Approximately Optimal Auction when Bidders have Private Budgets

Interim Report

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ABSTRACT
We tackle the problem of designing revenue maximizing auctions in the Bayesian framework, when bidders not only have private valuations but also private budgets. We consider the setting of selling divisible goods to multiple agents each with linear utilities, but agents cannot pay beyond their budget. We focus on the case when the auctioneer can check that bidders do not over-report their budgets, but bidders may under-report. We characterize all incentive-compatible auctions in this framework. Using this result and LP-rounding, we design 4 practically implementable auction mechanisms: proportional-sharing, conservative-sharing, fixed-queuing, and random queuing. The sharing auctions are similar to having a separate allocation table for each bidder, and scaling down appropriately to ensure that the total allocation does not exceed supply. The queuing auctions sort bidders according to some arbitrary priority, and allocate to bidders with higher priorities first. We show that the auctions achieve revenue approximation ratios of 4.08, 3.26, 5.83 and 3.73 respectively.

1. INTRODUCTION
One of the most positive results in Mechanism Design is Myerson’s 1982 closed-form characterization of the revenue maximizing auction, when bidders have private valuations drawn independently from some possibly different distributions [7]. Myerson showed that the optimal auction defines some virtual valuation function, performs a Vickery auction on virtual valuations to determine allocation, and charges according to the inverse of the virtual valuation of the Vickrey price [7]. More accessible proofs of this result can be found in [9, 5].

However, little is known in the case when bidders not only have private valuations but also private budget constraints. In practice, budget constraints are common, and they render Myerson’s auction infeasible. In fact, budget constraints so complicate the problem, that even solving the case with one bidder becomes highly non-trivial [4]. While there are characterizations of the optimal solution when bidders are identical [8], little is known when bidders are asymmetric or when we require a polynomial-time implementable auction. While the optimal auction under discretization can always be found by solving a linear program, this linear program has exponentially many variables and constraints. Currently the best result in the literature is a 5.83-approximation which follows from LP-rounding [2].

This paper contains a medley of new results. We first characterize all incentive compatible auctions in the Bayesian framework in the case when bidders cannot over-report their budget (Section 3). We focus on this case because in practice the auctioneer can enforce this by demanding bidders to post bonds. As in [2] our work is based on relaxing the feasibility constraint in the optimal LP, and we show that the relaxed LP can be solved by separately pre-computing for each bidder an “expected demand curve” $G_i(\lambda)$. At the time of the auction the LP can be solved by simple binary search for the parameter $\lambda$ that equates expected demand with supply (Section 4.1). However, the allocation may no longer be feasible, as it may allocate more goods than are available.

We then prove a general theorem stating that composing any incentive-compatible allocation function with a monotone concave function preserves its incentive-compatibility (Section 4.2). Using this we propose four practically implementable auctions: proportional-sharing, conservative-sharing, fixed-queuing, and random-queuing. Suppose we have $M$ units of the good. In each case we solve the LP efficiently as described before, hence creating a posted-price table for each bidder: if bidder $i$ reports valuation $v_i$ and budget $b_i$, the table tells us that we should aim to allocate $x_i^*(v_i, b_i)$ units of the good. When the $x_i^*$’s sum to more than $M$, the proportional-sharing auction scales allocations down to total exactly $M$, while the conservative-sharing auction scales down by slightly more. The fixed-queuing auction sorts bidders in some arbitrary priority and allocates to the bidders with higher priorities first, and the random-queuing auction executes fixed-queuing with a randomly generated
priority order. We show that the auctions achieve at least 1/4.08, 1/3.26, 1/5.83 and 1/3.73 of the optimal revenue, respectively. The analysis may be of independent interest.

1.1 Previous Work

There is a recent line of research by Economists on characterizing the optimal Bayesian auction given budget constraints. Che and Gale studies the one bidder case, and even then the optimal auction is highly non-trivial [4]. In the case when bidders can report arbitrary budgets, the optimal selling mechanism commits to a non-decreasing convex pricing function \( p(x) \), such that if the buyer wishes to purchase \( x \) units of good, she needs to pay \( p(x) \). The pricing function can be found by solving a linear program or a differential equation. In the case when bidders cannot overreport budget, the pricing function becomes two-dimensional.

Our characterization theorems are closely related to some of their results, with the generalization that instead of only describing the optimal allocation and pricing functions, we characterize all incentive-compatible ones.

When there are multiple bidders, Laffont and Robert characterize the optimal auction with identical bidders and public budget constraints [6], and Pai and Vohra do the same with private budgets [8]. However, their arguments crucially depend on symmetric priors for the bidders, and they do not cover the case when the auctioneer can enforce no-overreporting of budgets. Moreover, it is unclear whether Pai and Vohra's solution is polynomial-time solvable.

There is another line of research on revenue maximizing auctions in the worst-case scenario, when no priors on bidder distributions are known [1, 3]. This is very different from the Bayesian setting that we consider.

Our work is most similar to the approximately optimal Bayesian auction in [2]. We improve the approximation ratio and our auctions are more natural.

2. PROBLEM FORMULATION

Suppose there are \( n \) bidders and \( M \) identical, infinitely-divisible items to be auctioned. Bidder \( i \) wishes to purchase a maximum of \( s_i \leq M \) units of goods, and these size constraints are common knowledge. Each bidder also has private type \( t_i = (v_i, b_i) \), where \( v_i \geq 0 \) and \( b_i \geq 0 \) are her private valuation and budget respectively. We assume that \( t_i \) is drawn independently for each bidder from some distribution with density \( f(v, b) \). For convenience we sometimes factor this as \( f(v, b) = g_i(b)f_i(v|b) \). We assume that the bidder’s are risk neutral with linear utility \( u_i = v_i - p_i \) is bidder i’s utility under \( x_i \) units of allocation, except that they have utility of \( -\infty \) when their budget constraint is exceeded. (This is called a hard budget constraint and is the most common setting studied [4, 6, 8, 2].) We seek the auction implementable in dominant strategies that maximizes the auctioneer’s expected profit.

