Geometric Charaterization of Proper Scoring Rules and Hanson Market Makers

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ABSTRACT

One problem in implementing a prediction market is providing liquidity, and currently the most widely used approach is a type of market makers designed by Robin Hanson. In such a market, for each event to be predicted, there is a security corresponding to each possible outcome, and each security pays off \$1 if the corresponding outcome occurs. Hanson's market maker keeps a record of the total quantity of each security purchased to date, and defines some cost function C on the vector of quatities. To purchase some bundle of securities, the speculator pays the difference of the cost function. Once the event comes to pass, the market principal rewards the winning securities. One nice property of Hanson market makers is that the principal can bound her total loss, and also define a cost function based on any proper scoring rule. But defining a cost function based on a proper scoring rule can be a tedious process, and the relationship between Hanson market makers and proper scoring rules as presented in literature is complicated, involving computing the integral of multivariable price function. In this paper, we a nice geometric connection between proper scoring rules and Hanson market makers, based on a previously known characterization of all proper scoring rules in terms of some convex score function G. We show that in fact G and C are convex conjugates of each other. This illucidats the correspondence between scoring rules and Hanson market makers, and adds intuition to Hanson market makers. To the best of our knowledge, this is the first statement of the direct relationship between G and C.

1. INTRODUCTION

Proper scoring rules provide a method of eliciting a prediction, by paying an agent a certain sum depending on the agent's prediction and the the outcome. They are extensively used to elicit forecasts [8, 9, 12] and to compute statistical evaluations [1, 2].

Hanson demonstrated that proper scoring rules can be used to define automated market makers to provide liquidity in prediction markets [4, 3]. Given any proper scoring rule on a set of mutually exclusive outcomes, Hanson shows how to define market maker functions m_i for each outcome, as extensions of inverses of scoring rules. In such a market, for each event to be predicted, there is a security corresponding to each possible outcome, and each security pays off \$1 if the corresponding outcome occurs. Hanson's market maker keeps a record of the total quantity of each security purchased to date, and uses a cost function C defined by integrating the market maker functions m_i. To purchase some bundle of securities, the speculator pays the difference of the cost function. Once the event comes to pass, the market principal rewards the winning securities. Based on properties of proper scoring rules, one can show that it is in each agent's interest to move prices to the true underlying probabilities. Using the logarithmic scoring rule, this approach defines the Logarithmic Market Scoring Rule (LMSR), which is currently used as the market maker at InklingMarkets, Net Exchange, Wisdeom Hive, and Microsoft.

The current literature presents the relationship between proper scoring rules and Hanson market makers in a complicated and convoluted way. Using a known characterization of proper scoring rules in terms of an arbitrary convex functions G, we show a direct 1-1 correspondance between cost function C and G. In fact, C is the *convex conjugate* of G. This result adds geometric intuition to Hanson market makers.

In Section 2, we review definitions of proper scoring rules and present the geometric characterization, parts of which are due to Savage [11], McCarthy [7], Hendrickson and Buehler [5], and Gneiting and Raftery [2]. In Section 3.2, we show the 1 to 1 correspondance with Hanson market makers, using concepts from convexity theory.

2. REVIEW OF GENERAL THEORY

We first present the general theory of proper scoring rules and a characterization theorem, which was discovered at first by Savage [11] but the version shown here is due to Gneiting and Rafetery [2]. While proper scoring rules can be defined to elicit any statistical property [6], we will focus on proper scoring rules which elicit the whole distribution, represented

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by a probability for each possible outcome. None of the content in this section are new results, but some are presented in a new way.

2.1 Review of Convexity Theory

We review general theories of convex real-valued functions in R^n, as well as notions of subgradient vectors and subtangent hyperplanes. In the following, let U denotes an open domain $U \in \mathbb{R}^n$.

Definition 1. A function $G : U \to \mathbb{R}$ is *convex* if $\forall x, y \in U$, and $\forall \lambda \in (0, 1)$,

$$G(\lambda x + (1-\lambda)y) \leq \lambda G(x) + (1-\lambda)G(y)$$

G is called *strictly convex* if the inequality is strict.

