x^*_i \right| \right) \end{array} \right) n! da_1 ... da_n, \tag{2}$$
$$- \int \cdots \int \sum_{i=1}^{n} \left(\begin{array}{c} \sum_{j=1}^{(n+1)/2} \pi_j(n)\, u_{\beta'} \left(\left| x^*_{\psi(\alpha n)} - x^*_i \right| \right) + \\ \sum_{j=((n+1)/2)+1}^{n+1} \pi_j(n)\, u_{\beta'} \left(\left| x^*_{n+1-\psi(\alpha n)} - x^*_i \right| \right) \end{array} \right) n! da_1 ... da_n,$$

is nondecreasing in α for all $\beta' > \beta$. The first line of the above is identical to the second except for the differences in the function u. As in the proof of Theorem 1 an increasing in α makes the term $\sum_{j=1}^{(n+1)/2} \pi_j(n)\, u\left(\left| x^*_{\psi(\alpha n)} - x^*_i \right| \right)$ larger, since $\left| x^*_{\psi(\alpha n)} - x^*_i \right|$ increases and the probabilities $\pi_j = F^{(j-1)}\left(\hat{\theta}\right)\left[1 - F\left(\hat{\theta}\right)\right]^{(n-j+1)}$ are unchanged and similarly for the term

$$\sum_{j=((n+1)/2)+1}^{n+1} \pi_j(n)\, u\left(\left| x^*_{n+1-\psi(\alpha n)} - x^*_i \right| \right).$$

The magnitude of this change is larger for $u_{\beta'}$ than u_β by Jensen's inequality, and thus the result follows. ■