By the revelation principle, it suffices to consider direct-revelation mechanisms. In this case, a deterministic auction mechanism specifies an allocation function \( \pi(\bar{\ell}) \) and a payment function \( p(\bar{\ell}) \), where \( \bar{\ell} = ((v_1, b_1), (v_2, b_2), \ldots, (v_n, b_n)) \) consists of the bidders’ reports of types, and given this report the auctioneer allocates to bidder \( i \) \( x_i(\bar{\ell}) \) units of good and charge \( p_i(\bar{\ell}) \). (It suffices to consider deterministic mechanisms because each randomized mechanism that does not exceed bidders’ reported budgets induces some expected allocation and payment functions \( \pi(\cdot), p(\cdot) \), and yield the same expected utilities for the bidders and the auctioneer as the deterministic mechanism with these allocation and payment functions.) We require that the mechanism \( (\pi, p) \) satisfies the following:

- Incentive Compatibility (IC): It is a dominant strategy for each bidder \( i \) to report her true type \( t_i = (v_i, b_i) \). One may pose two versions of this constraint: when bidders can report arbitrary budgets we call the constraint (G-IC) (G for general); when bidders can never over-report budgets we call the constraint (P-IC) (P for partial). In this paper we focus on the (P-IC) case.
- Individual Rationality (IR): The strategy of not participating in the auction is dominated.
- Budget Feasibility (BF): The mechanism never requires a bidder to pay more than her reported budget. (This is implied by (IR) but it is convenient to treat it separately.)
- No Positive Transfers (NPT): The mechanism never subsidizes any bidder with a monetary amount.
- Allocation Feasibility (F): The mechanism allocates non-negative amounts, respects each bidder’s size constraint \( s_i \), and allocates at most \( M \) units in total.

2.1 Program Encoding the Optimal Auction

For convenience, define the probabilistic density of each type vector \( f(\bar{\ell}) = \prod_{i=1}^{n} f_i(v_i|b_i) \), and define the space of possible type vectors \( T \). Let \( t_{-i} \) denote the types reported by bidders other than \( i \). The optimal auction in the model in which bidders cannot overreport budgets can be encoded as a solution to the program in Figure 1.

The program is feasible because setting \( x_i = p_i = 0 \) for all constraints.

**Definition 1.** We call the auction valid if its allocation and payment functions \( \pi(\bar{\ell}) \) and \( p(\bar{\ell}) \) satisfy constraints (P-IC), (IR), (BF), (NPT), (F0) and (F1).

One corollary of all constraints being linear is that randomizing over several valid auctions yields a valid auction.

**Claim 1.** Given a finite set of valid auctions \( A_1, A_2, \ldots, A_m \), and a randomization strategy \( p_1, p_2, \ldots, p_m \), with \( \sum_j p_j = 1 \). After taking in bidders’ report \( \bar{\ell} \), the randomized auction defined by executing \( A_j \) with probability \( p_j \) is still a valid auction.

\(^2\)In this formulation, the (P-IC), (IR) constraints hold ex-post (regardless of other bidders’ reports), so the mechanism is implementable in dominant-strategies. One can relax this by modifying these constraints so that they hold ex-interim (in expectation over other bidders’ reports), in which case the mechanism becomes implementable in Bayes-Nash equilibrium.
We show that (F0), and (P1-3) implies

Suppose that functions

Condition (P1) follows from (C4), (C1) and (C2): many variables and constraints (exponential in \( n \)).

Myerson solved the optimal auction with only private valuations by first characterizing all valid auctions in terms of simple monotonicity conditions [9, 5]. We follow the same road map and characterize all valid auctions.

**Theorem 1 (Characterization of valid auctions).** Allocation and payment functions \( \bar{x} \) and \( \bar{p} \) satisfy ex-post (P-IC), (IR), (BF), (NPT), and (F0) (see Figure 1) if and only if for each \( i \) and each \( t_{-i} \), defining \( x(v,b) = x_i((v,b),t_{-i}), p(v,b) = p_i((v,b),t_{-i}) \), we have

- (F0) \( 0 \leq x(v,b) \leq s_i \).
- (C1) \( x(v,b) \) is non-decreasing in \( v \).
- (C2) \( vy(v,b) = \int_0^v x(q,b) dq \) is non-decreasing in \( b \).
- (C3) \( vx(v,b) - \int_0^v x(q,b) dq \leq b \ \forall v, b \).
- (C4) \( p(v,b) = vx(v,b) - \int_0^v x(q,b) dq \ \forall v, b \).

**Proof.** “If” direction: Suppose that functions \( x \) and \( p \) satisfy (F0), (C1-4), it suffices to check that

- (P1) \( vx(v,b) - p(v,b) \geq vx(v',b') - p(v',b') \) \( \forall v,v',b,b' \leq b \).
- (P2) \( vx(v,b) - p(v,b) \geq 0 \) \( \forall v,v',b,b' \).
- (P3) \( 0 \leq p(v,b) \leq b \) \( \forall v,b \).

Condition (P1) follows from (C4), (C1) and (C2):

\[
[vx(v,b) - p(v,b)] - [vx(v',b') - p(v',b')] = \int_0^v x(q,b) dq - \int_0^{v'} x(q,b') dq - (v - v')x(v',b') \geq 0 \quad \text{by (C4)}
\]

Condition (P2) follows from (C4) and (F0):

\[
vx(v,b) - p(v,b) = \int_0^v x(q,b) dq \geq 0
\]

LHS of (P3) follows directly from (C4) and (C1), while RHS of (P3) follows from (C4) and (C3).