One nice property of convex functions is that the supremum of a family of convex functions is still convex.

THEOREM 1. *Given a (possibly infinite) family J of convex functions defined on domain U, define the function $G : U \to \mathbb{R}$ as $G(x) = \sup_{G' \in J} G'(x)$, then G is convex.*

PROOF.

$$\begin{aligned} & G(\lambda x + (1-\lambda)y) \\ = \ & \sup_{G' \in J} G'(\lambda x + (1-\lambda)y) \\ \leq \ & \sup_{G' \in J} (\lambda G'(x) + (1-\lambda)G'(y)) \\ \leq \ & \sup_{G' \in J} \lambda G'(x) + \sup_{G' \in J} (1-\lambda)G'(y) \\ = \ & \lambda G(x) + (1-\lambda)G(y) \end{aligned}$$

□

By Definition 1, the domain U of a convex function is necessarily a convex set (meaning that if $x, y \in U$, the line segment from x to y is also in U.) On open domains, convex functions are continuous everywhere.

THEOREM 2. *A convex function G defined on an open domain U is everywhere continuous.*

Definition 2. Given convex function $G : U \to \mathbb{R}$, a *subgradient* is a vector function $\nabla^* G : U \to \mathbb{R}^n$ such that $\forall x, y \in U$,

$$G(y) \geq G(x) + \nabla^* G(x) \cdot (y - x)$$

Where $\cdot$ denotes the regular dot product. $\nabla^* G$ is called a *strict subgradient* if the above inequality is strict $\forall y \neq x$.

It is well known that subgradient vectors exist for every convex function defined on open domains. If G is differentiable at $x \in U$, then the subgradient at x has to be the gradient: $\nabla^* G(x) = (\Delta G)(x)$. Otherwise there might be many choices of $\nabla^* G(x)$.

A less conventional but more intuitive term is the subtangent hyperplane, which are closely tied to subgradients. Intuitively, these are hyperplanes touching but lying entirely below the convex function.

Definition 3. Given convex function $G : U \to \mathbb{R}$, a *subtangent hyperplane* at $x \in U$ is a linear function $H_x : U \to \mathbb{R}$ such that

$$\begin{aligned} H_x(x) &= G(x) \\ H_x(y) &\leq G(y) \qquad \forall y \in U \end{aligned}$$

H_x is called a *strictly subtangent hyperplane* if the above inequality is strict $\forall y \neq x$.

Given subgradient vectors $\nabla^* G(x)$, it is straightforward to check that we can obtain a subtangent hyperplane in the following way.

LEMMA 1. *Given convex function G and (strict) subgradient vector $\nabla^* G(x)$ at point $x \in U$. Define $H_x(y) = G(x) + \nabla^* G(x) \cdot (y - x)$, then $H_x(y)$ is a (strictly) subtangent hyperplane of G at x.*

2.2 Definition of Proper Scoring Rules

Let Ω be a set of n possible disjoint outcomes; label $\Omega = \{1, 2, \cdots, n\}$. Let $\mathbb{P} = \{P \in \mathbb{R}^n : 0 < P_i < 1, \sum_{i=1}^n P_i = 1\}$ be the set of possible probabilistic distributions over the outcomes, in which every probability must be non-zero. (This condition helps us handle peculiar cases with discontinuities at the boundary, and makes the ensuing math much nicer. After building our theory on distributions with full support, we can handle the edge cases by taking limits.) Define the standard basis $B(i) \in \mathbb{P}$, $B_i(i) = 1$, $B_j(i) = 0 \ \forall j \neq i$.

