12.810 Dynamics of the Atmosphere

Internal gravity waves in the atmosphere
ERA40 reanalysis data 1980-2001

Potential temperature (K)

Increase with height (implies dry static stability)

(ERA40 reanalysis data 1980-2001)
Positive static stability allows for internal gravity waves: here forced by mountain

Vertically propagating

Trapped

Also forced by moist convection, geostrophic adjustment, surface warming/cooling...
Trapped lee waves downwind from Hawaiian Islands
Internal gravity waves

• Basic theory of internal gravity waves will first be introduced: see *handout*

• Then discuss mountain waves, compressible gravity waves, and interaction of gravity waves with mean flow
Internal gravity waves: Introductory material

Governing equations for non-rotating, inviscid, adiabatic flow in Boussinesq approximation:
- Holton section 2.8
- Vallis section 2.5 for anelastic (Boussinesq a special case)

\[
\begin{align*}
\frac{Du}{Dt} &= - \frac{\partial \phi}{\partial x} \\
\frac{Dv}{Dt} &= - \frac{\partial \phi}{\partial y} \\
\frac{Dw}{Dt} &= - \frac{\partial \phi}{\partial z} + b \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{Db}{Dt} &= 0
\end{align*}
\]

Rewrite for convenience
\[\alpha \frac{Dw}{Dt} = - \frac{\partial \phi}{\partial z} + b, \quad \alpha = 1: \text{full equations} \quad \alpha = 0: \text{hydrostatic}\]
Waves on a basic state

- Basic state is at rest: \( u = v = w = 0 \)

- Basic state is stably stratified:
  \[ b_0 = N^2 z \]
  \[ \phi_0 = \int b_0 \, dz \]

- \( N \) is the buoyancy frequency (the angular frequency at which a parcel moving vertically would oscillate)
Assume small amplitude perturbations and linearize the equations (drop terms that are squared in wave amplitude)

\[\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x}\]
\[\frac{\partial v'}{\partial t} = -\frac{\partial \phi'}{\partial y}\]
\[\alpha \frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b'\]
\[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0\]
\[\frac{\partial b'}{\partial t} + N^2 w' = 0\]
Look for wavelike solutions

\[
\begin{pmatrix}
  u' \\
v' \\
w' \\
\phi' \\
b'
\end{pmatrix} = \text{Re} \begin{pmatrix}
  U \\
  V \\
  W \\
  \Phi \\
  B
\end{pmatrix} e^{i(kx+ly+mz-\omega t)}
\]

where \( \mathbf{k}=(k,l,m) \) is the wavenumber vector and \( \omega \) is the angular frequency.

\[
\begin{align*}
\omega U - k \Phi &= 0 \\
\omega V - l \Phi &= 0 \\
\omega \alpha W - m \Phi - iB &= 0 \\
kU + lV + mW &= 0 \\
-i\omega B + N^2 W &= 0
\end{align*}
\]
Dispersion relation for non-hydrostatic (α=1) waves

\[ \omega = \pm N \sqrt{\frac{k^2 + l^2}{k^2 + l^2 + m^2}} \]

Or put more simply

\[ \omega = \pm N \sin \gamma \]

Which implies that

\[ |\omega| \leq N \quad \text{(no propagation otherwise!)} \]
Propagation: Phase and group velocities

The phase speed in the direction of $k$ is given by

$$c = \frac{\omega}{|k|}$$

and the group velocity is

$$c_g = \left( \frac{\partial \omega}{\partial k'}, \frac{\partial \omega}{\partial l'}, \frac{\partial \omega}{\partial m} \right) = \frac{\omega m}{(k^2 + l^2 + m^2)} \left[ \begin{array}{ccc} \frac{km}{k^2 + l^2} & \frac{lm}{k^2 + l^2} & -1 \end{array} \right]$$
A wave with group velocity upwards and to the right

\[ \text{Wavy lines are isolines of } b = b' + b_0(z) \]
\[ \text{Black arrows show velocity} \]
\( \mathbf{c}_g \cdot \mathbf{k} = 0: \) Group propagation is along phase lines!

From a localized source oscillating with a single frequency, the waves form rays (the "St Andrews' cross") at angles \( \gamma = \sin \left( \frac{\mathbf{c}_g}{N} \right) \) to the horizontal, with the phase propagation across the rays:

Hydrostatic case (\( \mathbf{c}_g = 0 \) when \( \mathbf{c}_g = 0 \), the dispersion relation becomes\):

\[
\mathbf{c}_g^2 + \mathbf{k}^2 = \pm \mathbf{N} \tan \gamma 
\]

There is no longer any restriction \( \gamma \), so the hydrostatic approximation is not valid for high frequency waves for which this approximation predicts \( \gamma \approx \mathbf{N} \), but should be good for \( \tau \ll (\mathbf{c}_g \mathbf{N}) \). Equivalently, it requires \( \mathbf{k}^2 \tau_0 ^2 \), i.e., vertical scales much less than horizontal scales.
Group velocity is upwards if phase propagation downwards! (but both phase and group propagate to the right)

From a localized source oscillating with a single frequency, the waves form rays (the “St Andrews’ cross”) at angles $\gamma = \sin^{-1}(\frac{\omega}{N \tau_{\text{total}}})$ with the horizontal, with the phase propagation across the rays:

Hydrostatic case ($\varepsilon = 0$): When $\varepsilon = 0$, the dispersion relation becomes $\varepsilon = \pm \frac{N \tan \gamma}{k^2 + l^2} = \pm \frac{N}{m^2}$.