“Only if” direction: We show that (F0), and (P1-3) implies (C1-4).

To show (C1), we use (P1) twice:

\[
vx(v,b) - p(v,b) \geq vx(v',b') - p(v',b') \geq vx(v',b) - p(v',b)
\]

Adding, we get

\[
(v - v')[x(v,b) - x(v',b)] \geq 0
\]

Which implies (C1).

To show (C4), we note that equations (1) and (2) also imply

\[
vy(v,b) - vy(v',b) \leq vy(v',b) - vy(v',b) \leq vy(v,b) - vy(v',b)
\]

And because \( v', v \geq 0 \) and because of (C1), we get that fixing \( b, p(v,b) \) is monotonically non-decreasing. For fix \( b, \) define \( x(v) = x(v,b) \) and \( p(v) = p(v,b) \). Because \( x(v), p(v) \) are monotonically non-decreasing, so we can define the Lebesgue-Stieltjes integrals \( \int_v dq x(q,v) \) and \( \int_v dp(q) \). The above implies that

\[
dp(q) = dq x(q,v)
\]

integrating by parts, we get

\[
p(v,b) - p(0,b) = \int_0^v dq x(q,v)
\]

Using (P1) with \( v = b' = 0 \), we get \( p(0,b) \leq p(0,0) \). By (P3),

\[
0 \leq p(0,b) \leq p(0,0) \leq 0
\]

Which implies that \( p(0,b) = 0 \), and this combined with above yields (C4).

(C3) immediately follows from (C4) and (P3).

To show (C2), let \( v' = v \) in (P1), and substituting in (C4), we get

\[
\int_0^v x(q,b) dq \geq \int_0^v x(q,b') dq \quad \forall b' \leq b
\]

Which implies (C2). This completes the proof of our characterization theorem for P-valid auctions.
**Definition 2.** For bidder $i$, we call the function $x(v, b)$ a **valid allocation function** if it satisfies conditions (F0), (C1-3) of Theorem 1. We call the function $p(v, b)$ a **valid payment function** if it satisfies condition (C4) of Theorem 1 for some valid allocation $x(v, b)$.

### 3.1 Alternate Formulation of Optimal Program

As in Myerson [7], we can simplify the optimal program by writing the objective in terms of virtual valuations.

**Definition 3.** Define bidder $i$’s **virtual valuation function** as

$$
\phi_i(v, b) = v - \frac{1 - F_i(v)}{f_i(v)}
$$

where $F_i(v) = \int_0^v f_i(q) dq$. (Recall that we factor the density function of the bidder’s valuation/budget distribution as $f(v, b) = g_b(b) f(v|b)$).

**Theorem 2 (Revenue Equivalence).** In any valid auction, the expected revenue

$$
\sum_{i=1}^n \int_{\mathcal{F}} p_i(\tilde{t}) f(\tilde{t}) d\tilde{t} = \sum_{i=1}^n \int_{\mathcal{F}} x_i(\tilde{t}) \phi_i(v_i, b_i) f(\tilde{t}) d\tilde{t}
$$

where $\phi_i(v, b)$ is bidder $i$’s virtual valuation function.

**Proof.** Define $f(t_{-i}) = \prod_{j \neq i} f(v_j, b_j)$. Redefine

$$
p_i(v, b) = \int_{t_{-i}} p_i((v, b), t_{-i}) f(t_{-i}) dt_{-i},
$$

$$
x_i(v, b) = \int_{t_{-i}} x_i((v, b), t_{-i}) f(t_{-i}) dt_{-i}
$$

Regardless of the version of incentive compatibility we choose, we get that for each fixed $t_{-i}$, $p_i(v, b, t_{-i}) = p_i((v, b), t_{-i})$ and $x_i(v, b, t_{-i})$ satisfies condition (C4) in Theorem 1, which by linearity implies that $p(v, b)$ and $x(v, b)$ also satisfies (C4). Hence,

$$
\int_{\mathcal{F}} p_i(\tilde{t}) f(\tilde{t}) d\tilde{t} = \int_{\mathcal{F}} \int_{t_{-i}} p(v, b) f_i(v|b) g_b(b) dv db
$$

$$
= \int_{\mathcal{F}} \int_{b_0} \int_{t_{-i}} x(v, b) f_i(v|b) g_b(b) dv db = \int_{\mathcal{F}} \int_{b_0} \int_{t_{-i}} x(v, b) f_i(v|b) g_b(b) dv db
$$

$$
= \int_{\mathcal{F}} \int_{b_0} \int_{t_{-i}} x(v, b) f_i(v|b) g_b(b) dv db = \int_{\mathcal{F}} x_i(\tilde{t}) \phi_i(v_i, b_i) f(\tilde{t}) d\tilde{t}
$$

Summing, we get the desired result. ☐

The optimal program for the case when bidders cannot over-report budgets now becomes the program (OPT2) in Figure 2.

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### 4. APPROXIMATELY OPTIMAL AUCTION

#### 4.1 LP Rounding and Lagrangian

Currently, the feasibility constraint in (OPT2) (Figure 2) has combinatorially many constraints. To make the program tractable, we define new variables

$$
x_i(v, b) = \int_{t_{-i}} x_i((v, b), t_{-i}) f(t_{-i}) dt_{-i},
$$

$$
p_i(v, b) = \int_{t_{-i}} p_i((v, b), t_{-i}) f(t_{-i}) dt_{-i}
$$

**Figure 2: The optimal program (OPT2).**

Also we relax the feasibility constraints and only require that we allocate at most $M$ items in expectation. The relaxed program is in Figure 3.