Definition 4. A *scoring rule* is a function $S : \mathbb{P} \times \Omega \to \mathbb{R}$. Given a scoring rule, we can define the expected payment function $\tilde{S} : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ as

$$\tilde{S}(P, Q) = \sum_{i=1}^{n} S(P, i) Q_i$$

A scoring rule can be thought of as a reward system in which we ask an agent to predict the distribution P of outcomes in Ω, and we reward the agent $S(P, i)$ if outcome i occurs. Since the outcomes are disjoint the agent's expected utility assuming the true distribution is $Q \in \mathbb{P}$ is $\sum_{i=1} S(P, i) Q_i = \tilde{S}(P, Q)$.

We want to use the scoring rule to incentivize the agent to report her best estimate of the true distribution Q.

Definition 5. A scoring rule $S : \mathbb{P} \times \Omega \to \bar{\mathbb{R}}$ is *proper* if $\forall Q \in \mathbb{P}$,

$$\tilde{S}(Q, Q) \geq \tilde{S}(P, Q) \qquad \forall P \in \mathbb{P}$$

It is *strictly proper* if equality occurs iff $Q = P$.

2.3 General Characterization of Proper Scoring Rules

Essentially, every proper scoring rule is equivalent to a family of subtangent hyperplanes to some convex function. This general, geometric characterization is due to Gneiting and Raftery [2].

Theorem 3 (Gneiting & Rafetery). *Given a convex function $G : \mathbb{P} \to \mathbb{R}$ and a family of subtangent hyperplanes $H_P : \mathbb{P} \to \mathbb{R}$, $P \in \mathbb{P}$, then setting $S(P, i) = H_P(b(i))$ defines a proper scoring rule. Equivalently expressed in terms of subgradient vectors, if $H_P(Q) = G(P) + \nabla^* G(P) \cdot (Q - P)$,*

$$S(P, i) = G(P) - \nabla^* G(P) \cdot P_i + \nabla^* G_i(Q)$$

Conversely, every proper scoring rule can be written in the above form. The above result is true if proper is replaced by strictly *proper and subtangent by* strictly *subtangent.*

Proof. Suppose that H_P is a subtangent hyperplane (recall Definition 3), $\forall Q \in \mathbb{P}$,

$$\tilde{S}(P, Q) = \sum_{i=1}^{n} H_P(B(i))Q_i = H_P(Q)$$

(Since H_P is a linear function constrained a $n - 1$ dimensional space, it is determined by a weighted average of the n intercepts we specify.) Therefore, because H_P lies below $G(Q)$,

$$\tilde{S}(P, Q) = H_P(Q) \leq G(Q) = H_Q(P) = \tilde{S}(Q, Q)$$

Hence, S is a proper scoring rule. If H_P is strictly subtangent, then the above inequality is strict $\forall Q \neq P$ and S is strictly proper.

Conversely, suppose S is a proper scoring rule, define $G(Q) = \tilde{S}(Q, Q)$. Note that $G(Q) = \sup_{P \in \mathbb{P}} \tilde{S}(P, Q)$. For fixed P, $\tilde{S}(P, Q) = \sum_i S(P, i)Q_i$ is a linear and therefore convex function of $Q \in \mathbb{P}$. By Theorem 1, G is convex. Now define $H_P = \tilde{S}(P, Q)$. H_P is a linear function in Q, and $H_P(P) = G(P)$. Moreover, $\forall Q \in \mathbb{P}$,

$$H_P(Q) = \tilde{S}(P, Q) \leq \tilde{S}(Q, Q) = G(Q)$$

Hence, H_P is a subtangent hyperplane to G. (It is strictly subtangent if S is strict.) □

Geometric Interpretation: Given the agent's report P and the true distribution Q, the agent's expected reward is $\sum_i S(P, i)Q(i)$, which is equal to the subtangent hyperplane $H_P(Q)$ with intercepts at $B(i)$ equal to $S(P, i)$. Since $H_P(P) = G(P)$ the agent can attain expected reward of $G(P)$ if the true distribution $Q = P$. But if the true distribution is $Q \neq P$, the agent gets reward $H_P(Q) \leq G(Q)$ by reporting P, but she could have gotten $G(Q)$ if she had reported the true distribution Q.