There is no longer any restriction $\varepsilon \ll N$, so the hydrostatic approximation is not valid for high frequency waves for which this approximation predicts $\varepsilon \gg N$, but it should be good for $\varepsilon \ll \frac{\tau_{1} \omega}{N}$. Equivalently, it requires $k^2 + l^2 \leq \frac{m^2}{\varepsilon}$, i.e., vertical scales much less than horizontal scales.

The $z$ direction is special because of gravity.
\( \mathbf{k} \cdot \mathbf{u} = 0: \) Fluid motions are along phase lines

From a localized source oscillating with a single frequency, the waves form rays (the "St Andrews' cross") at angles \( \gamma = \sin(\lambda/N) \) to the horizontal, with the phase propagation across the rays:

Hydrostatic case (\( \varepsilon = 0 \)) When \( \varepsilon = 0 \), the dispersion relation becomes

\[
\pm \sqrt{N^2 \tan^2 \gamma} \pm \frac{k^2 + l^2}{m^2} = \pm \sqrt{N^2},
\]

There is no longer any restriction \( \varepsilon \leq N \), so the hydrostatic approximation is not valid for high frequency waves for which this approximation predicts \( \varepsilon \gg N \), but should be good for \( \varepsilon \ll \lambda \). Equivalently, it requires

\[
k^2 + l^2 \ll m^2,
\]

i.e., vertical scales much less than horizontal scales.

Implies that no advection of wave properties such as \( b \): plane gravity wave is a nonlinear solution!
From a localized source oscillating with a single frequency $\omega$, the waves form rays (the “St Andrews’ cross”) at angles $\gamma = \sin^{-1}(\omega/N)$ to the horizontal, with the phase propagation across the rays:

\[ \frac{\omega}{N} \tan \gamma = \pm \frac{N}{m}k^2 + \ell^2 \]

There is no longer any restriction $\pi N$, so the hydrostatic approximation is not valid for high frequency waves for which this approximation predicts $\pi N$, but it should be good for $\tau_1(\pi N)$. Equivalently, it requires $k^2 + \ell^2 \geq m^2$, i.e., vertical scales much less than horizontal scales.
\( \mathbf{k} = (k, m) = (2, 2); \quad \mathbf{c}_g = (0.18, -0.18) \)
Relation of frequency to buoyancy frequency $N$

$$\omega = \pm N \sin \gamma$$

Implications of $\mathbf{k} \cdot \mathbf{u} = 0$
(fluid motions perpendicular to wavevector)

- $\gamma \rightarrow \pi/2$ motions are vertical and $\omega \rightarrow N$
- $\gamma \rightarrow 0$ motions are horizontal $\omega \rightarrow 0$
Relation of frequency to buoyancy frequency $N$

$$\omega = \pm N \sin \gamma$$

**Implications of $\mathbf{k} \cdot \mathbf{u} = 0$**

(fluid motions perpendicular to wavevector)

- $\gamma \to \pi/2$ motions are vertical and $\omega \to N$
- $\gamma \to 0$ motions are horizontal $\omega \to 0$

No resistance from stratification!
Hydrostatic case (set $\alpha=0$)

\[ \omega = \pm \frac{N}{m} \sqrt{k^2 + l^2} = \pm N \tan \gamma \]

Only a good approximation to

\[ \omega = \pm N \sin \gamma \]

when $\gamma$ is small i.e. $k^2 + l^2 \ll m^2$

This is true when vertical scales are small compared to horizontal scales
Mountain waves
Fig 1  Streamlines over periodic mountains

Durran, AMS, 1990
Evanescent waves (e.g. weak stratification)

Vertically propagating waves (e.g. strong stratification)

Fig 1 Streamlines over periodic mountains

Durran, AMS, 1990
Evanescent waves (e.g. weak stratification)

No phase tilt with height

Vertically propagating waves (e.g. strong stratification)

Phase tilt with height

Fig 1  Streamlines over periodic mountains
Fig 2  Trapped lee waves

(e.g. weaker stratification)

(e.g. stronger stratification)

Durran, AMS, 1990
Fig 2  Trapped lee waves

(e.g. weaker stratification)

No phase tilt with height as not propagating upwards in net
(e.g. stronger stratification)

Fig. 4.4. Streamlines in steady airflow over an isolated ridge when the vertical variation in the Scorer parameter permits trapped waves.
Fig 3  Waves over a *broad* isolated ridge
Because broad ridge, flow is periodic in the vertical (where does the ridge repeat itself?)