**Figure 3: The relaxed program (LP).**

The optimal objective to (LP) upper-bounds the optimal revenue, because for any feasible setting $x_i(\tilde{t})$ and $p_i(\tilde{t})$ to the original exponentially sized LP, we can define $x_i(v, b)$ and $p_i(v, b)$ as above, and they are feasible for (LP).

We multiply the feasibility constraint with Lagrange multiplier $\lambda \geq 0$ and move it into the objective, yielding the Lagrangian program in Figure 4.

**Figure 4: The Lagrangian program (L(\lambda)).**

Because $\forall \lambda > 0$, a feasible setting of $\bar{x}$ in the original program (LP) is feasible in the Lagrangian program ($L(\lambda)$), with the objectives equal, we have $\forall \lambda > 0$, the optimal objective $L(\lambda) \geq LP$. They are equal when some $\lambda$ induces an optimal solution with the feasibility constraint tight.

Note that the Lagrangian program can be broken into $n$ separate maximization programs. For $\lambda > 0$ define the program ($L_i(\lambda)$) in Figure 5.

**Figure 5: The Lagrangian program ($L_i(\lambda)$).**

Because $\forall \lambda > 0$, a feasible setting of $\bar{x}$ in the original program (LP) is feasible in the Lagrangian program ($L(\lambda)$), with the objectives equal, we have $\forall \lambda > 0$, the optimal objective $L(\lambda) \geq LP$. They are equal when some $\lambda$ induces an optimal solution with the feasibility constraint tight.
Maximize \( \int_{v,b} x_i(v,b)(\phi_i(v,b) - \lambda)f_i(v,b)dvdb \) \( (L_\lambda) \)

Such that \( \forall i \), the function \( x_i(v,b) \) is a valid allocation (see Definition 2).

**Figure 5: The single bidder program \( (L_\lambda) \)**

\[
x^*_\lambda(x, \cdot) = \text{argmin}_{x(\cdot)} \{ \int_{v,b} x(v,b)f_i(v,b)dvdb \mid x(\cdot) \text{ maximizes } L_i(\lambda) \}
\]

\[
G_\lambda(x) = \int_{v,b} x^*_\lambda(v,b)f_i(v,b)dvdb
\]

\[
\phi_{max} = \sup_{v,b} \{ \phi_i(v,b) \}
\]

**Lemma 1.** \( G_\lambda(x) \) is monotonically non-increasing in \( \lambda \).

**Proof.** For some \( \lambda_1, \lambda_2 \), let \( x^*_\lambda_1 \) and \( x^*_\lambda_2 \) be the optimal solution of programs \( (L_\lambda(1)) \) and \( (L_\lambda(2)) \) respectively, as defined before. We have,

\[
\text{Lemma 2. } \forall i, G_i(\phi_{max}) = 0.
\]

**Proof.** This follows from the observation that when \( \lambda \geq \phi_{max}, \) each term in the integral defining \( L_i(\lambda) \) is negative, and since \( x \) is non-negative, this objective is maximized when \( x(v,b) = 0 \). It is easy to check that this is a valid allocation.

**Theorem 3.** (Approximately solving \( (LP) \)). Define \( G(\lambda) = \sum_i G_i(\lambda) \). If \( G(0) < M \), then the program \( (LP) \) is solved by setting \( x_i = x^*_0 \). Otherwise \( \epsilon > 0 \), we can use binary search to find \( \lambda^- < \lambda^+ \) such that

- \( G(\lambda^-) \geq M \)
- \( G(\lambda^+) < M \)
- \( \lambda^+ - \lambda^- \leq \epsilon \phi_{max} \)

Set

\[
x_i = \frac{M - G(\lambda^+)}{G(\lambda^-) - G(\lambda^+)} x^*_\lambda^- + \frac{G(\lambda^-) - M}{G(\lambda^-) - G(\lambda^+)} x^*_\lambda^+
\]

Then \( x \) satisfies \( (LP) \) and achieves at least \( 1 - \epsilon \) of the optimal objective.

Hence, solving \( (LP) \) to any degree of accuracy becomes practical because we can precompute \( x^*_\lambda \) for each bidder for various \( \lambda \)'s, which can be done by discretizing \( x \), in time polynomial in the degree of discretization. The corresponding \( G_r(\lambda) \) yields an “expected demand curve” for each bidder. At the time of the auction, the auctioneer can simply look up precomputed values of \( G_r(\lambda) \) and binary search to find a parameter \( \lambda \) that matches total expected demand with supply. This along with the precomputed values determine a near-optimal solution to \( (LP) \).

### 4.2 Making the Relaxed LP Solution Feasible

Using our characterization of valid allocation functions (Definition 2), we show several ways of making arbitrary combinations of valid single bidder allocations also satisfy the true feasibility constraint \( x(t) \leq M \forall t \in T \), while remaining valid. The main tool is the following theorem.

**Theorem 4.** Given an allocation function \( x(v,b) \) valid for bidder \( i \) (Definition 2), and a non-negative, non-decreasing concave function \( h(x) \leq x \), the allocation function \( x'(v,b) = h(x(v,b)) \) remains valid for bidder \( i \).

**Proof.** It suffices to check that \( x' \) satisfies (P0), (C1-3) in Theorem 1.

(P0) is satisfied because \( h \) is non-negative and \( h(x) \leq h(s_i) \leq s_i \forall x \leq s_i \).

(C1) is satisfied because \( x(v,b) \) satisfies (C1) and \( h \) is non-decreasing.

To show (C2), for any \( b_1 \leq b_2 \), define \( y_1(v) = s_i - x(v,b_1), y_2(v) = s_i - x(v,b_2) \). Then \( y_1 \) and \( y_2 \) are both non-decreasing. Define \( h' = s - h(s - x) \), then \( h' \) is non-decreasing and convex. Moreover, \( \forall v \leq s_i \), because \( x \) satisfies (C2),

\[
\int_0^v y_1(q) dq - \int_0^v y_2(q) dq = \int_0^v x'(q,b_2) dq - \int_0^v x'(q,b_1) dq \geq 0
\]

We use the following variant of Karamata’s inequality, proven in Appendix A.