For each proper scoring rule S, there exists unique convex function $G(P)$ which represents the maximum expected reward the agent can obtain by reporting P. Conversely, any convex function corresponds to some proper scoring rule[1]

2.4 Connections to the Financial Asset Interpretation

For one-dimensional proper scoring rules (two outcomes 0 and 1), one well-known and popular interpretation deals with buying shares of an asset which pays \$1 if the event happens, and 0 if the event does not happen. We are allowed to buy shares for various prices, and the starting price is 0. From price level p_1 to p_2, we are allowed to by $\int_{p_1}^{p_2} N(q)dq$ shares for some predefined density function N. By participating in this market, we get the lump-sum reward A. Hence, our total reward if the event does not happen, $S(p, 0) = A - \int_0^p qN(q)dq$ and our total reward if the event does happen is $S(p, 1) = A + \int_0^p (1-q)N(q)dq$. See Figure 1 for an illustration.

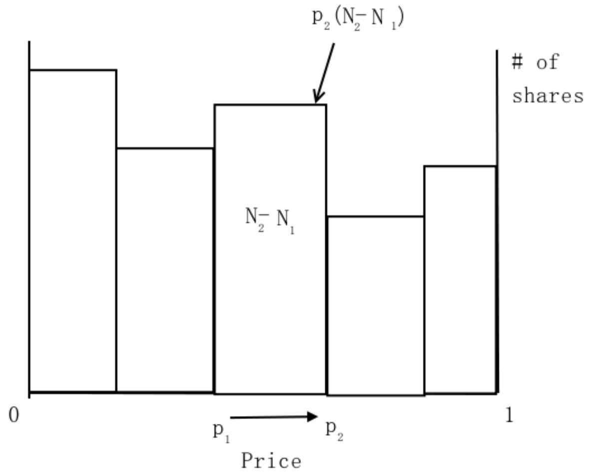

Figure 1: The financial asset interpretation of a one-dimensional proper scoring rule. Suppose that from price p_1 to p_2, we are allowed to buy $N_2 - N_1$ securities. As we move from probability p_1 to p_2, the number of shares purchased changes from N_1 to N_2. $S(p, 0)$ (A - (how much we have to pay)) goes down by $p_2(N_2 - N_1)$, and $S(p, 1)$ (reward if 1 occurs - (how much we have to pay)) goes up by $N_2 - N_1 - P_2(N_2 - N_1)$.

It is well-known that every one-dimensional proper scoring rule can be expressed in this form. Intuitively, this also explains why proper scoring rules are important for prediction markets, as the interpretation resembles such a market.

By Theorem 3, we can also interpret a one-dimensional proper scoring rule as the intercepts of subtangents to some convex function G, with the convex function being the expected reward of reporting correctly at each point. See Figure 2 for an illustration.

These two interpretations turn out to be intricately intertwined. If we interpret $N(p)$ as the slope of the convex function G, then buying shares at price p in the financial asset model corresponds to pivotting by point p in the convex function interpretation. By comparing Figures 1 and 2, one can easily check that the induced payment functions $S(p, 0)$ and $S(p, 1)$ are identical in these interpretations.

In Section 3.2, we generalize this geometric relationship between a scoring rule and a security market to allow multiple-dimensions.

[1] If G is not differentiable, then many families of subtangent hyperplanes can be specified, each of which corresponds to a proper scoring rule.