Fig 3 Waves over a *broad* isolated ridge
Contours of potential temperature in a breaking gravity wave

Figure 4: Schematic of isentropes in a breaking gravity wave.
The figure shows contours of potential temperature in a breaking gravity wave. The waves break by convective instability. Consider Fig. 4. An upward propagating wave is increasing in amplitude with height. In reality, such an increase (as expressed by $\frac{\partial}{\partial z} j = \frac{\partial}{\partial x} j$), which affects matter, can occur because of decreasing $A$ (in the mesosphere, generally), or also because of decreasing $u_0! c$ (such as near the tropopause, above the main jet). The isentropes ($\frac{\partial}{\partial z} = \frac{\partial}{\partial x} (z; x; t)$) are perturbed by the wave; at small wave amplitude, the isentropes are just wavy but when the amplitude gets very large, the distortion of the isentropes is such that they may overturn ($\frac{\partial}{\partial z} < 0$), at which point convective instability sets in and rapidly mixes the wave, thereby dissipating the wave and, among the things, reducing its momentum. Thus, there is a convergence of the wave’s momentum where breaking occurs, and the mean flow is affected, as illustrated in Fig. 5. In the case shown, the wave has $c > u_0$, so by (34), it sup若干 momentum is positive below the breaking region. Above the breaking region, the momentum is assumed to vanish (i.e. the wave is completely dissipated in the breaking region). Hence, integrated over the breaking region, $\int_{\text{above}}^{\text{below}} \frac{\partial}{\partial z} u \, dz = \int_{\text{above}}^{\text{below}} \frac{\partial}{\partial z} (u_0 w_0) \, dz = u_0 w_0 \int_{\text{below}}^{\text{above}} \frac{\partial}{\partial z} (u_0 w_0) \, dz$.

**Fig 4** Contours of potential temperature in a breaking gravity wave
Fig. 1. Potential temperature cross section for 17 February 1970. Solid lines are isentropes (°K), dashed lines aircraft or balloon flight trajectories. The cross section is along a 275°-095° true azimuth line, crossing the Kremmling, Colo., and Denver VOR aircraft navigation stations.

Fig 5
Example of mountain wave over Rockies

Lilly et al, JAS, 1973
More generally, if $6h$ is the amplitude of the vertical displacement of an isentropic surface, then $z = \frac{4kpU}{6h^2}$.

From the Eliassen-Palm theorem (Eliassen and Palm 1961; McIntyre 1980), vertically propagating waves in the absence of transience and dissipation obey the condition $t = t_s$ at all levels. Using the hydrostatic dispersion relation Eq. (3), the wave's impact on the local static stability and vertical shear can be written as

$$NT_{total} = N^2 \left[ 1 + \left( \frac{NH}{U} \right)^2 \cos \theta \right]$$

$$q_{total} = q \left[ 1 + \frac{R}{\frac{\pi}{2}} \left( \frac{\theta}{U} \right) \sin \theta \right]$$

where $6h$ is the amplitude of the displacement of the isentropic surface, $\theta$ the wave phase, $r = \frac{8}{U} \frac{\partial z}{\partial r}$ and $R_i = \frac{\rho}{\rho_i}^2$ is the Richardson number. The subscript 'total' on the left-hand side of Eqs. (6) and (7) refers to the sum of the wave and background flow contributions.

Equations (6) and (7) suggest that a sufficiently large isentropic displacement could induce local Kelvin-Helmholtz instability. This forms the basis for our wave-breaking mechanism and incorporates ideas both from the billow instability mechanism discussed by Scorer (1978), and Lindzen's (1981) convective overturning parametrization for mesospheric gravity wave breaking. Further details are given in section 5.

It should be remarked that Eq. (7) can also be derived from a finite amplitude conservation equation formulated by Long (1953). Details of this derivation are given in Shutts (1986).

Figure 3. Mean observed profile of momentum flux over the Rocky mountains on 17 February 1970 (after Lilly and Kennedy 1973).

**Fig 6** Observed vertical momentum flux in mountain wave over Rockies

Palmer et al, QJRMS, 1986
Fig 7  Zonal-mean zonal winds (m/s) in control simulation of general circulation model (GCM) without gravity-wave drag parameterization (season is DJF)

McFarlane, JAS, 1987
Fig 8

Deceleration of zonal-mean zonal wind by orographic gravity-wave drag parameterization

McFarlane, JAS, 1987
Units: m/s/day
Fig 9

Change in zonal-mean zonal wind (m/s)
(simulation with gravity-wave drag minus control)

McFarlane, JAS, 1987
Fig 9  
Change in zonal-mean zonal wind (m/s)  
(simulation with gravity-wave drag minus control)  

McFarlane, JAS, 1987
Fig 9

Change in zonal-mean zonal wind (m/s)
(simulation with gravity-wave drag minus control)

Probably just internal variability

McFarlane, JAS, 1987
Change in zonal-mean temperature (K) (simulation with gravity wave drag minus control)

McFarlane, JAS, 1987
Warming to keep thermal wind balance

Fig 10

Change in zonal-mean temperature (K)
(simulation with gravity wave drag minus control)

McFarlane, JAS, 1987
Fig 11. Magnitude of orographic gravity wave drag stress on the atmosphere (Pa)

McFarlane, JAS, 1987