**Lemma 3.** Given functions \( f_1, f_2 : [0,t] \rightarrow \mathbb{R} \) such that they are both non-increasing and \( \forall 0 \leq s \leq t \)

\[
\int_0^s f_1(x) dx \geq \int_0^s f_2(x) dx
\]

Then if \( g \) is a non-decreasing convex function, we have

\[
\int_0^s g(f_1(x)) dx \geq \int_0^s g(f_2(x)) dx
\]

Applying this lemma with \( f_1 = y_1, f_2 = y_2, \) and \( g = h' \), we have \( \forall v \)

\[
\int_0^v s_i - h(x_1(q)) dq = \int_0^v h'(y_1(q)) dq \geq \int_0^v s_i - h(x_2(q)) dq
\]
And this is equivalent to
\[
\int_0^\omega h(x'(q, b_2))dq \geq \int_0^\omega h(x'(q, b_1))dq
\]
Which is (C3).

To show (C3), note that
\[
\begin{align*}
v_x'(v, b) - \int_0^\omega x'(q, b)dq &= \int_0^\omega h(x'(q, b)) - h(x(q, b))dq \\
&\leq \int_0^\omega h(x'(q, b) - x(q, b)) - h(0)dq \\
&\leq \int_0^\omega x(v, b) - x(q, b) dq \\
&\leq \int_0^\omega v_x(v, b) - x(q, b) dq \\
&= b
\end{align*}
\]
Since \(x\) satisfies (C3). \(\square\)

**Corollary 1.** For bidder \(i\), if \(x(v, b)\) is a valid allocation function (Definition 2), then for \(0 \leq \alpha \leq 1\), the following are also valid allocation functions:

1. \(x'(v, b) = x(v, b) \min(\alpha, \frac{M}{k + x(v, b)})\).
2. \(x'(v, b) = x(v, b) \min(\alpha, \frac{M}{k + x(v, b)})\).
3. \(x'(v, b) = \min(\alpha x(v, b), M - k)\)

This leads to several natural ways of implementing a feasible auction given the LP solution. Each of our auctions has a parameter \(\alpha \leq 1\) to be specified later.

**Definition 4. [Sharing Auctions]** Define the proportional-sharing auction with parameter \(\alpha\) as follows. First solve the program (LP) efficiently as in Section 4.1, yielding (near) optimal solution \(\{x_i'(v, b)\}\). Then take in bidders’ reports \(\bar{t}_i\), and allocate to bidder \(i\)
\[
x_i(\bar{t}) = x_i'(v, b_1) \min(\alpha, \frac{M}{\sum_j x_j'(v, b_j)})
\]
Define the conservative-sharing auction similarly, except allocate
\[
x_i(\bar{t}) = x_i'(v, b_1) \min(\alpha, \frac{M}{\sum_{j \neq i} x_j'(v, b_j)})
\]
In both cases, charge \(p_i(\bar{t}) = v_i x_i(\bar{t}) - \int_0^\omega x_i((q, b_i), t_i) dq\).

**Definition 5. [Queueing Auctions]** Define the fixed-queueing auction with parameter \(\alpha\) as follows. First solve the program (LP) efficiently as in Section 4.1, yielding (near) optimal solution \(\{x_i'(v, b)\}\). Then sort the bidders according to some permutation \(\pi\), and allocate to bidder \(i\),
\[
x_i(\bar{t}) = \min(\alpha x_i'(v, b_i), M - \sum_{\pi(j) < \pi(i)} x_j'(v, b_j))
\]
And charge \(p_i(\bar{t}) = v_i x_i(\bar{t}) - \int_0^\omega x_i((q, b_i), t_i) dq\).

Define the random-queueing auction as the randomized mechanism of executing fixed-queueing with a uniformly random permutation.

**Theorem 5.** The proportional and conservative-sharing auctions, as well as the fixed and random-queueing auctions, are all valid auctions (Definition 1), satisfying (P-IC), (IR), (BF), (NPT) and (F) ex-post.

**Proof.** Let the (near) optimal solution to (LP) be \(\{x_i^*(v, b)\}\).
We first show that the sharing and fixed-queueing auctions are valid.

For each of the three auctions and each \(i\), fix \(t_i\), and define \(x(v, b) = x_i((v, b), t_i)\). For the sharing auctions, \(x\) is exactly functions 1 and 2 respectively in Corollary 1 with \(k = \sum x_i^*(v, b_i)\). For the fixed-queueing auction, \(x\) is function 3 in Corollary 1 with \(k = \sum \pi(j) < \pi(i)x_j\). In each case, when \(t_i\) is fixed \(k\) is fixed. By Corollary 1, the allocation functions are valid (in the sense of Definition 2), which implies by Theorem 1 that the auctions satisfy (P-IC), (IR), (BF), (NPT) and (F) ex-post. It remains to show that these auctions satisfy (F1).

Proportional-sharing satisfies (F1) because in
\[
\sum_i x_i(\bar{t}) \leq \sum_i x_i^*(v, b) \sum_i x_i'(v, b) = M
\]
Conservative-sharing satisfies (F1) because in each case and for each bidder it allocates no more than in proportional-sharing, since \(x_i'(v, b) \leq x_i^*(v, b)\) by requirement.

Fixed-queuing satisfies (F1) by definition.

Hence, the sharing auctions and fixed-queueing are all valid auctions (Definition 1). By claim 1, random-queueing is also a valid auction. \(\square\)

**4.3 Proving Revenue Near-Optimality**

We now prove constant competitive ratios for the auctions we defined in Section 4.2.