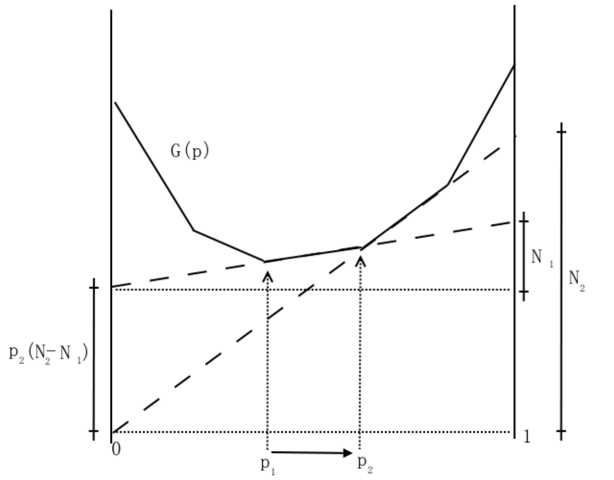

Figure 2: The convex function interpretation of a one-dimensional proper scoring rule. The slope before p_2 is N_1 and the slope after p_2 is N_2. As we move from probability p_1 to p_2, we pivot around point p_2. $S(p,0)$ (the intercept of the subtangent at 0) goes down by $p_2(N_2 - N_1)$, and $S(p,1)$ (the intercept at 1) goes up by $N_2 - N_1 - P_2(N_2 - N_1)$.

3. HANSON'S MARKET MAKERS AND CONVEX CONJUGATION

3.1 Hanson's Market Marker

Hanson's market makers provide liquidity to a prediction market with multiple agents. They have the following structure: There are $n = |\Omega|$ exhausive and mutually exclusive outcomes, each associated with its won security. The market maker keeps track of the total # of shares purchased for each security, reprsented by a vector $\vec{N}$. There is a cost function $C : \mathbb{R}^n \to \mathbb{R}$, s.t. if some agent wishes to to buy a bundle $\vec{\delta}$ of shares (δ_i is positive for buying security i, and negative for selling), the agent pays $C(\vec{N} + \vec{\delta}) - C(\vec{N})$. After the event occurs, suppose that outcome i comes true, the market maker pays \$1 for every share of security i.

If the true underlying distribution is $\vec{P}$. the agent's expected profit is $\vec{P} \cdot \vec{\delta} - (C(\vec{N} + \vec{\delta} - C(\vec{N})$. In order for this to define a reasonable market, C must satisfy two conditions

1. No simple arbitrage: Since exactly one of the n outcomes occurs, the agent can guarantee a payout of \$1 by buying 1 share of each security. Hence, we want $C(\vec{N} + \alpha\vec{1}) = \alpha + C(\vec{N})$, where $\alpha \in \mathbb{R}$ and $\vec{1}$ is an n-dimensional vector of 1's.

2. Buying a bundle does not decrease its price: This corresponds to requiring $\frac{1}{t}(C(\vec{N}+(s+t)\vec{\delta}) - C(\vec{N}+s\vec{\delta})) \geq \frac{1}{s}(C(\vec{N} + s\vec{\delta}) - C(\vec{N}))$ where $0 \leq s, t$, which is equivalent to requiring C to be *convex*.

Given such a cost function C, each will want to purchase securities to change $\vec{N}$ to maximize $\vec{P} \cdot \vec{N} - C(\vec{N})$, which is a concave function since N is convex. This suggests that once the agents communicate information through prices and agree on an equilibrium, the market will converge to some $\vec{N}^*$ at least up to an additive factor $\alpha\vec{1}$. This scheme defines a fairly reasonable automated market maker.

Hanson showed that one can construct such a C from a proper scoring rule using the following approach [10, 4, 3]. Suppose the proper scoring rule is S, let $S_i(\cdot) = S(\cdot, i)$, then this defines a vector function $\vec{S} : \mathbb{P} \to \mathbb{R}^n$. Define price function $\vec{m} : \mathbb{R}^n \to \mathbb{R}$ as the inverse of $\vec{S}$, such that $\forall \vec{P} \in \mathbb{P}$, $\vec{m}(\vec{S}(\vec{P})) = \vec{P}$, and extend this to all of $\mathbb{R}^n$ by requiring $\vec{m}(\vec{x} + \alpha\vec{1}) = \vec{m}(\vec{x})$, $\forall \vec{x}$. Hanson showed that under some regularity conditions $\vec{m}$ is well-defined. Moreover, by showing that some multi-variable integral is path-independent, Hanson showed that one can integrate $\vec{m}$ to obtain cost function C, such that $\nabla_i C = m_i$. This implies that the market maker will always price securities at marginal price $\vec{m}$ and buying a bundle costs the agent according the difference in C.