We will use the following probabilistic bounds, both of which are proven in Appendix A.

**Lemma 4.** (Cantelli’s Inequality). If \(Y \geq 0\) is a random variable with \(E[|R|] \leq \alpha\) and \(\text{Var}[Y] \leq V\), then \(\forall \beta \geq \alpha\),
\[
Pr[Y \geq \beta] \leq \frac{V}{(\beta - \alpha)^2 + V}
\]
**Lemma 5.** If \(Y \geq 0\) is a random variable with \(Pr[Y \leq g(b)] \leq f(b)\) \(\forall 0 \leq b \leq \infty\) where \(g\) is a monotonically non-increasing function with \(g(0) = Y_{\text{max}} = \sup Y\), \(g(\infty) = 0\), and \(f\) is a continuous function, then
\[
E[Y] \geq Y_{\text{max}} + \int_0^\infty f(b)dg(b)
\]
where we use Riemann-Stieljes integration.

**Theorem 6.** The conservative-sharing auction with parameter \(\alpha = 1/2\) achieves at least \(1 - \ln 2 \geq \frac{1}{12}\) of the optimal revenue.
where

For convenience, denote \( r_{i,t} = \{ \begin{array}{ll} \frac{1}{\alpha + b} & \text{if } k_i \leq 1 - \alpha \\ \frac{k_i}{\alpha + b} & \text{otherwise} \end{array} \) for all \( t, i \). Define the functions

\[ g(b) = \begin{cases} \frac{1}{\alpha + b} & \text{for } 0 \leq b \leq 1 - \alpha \\ \frac{k_i}{\alpha + b} & \text{otherwise} \end{cases} \]

and

\[ f(b) = \begin{cases} \frac{1}{\alpha + b} & \text{for } 0 \leq b \leq 1 - \alpha \\ \frac{k_i}{\alpha + b} & \text{otherwise} \end{cases} \]

Note that \( g(0) = 1 = r_{\max} = \sup\{r\}, g(\infty) = 0 \), \( g \) is monotonically non-increasing, and \( f \) is continuous.

For \( 0 \leq b \leq 1 - \alpha \), \( g(b) = 1 \), so by Markov’s inequality,

\[ Pr\{r \leq g(b)\} = Pr\{k_i \geq 1 - \alpha\} \leq \frac{E[k_i]}{1 - \alpha} = f(b) \]

Similarly for \( b \geq 1 - \alpha \)

\[ Pr\{r \leq g(b)\} = Pr\{k_i \geq b\} \leq \frac{E[k_i]}{b} = f(b) \]

Hence, we can apply Lemma 5,

\[ E[r] \geq 1 + \int_0^\infty f(b)db = 1 + \int_0^{1-\alpha} \frac{1}{\alpha + b}db = 1 + \frac{\alpha}{\alpha + b} \bigg|_0^{1-\alpha} = 2 + \frac{2}{\alpha} \ln(1 - \alpha) \]

\[ \square \]

Using this lemma and plugging in \( \alpha = 1/2 \) we get that the total expected revenue,

\[ \sum_i R_i \geq (2\alpha + \ln(1 - \alpha)) \sum_i R_i^* \geq (1 - \ln(2)) R_{\text{opt}} \]

where \( R_{\text{opt}} \) is the optimum revenue.

**Theorem 7.** The proportional-sharing auction with \( \alpha = \frac{3}{2} - \sqrt{2} \) achieves at least \( \frac{1}{108} \) of the optimal revenue.

**Proof.** The proof is completely analogous to the proof of Theorem 6.

Let \( \bar{x}(v, b) \) be the optimal solution to the relaxed program (LP) and let \( p_i(v, b) \) be the corresponding payment functions. Analogous to before, define \( k_i = \alpha \frac{1}{M} \sum_{j \neq i} x_j(v_j, b_j) \),

\[ r_{i,t} = \{ \begin{array}{ll} \frac{1}{\alpha + b} & \text{if } k_i \leq 1 - \alpha \\ \frac{k_i}{\alpha + b} & \text{otherwise} \end{array} \] for all \( t, i \). Define the functions

\[ g(b) = \begin{cases} \frac{1}{\alpha + b} & \text{for } 0 \leq b \leq 1 - \alpha \\ \frac{k_i}{\alpha + b} & \text{otherwise} \end{cases} \]

and

\[ f(b) = \begin{cases} \frac{1}{\alpha + b} & \text{for } 0 \leq b \leq 1 - \alpha \\ \frac{k_i}{\alpha + b} & \text{otherwise} \end{cases} \]

Note that \( g(0) = 1 = r_{\max} = \sup\{r\}, g(\infty) = 0 \), \( g \) is monotonically non-increasing, and \( f \) is continuous.

For \( 0 \leq b \leq 1 - \alpha \), \( g(b) = 1 \), so by Markov’s inequality,

\[ Pr\{r \leq g(b)\} = Pr\{k_i \geq 1 - \alpha\} \leq \frac{E[k_i]}{1 - \alpha} = f(b) \]

Similarly for \( b \geq 1 - \alpha \)

\[ Pr\{r \leq g(b)\} = Pr\{k_i \geq b\} \leq \frac{E[k_i]}{b} = f(b) \]
Hence, applying Lemma 5,

\[ g(b) = \begin{cases} 
\frac{1}{b} & \text{for } b \leq 1 - \alpha \\
\frac{1}{b} & \text{otherwise}
\end{cases} \]

And

\[ f(b) = \begin{cases} 
\frac{\alpha}{1 - \alpha} & \text{for } 0 \leq b \leq 1 - \alpha \\
\frac{1}{b} & \text{otherwise}
\end{cases} \]

As in the proof of Lemma 6, it is straightforward to check that regardless of \( b \),

\[ \Pr[r \leq g(b)] \leq f(b) \]

Hence, applying Lemma 5,

\[
E[r] \geq 1 + \int_0^\infty f(b)dg(b) \\
= 1 + \int_0^\infty f(1 - \alpha)[g^-(1 - \alpha) - g^-(1 - \alpha)] + \int_0^\infty f(b)g'(b)db \\
= 1 + \frac{\alpha}{1 - \alpha}[1 - (1 - \alpha)] + \int_0^\infty \frac{\alpha}{(b\alpha + 1)^2}db \\
= 1 - \alpha^2 + \frac{\alpha}{1 - \alpha} + \frac{\alpha}{(b\alpha + 1)^2} + (b\alpha + 1)1 - \alpha \\
= -\frac{\alpha}{1 - \alpha} - \frac{\alpha}{1 - \alpha} \ln(1 - \alpha)
\]

Which proves the lemma. \( \square \)

Using \( \alpha = \frac{1 - \sqrt{3}}{3} \), we get

\[
\sum_i R_i \geq \left( -\frac{\alpha^2}{1 - \alpha} - \ln(1 - \alpha) \right) \sum_i R_i^* \geq \frac{1}{4.08}R_{\text{opt}}
\]

**Theorem 8.** In the fixed-queuing auction with \( \alpha = 1 - \frac{1}{\sqrt{3}} \), we obtain at least \( \frac{1}{3.73} \) of the optimal revenue. In random-queuing, with \( \alpha = 1 - \frac{1}{\sqrt{3}} \), we obtain \( \frac{1}{3.73} \) for the optimum.

**Proof.** Let \( \{x_i^*(v, b)\} \) be the optimal solution to (LP) (Figure 3). In both queuing auctions define

\[
k_i = \frac{1}{M} \sum_{v(j) < i} x_j^*(v_j, b_j)
\]

(In random-queuing, \( k_i \) not only depends on \( t_i \), but also on the random permutation \( \pi(\cdot) \). Define the random variable

\[ r = \begin{cases} 
1 & \text{if } k_i \leq 1 - \alpha \\
0 & \text{otherwise}
\end{cases} \]

For each \( t \) and each permutation \( \pi(\cdot) \), the payment

\[ p_i(t) \geq \alpha r p_i^*(v_i, b_i) \]

But \( t_i \) and the randomization over permutation \( \pi(\cdot) \) for random queuing are both independent of \( v_i, b_i \), so as in the proof of Theorem 6, the expected revenue,

\[ R_i \geq \alpha E[r]R_i^* \]

In fixed-queuing, \( E[k_i] \leq \alpha \), so by Markov’s inequality,

\[ E[r] = \Pr[k_i \leq 1 - \alpha] \geq 1 - \frac{E[k_i]}{1 - \alpha} \geq 1 - \frac{\alpha}{1 - \alpha} \]

In random queuing,

\[ E[k_i] = \frac{\alpha}{1 - \alpha} \sum_{j \neq i} \Pr[\pi(j) < \pi(i)]E[x_j^*] \]

\[ \leq \frac{\alpha}{1 - \alpha} \sum_{j \neq i} E[x_j^*] \]

Hence, Markov’s inequality yields the improved bound

\[ E[r] \geq 1 - \frac{\alpha}{2(1 - \alpha)} \]

Plugging this into the revenue bound, for the fixed-queuing case, with \( \alpha = 1 - \frac{1}{\sqrt{3}} \), we get a 5.83 approximation; for random-queuing, we get a 3.73 approximation. \( \square \)

In [3] and [1], it is shown that when each bidder’s relative purchasing power decreases, their auctions become asymptotically optimal. We show a result of a similar flavor here.

**Theorem 9.** All \( \ell \) auctions—consecutive and proportional sharing, fixed and random queuing—are asymptotically optimal when each bidder wants only a small proportion of the goods: If \( s_i \leq \epsilon M v_i \), the auctions with parameter \( \alpha = 1 - \epsilon^{1/3} - \epsilon \) achieve \( 1 - 2\epsilon^{1/3} \) of the optimal revenue.

**Proof.** Let \( \{x_i^*(v, b)\} \) be the optimal solution to (LP) (Figure 3). Define \( k_i = \frac{\alpha}{M} \sum_{j \neq i} v_j^* \). Note that \( k_i \) is independent of bidder \( i \)th type \( t_i = \{v_i, b_i\} \). Moreover, in all four auctions, one can check that if \( k_i \leq 1 - \epsilon \), then \( x_i(t) = x_i^*(v_i, b_i) \forall v_i, b_i \). By condition (C4) in Theorem 1, we have in these cases \( p_i(t) = p_i^*(t) \). Lower-bounding the payments in all other cases by 0, we get that the expected revenue from \( i \),

\[ R_i \geq \alpha \Pr[k_i \leq 1 - \epsilon]R_i^* \]

Now, by independence

\[ \text{Var}[k_i] \leq E[k_i^2] = \frac{\alpha^2}{M^2} \sum_{j \neq i} E[(x_j^*)^2] \]

\[ \leq \frac{\alpha^2}{M^2} \sum_{j \neq i} E[x_j^*] \]

\[ \leq \frac{\alpha^2}{M^2} \sum_{j \neq i} E[x_j^*] \]

We now apply Lemma 4 on \( k_i \) and get

\[ \Pr[k_i \geq 1 - \epsilon] \leq \frac{\alpha^2}{\epsilon} \leq \frac{\alpha^2}{\epsilon} \]

\[ \leq \frac{\epsilon^{1/3}}{\epsilon^{1/3}} \]

Hence, we get that \( (\sum_i R_i)/R_{\text{opt}} \) is at least

\[ \alpha \Pr[k_i \leq 1 - \epsilon] \geq (1 - \epsilon - \epsilon^{1/3})(1 - \epsilon^{1/3}) \geq 1 - 2\epsilon^{1/3} \]

\( \square \)
Remark 1. Using the variance bound in the proof of Theorem 9 in the previous analysis improves the general approximation ratio slightly for all four auctions. Because the improvements are small and the analyses become much messier, the proofs are omitted.