3.2 Duality of C and G

We provide an alternate, more direct proof of Hanson's result, using ideas from convexity theory. Recall from Section 2.3 that any proper scoring rule corresponds to some convex function $G : \mathbb{P} \to \mathbb{R}$. We show that we can find the cost function C by setting it to be the convex conjugate of G, defined as follows.

Definition 6. Given $X \subseteq \mathbb{R}^n$ and a function $F : X \to \mathbb{R}$. The *convex conjugate* of F is a function $F^* : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F^*(\vec{y}) = \sup\{\vec{x} \cdot \vec{y} - F(\vec{x}) | \vec{x} \in X\}$$

This is also known as the *Legendre-Fenchel transform* or the Fenchel transform.

Convex conjugation is a generalization of Legendre transforms from Physics [13]. Below is a list of the properties we will use.

FACT 1 (WELL-KNOWN). *1. The convex conjugate F^* of any function is convex.*

2. For convex function $F : \mathbb{R}^n \to \mathbb{R}$, convex conjugation is self-inversive: $F^{**} = F$.

3. For convex function F, if $\nabla^* F(\vec{x})$ is a subtangent to F at $\vec{x}$, then $\vec{x}$ is a subtangent to F^* at $\nabla^* F(\vec{x})$.

4. If convex function F is differentiable, then F^* is differentiable and $\nabla F^*(\nabla F(\vec{x})) = \vec{x}$, where ∇ is the gradient operator.

THEOREM 4. *Given a proper scoring rule corresponding to convex function $G : \mathbb{P} \to \mathbb{R}$, extend G to all of $\mathbb{R}^n$ by defining $G(\vec{P}) = \infty \ \forall \vec{P} \notin \mathbb{P}$. Then setting $C = G^*$ defines a valid market maker cost function*[2] .

[2]One can show that the cost function C defined this way is only finite for $\vec{N} \in \{S(\vec{P}) + \alpha\vec{1} | \vec{P} \in \mathbb{P}\}$. The same result holds under Hanson's original definition.

PROOF. Using item 1 of Fact 1, because G is convex, we have that C is convex.

Because G is infinity outside of $\mathbb{P}$, we have

$$C(\vec{N}) = \sup_{\vec{P}\in\mathbb{R}^n} \{\vec{N}\cdot\vec{P} - G(\vec{P})\} = \sup_{\vec{P}\in\mathbb{P}} \{\vec{N}\cdot\vec{P} - G(\vec{P})\}$$

Which implies that $C(\vec{N}+\alpha\vec{1}) = C(\vec{N}) + \alpha$. Hence, C sastifies both requirements for a valid cost function. □

In fact, our method of generating C is more general than Hanson's, because we do not assume that C is differentiable. In fact, by the self-inversive property of convex conjugation, out method also shows that any valid cost function satisfying convexity and $C(\vec{N}+\alpha\vec{1}) = C(\vec{N})+\alpha$, comes from a proper scoring rule corresponding to the function $G = C^*$. The duality relationship between C and G is one-to-one.

We now show that when C is differentiable, our method of obtaining C is equivalent to Hanson's.

THEOREM 5. *Hanson's method of generating market maker with differentiable cost function $C : \mathbb{R}^n \to \mathbb{R}$ using proper scoring rule $\vec{S} : \mathbb{P} \to \mathbb{R}^n$ is equivalent to defining $G(\vec{P})$ as expected payout $\sum_i P_i S_i(\vec{P})\ \forall \vec{P} \in \mathbb{P}$ and $G(\vec{P}) = \infty\ \forall \vec{P} \notin \mathbb{P}$, and setting $C = G^*$. (The equivalence is up to some additive constant.)*

PROOF. Note that $C = G^*$ satisfies

$$\nabla C(\vec{S}(\vec{P})) = \vec{P}$$

Using items 3 and 4 in Fact 1, and the fact from Section 2.3 that $\vec{S}$ is a subtangent hyperplane function of G.