5. NEXT STEPS

This is only an interim report, and much more remains to be done. In particular, we want to

- Given the characterization theorem, perhaps we can show under some regularity conditions what the optimal auction is, or perhaps an auction with an approximation ratio under 2.

- We can solve for the optimal auction in small cases. We want to implement this and gain intuition.

- Although the current auctions are reasonably implementable, there is still much precomputation that needs to be done. Perhaps there is a simpler, computationally less-intensive auction that achieves similar approximation ratios.

- The case when users can report arbitrary budgets need to be worked out. This case is not as nice because Claim 1 and Theorem 4 no longer hold. Currently, we have a characterization of the single-bidder Lagrangian solution as a single payment function that allocates a certain amount when user pays a certain amount (much like a posted-price auction). This result follows from [4]. Using this, the analysis of fixed-queueing auction goes through and achieves a 4.98 approximation. Can we do better?

6. REFERENCES


APPENDIX

A. OMITTED PROOFS

Lemma 8 (Lemma 3). Given functions $f_1, f_2 : [0, t] \rightarrow \mathbb{R}$ s.t. they are both non-increasing and $0 \leq s \leq t$

$$\int_0^s f_1(x)dx \geq \int_0^s f_2(x)dx$$

Then if $g$ is a non-decreasing convex function, we have

$$\int_0^s g(f_1(x))dx \geq \int_0^s g(f_2(x))dx$$

Proof of Lemma 3. This is similar to Karamata’s inequality, proofs of which can be found in [10].

We use the following easy-to-check, standard lemma:

Lemma 9. For any convex function $g$, define the divided difference

$$\Delta g(a, b) = \begin{cases} \frac{g(a) - g(b)}{a - b} & \text{if } a \neq b \\ 0 & \text{otherwise} \end{cases}$$

Then $\Delta g$ is non-decreasing in both $a$ and $b$.

Define $c(x) = \Delta g(f_1(x), f_2(x))$, then by Lemma 9 and the non-increasing properties of $f_1$ and $f_2$, $c$ is non-increasing. Moreover, since $g$ is also non-decreasing, $c(x) \geq 0 \forall x$. Define $F_j(x) = \int_0^x f_j(q)dq$ for $j \in \{1, 2\}$. For any $s \leq t$, we have

$$\int_0^s c(x)f_1(x)dx - \int_0^s c(x)f_2(x)dx$$

$$= \int_0^s c(x)(f_1(x) - f_2(x))dx$$

$$= \int_0^s (F_1 - F_2)(f_1(x) - f_2(x))dx$$

$$\geq 0$$

where the last inequality follows from $F_1(0) = F_2(0) = 0$, $F_1(s) \geq F_2(s)$, $c(s) \geq 0$, and $(F_2 - F_1)dc(x) \geq 0$. \hfill \Box  

Lemma 10 (Lemma 4). If $Y \geq 0$ is a random variable with $E[R] \leq a$ and $Var[Y] \leq V$. Then $\forall b \geq a$,

$$Pr[Y \geq b] \leq \frac{V}{(b-a)^2 + V}$$
PROOF OF LEMMA 4. Let $p = Pr[Y \geq b]$. Define $a' = E[Y] \leq a$, $a_1 = E[Y|Y < b]$, $a_2 = E[Y|Y \geq b]$. Define the random variable

$$X = \begin{cases} a_1 & \text{if } Y < b \\ a_2 & \text{otherwise} \end{cases}$$

Note that $E[X] = E[Y] = a'$, and


Hence, we have $a_2 \geq b \geq a_1$,

$$\begin{align*}
1 - p) a_1 + p a_2 &= a' \\
1 - p) a_1^2 + p a_2^2 &\leq V + a'^2
\end{align*}$$

where the second inequality follows from the identity $E[x^2] = Var[x] + (E[x])^2$.

Substituting the first equation into the second and simplifying, we get

$$p(1 - p)(a_1 - a_2)^2 \leq V$$

But since $a_2 \geq b$,

$$a_2 - a_1 \geq b - a' - pb \geq b - a \frac{1 - p}{1 - p}$$

Substituting this into the above, we get the desired bound

$$p \leq \frac{V}{(b - a)^2 + V}$$

\[\square\]

LEMMA 11 (LEMMA 5). If $Y \geq 0$ is a random variable with

$$Pr[Y \leq g(b)] \leq f(b) \quad \forall 0 \leq b < \infty$$

where $g$ is a monotonically non-increasing function with $g(0) = Y_{\text{max}} = \sup(Y)$, $g(\infty) = 0$, and $f$ is a continuous function. Then

$$E[Y] \geq Y_{\text{max}} + \int_0^\infty f(b)dg(b)$$

where we use Riemann-Stieljes integration.

\[\square\]

PROOF OF LEMMA 5. The Riemann-Stieljes integral is defined because $g$ is monotonic and $f$ is continuous. The desired bound follows from simple manipulation and integration by parts:

$$\begin{align*}
E[Y] &= \int_0^{Y_{\text{max}}} Pr[Y \geq x]dx \\
&= -\int_0^{\infty} Pr[Y \geq g(b)]dg(b) \quad (\text{changing variable}) \\
&\geq -\int_0^{\infty} (1 - f(b))dg(b) \quad (\text{using our bound}) \\
&= -g(b) \bigg|_0^{\infty} + \int_0^{\infty} f(b)dg(b) \\
&= Y_{\text{max}} + \int_0^{\infty} f(b)dg(b)
\end{align*}$$

\[\square\]