Moreover, $C(\vec{N}+\alpha\vec{1}) = C(\vec{N})$.

Hence, using Hanson's method of inverting $\vec{S}$ to find $\vec{m}$ and integrating $\vec{m}$ to find the cost function would yield exactly cost function $C = G^*$, up to some additive constant. □

EXAMPLE 1. *We list some commonly used proper scoring rules and associated Hanson market makers.*

- *Using $G(\vec{P}) = b\sum_i P_i \log(P_i)$, we arrive at the logarithmic scoring rule $S_i(\vec{P}) = b\log(P_i)$, and the Logarithmic Market Scoring Rule (LMSR) with cost function*

$$C(\vec{N}) = b\log(\sum_i e^{N_i/b}).$$

- *Using $G(\vec{P}) = b\sum_i P_i^2$, we arrive at the quadratic or Brier scoring rule $S_i(\vec{P}) = 2b(P_i) - b\sum_j P_j^2$, and the cost function* [3]

$$C(\vec{N}) = \frac{\sum_i N_i}{n} + \frac{\sum_i N_i^2}{4b} - \frac{(\sum_i N_i)^2}{4bn} - \frac{b}{n}.$$

4. REFERENCES

[1] Stuetzle W. Buja A. and Shen Y. Loss functions for binary class probability estimation and classification: Structure and applications. manuscript, available at `http://www-stat.wharton.upenn.edu/~buja/`, 2005.

[2] Tilmann Gneiting and Adrian E. Raftery. Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102(477):359–378, March 2007.

[3] Robin Hanson. Combinatorial information market design. *Information Systems Frontiers*, 5(1):107–119, 2003.

[4] Robin Hanson. Logarithmic market scoring rules for modular combinatorial information aggregation. *Journal of Prediction Markets*, 1(1):1–15, 2007.

[5] A. Hendrickson and R. Buehler. Proper scores for probability forecasters. *The Annals of Mathematical Statistics*, 42(6):1916–1921, 1971.

[6] Nicolas Lambert, David M. Pennock, and Yoav Shoham. Eliciting properties of probability distributions: the highlights. *SIGecom Exch.*, 7(3):1–5, 2008.

[7] J McCarthy. Measures of the value of information. In *Proceedings of the National Academy of Sciences*, volume 42, pages 654–655, 1956.

[8] A. Murphy and R. Winkler. Probability forecasting in meterology. *Journal of the American Statistical Association*, 79(387):489–500, 1984.

[9] R. Nelson and D. Bessler. Subjective probabilities and scoring rules: Experimental evidence. *American Journal of Agricultural Economics*, 71(2):363–369, 1989.

[10] David M. Pennock and Rahul Sami. *Computational Aspects of Prediction Markets*, chapter 26, pages 651–674. Cambridge University Press, 2007.

[11] Leonard J. Savage. Elicitation of personal probabilities and expectations. *Journal of the American Statistical Association*, 66(336):783–801, 1971.

[12] D. Spiegelhalter. Probabilistic prediction in patient management and clinical trials. *Stat Med*, 5(5):421–433, 1986.

[13] Hugo Touchette. Legendre-fenchel transforms in a nutshell. Available at `www.maths.qmw.ac.uk/~ht/archive/lfth2.pdf`.

[3] Because the subgradient function to G can only take values of $2b\vec{P}$, the quadratic market maker is limited in that $\vec{N}$ must be constrained in the range $\{2b\vec{P}+\alpha vec1 | \vec{P} \in \mathbb{P}\alpha \in \mathbb{R}\}$. The market maker cannot accept any orders that takes $\vec{N}$ outside of this range, and this is one reason the logarithmic market maker is preferred in practice.