Revenue Maximizing Auction when Bidders have Private Budgets

Interim Report

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ABSTRACT

We tackle the problem of designing revenue maximizing auctions in the Bayesian framework, when bidders not only have private valuations but also private budgets. We consider the setting of selling divisible goods to multiple agents each with linear utilities, but agents cannot pay beyond their budget. We focus on the case when the auctioneer can check that bidders do not over-report their budgets, but bidders may under-report. We use a relaxed linear program to bound the optimal auction, and use it to construct a "greedy-queuing" auction that guarantees $1 - 1/e$ of the optimal revenue. In fact, the integrality gap of the program is exactly $e/(e-1)$ so the analysis is tight.

1. INTRODUCTION

One of the most positive results in Mechanism Design is Myerson's 1982 closed-form characterization of the revenue maximizing auction, when bidders have private valuations drawn independently from some possibly different distributions [7].[1] Myerson showed that the optimal auction defines some virtual valuation function, performs a Vickery auction on virtual valuations to determine allocation, and charges according to the inverse of the virtual valuation of the Vickrey price [7]. More accessible proofs of this result can be found in [9, 5].

However, little is known in the case when bidders not only have private valuations but also private budget constraints. In practice, budget constraints are common, and they render Myerson's auction infeasible. In fact, budget constraints so complicate the problem, that even solving the case with one bidder becomes highly non-trivial [4]. While there are characterizations of the optimal solution when bidders are identical [8], little is known when bidders are asymmetric or when we require a polynomial-time implementable auction. While the optimal auction under discretization can always be found by solving a linear program, this linear program has exponentially many variables and constraints. Currently the best result in the literature is a 5.83-approximation which follows from LP-rounding [2].

In this paper, we create a feasible auction by rounding the relaxed LP as in [2]. The difference is that our resultant "greedy-queuing" auction achieves a much better approximation ratio of $e/(e-1) \leq 1.59$, which matches the integrality gap of the relaxed LP. Hence our result is the best achievable using the LP.

1.1 Previous Work

There is a recent line of research by Economists on characterizing the optimal Bayesian auction given budget constraints. Che and Gale studies the one bidder case, and even there the optimal auction is highly non-trivial [4]: In the case when bidders can report arbitrary budgets, the optimal selling mechanism commits to a non-decreasing convex pricing function $p(x)$, such that if the buyer wishes to purchase x units of good, she needs to pay $p(x)$. The pricing function can be found by solving a linear program or a differential equation. In the case when bidders cannot overreport budget, the pricing function becomes two-dimensional. Our characterization theorems are closely related to some of their results, with the generalization that instead of only describing the optimal allocation and pricing functions, we characterize *all* incentive-compatible ones.

When there are multiple bidders, Laffont and Robert characterize the optimal auction with identical bidders and public budget constraints [6], and Pai and Vohra do the same with private budgets [8]. However, their arguments crucially depend on symmetric priors for the bidders, and they do not cover the case when the auctioneer can enforce no-overreporting of budgets. Moreover, it is unclear whether Pai and Vohra's solution is polynomial-time solvable.

There is another line of research on revenue maximizing auctions in the worst-case scenario, when no priors on bidder distributions are known [1, 3]. This is very different from the Bayesian setting that we consider.

Our work is most similar to the approximately optimal Bayesian

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[1]For this and similar works, Myerson won the Nobel prize in Economics in 2007

auction in [2]. We improve the approximation ratio and our auctions are more natural.

2. PROBLEM FORMULATION

Suppose there are n bidders and some infinitely-divisible goods to be auctioned [2]. Without loss of generality, normalize the amount of goods to 1. Bidder i wishes to purchase a maximum of $s_i \leq 1$ units of goods, and these size constraints are common knowledge. Each bidder also has private type $t_i = (v_i, b_i)$, where $v_i \geq 0$ and $b_i \geq 0$ are her private valuation and budget respectively. We assume that t_i is drawn independently for each bidder from some distribution with density $f_i(v, b)$. For convenience we sometimes factor this as $f_i(v, b) = g_i(b) f_i(v|b)$. We assume that the bidder's are risk neutral with linear utility ($u_i = v_i x_i - p_i$ is bidder i's utility under x_i units of allocation), except that they have utility of $-\infty$ when their budget constraint is exceeded. (This is called a hard budget constraint and is the most common setting studied [4, 6, 8, 2].) We seek the auction implementable in dominant strategies that maximizes the auctioneer's expected profit.

By the revelation principle, it suffices to consider direct-revelation mechanisms. In this case, a deterministic auction mechanism specifies an allocation function $\vec{x}(\vec{t})$ and a payment function $\vec{p}(\vec{t})$, where $\vec{t} = ((v_1, b_1), (v_2, b_2), \cdots, (v_n, b_n))$ consists of the bidders' reports of types, and given this report the auctioneer allocates to bidder i $x_i(\vec{t})$ units of good and charge $p_i(\vec{t})$. (It suffices to consider deterministic mechanisms because each randomized mechanism that does not exceed bidders' reported budgets induces some expected allocation and payment functions $\vec{x}(\cdot)$, $\vec{p}(\cdot)$, and yield the same expected utilities for the bidders and the auctioneer as the deterministic mechanism with these allocation and payment functions.) We require that the mechanism $(\vec{x}, \vec{p})$ satisfies the following:

- Incentive Compatibility (IC): It is a dominant strategy for each bidder i to report her true type $t_i = (v_i, b_i)$. We focus on the case when bidders can never over-report budgets, which we call (P-IC).
- Individual Rationality (IR): The strategy of not participating in the auction is dominated. This implies Budget Feasibility (BF).
- No Positive Transfers (NPT): The mechanism never subsidizes any bidder with a monetary amount.
- Allocation Feasibility (F): The mechanism allocates non-negative amounts, respects each bidder's size constraint s_i, and allocates at most 1 in total.

2.1 Program Encoding the Optimal Auction

For convenience, define the probabilistic density of each type vector $f(\vec{t}) = \prod_{i=1}^{n} f_i(v_i, b_i)$, and define the space of possible type vectors T. Let t_{-i} denote the types reported by bidders other than i. The optimal auction in the model in which bidders cannot overreport budgets can be encoded as a solution to the program in Figure 1.[3]

The program is feasible because setting $x_i = p_i = 0\ \forall i$ satisfies all the constraints.

Definition 1. We call the auction *valid* if its allocation and payment functions $\vec{x}(\vec{t})$ and $\vec{p}(\vec{t})$ satisfy constraints (P-IC), (IR), (BF), (NPT), (F0) and (F1).

In practice, one may discretize the type space and define a finite version of (OPT P-IC), which can be solved to find the optimal auction. However, these programs have intractably many variables and constraints (exponential in n).

3. STRUCTURE OF THE LINEAR PROGRAM

Myerson solved the optimal auction with only private valuations by first characterizing all valid auctions in terms of simple monotonicity conditions [9, 5]. We follow the same road map and characterize all valid auctions.

THEOREM 1 (CHARACTERIZATION OF VALID AUCTIONS). *Allocation and payment functions $\vec{x}$ and $\vec{p}$ satisfy ex-post (P-IC), (IR), (BF), (NPT), and (F0) (see Figure 1) if and only if for each i and each t_{-i}, defining $x(v, b) = x_i((v, b), t_{-i})$, $p(v, b) = p_i((v, b), t_{-i})$, we have*

- *(F0)* $0 \leq x(v, b) \leq s_i$.
- *(C1)* $x(v, b)$ *is non-decreasing in* v.
- *(C2)* $y(v, b) = \int_0^v x(q, b) dq$ *is non-decreasing in* b.
- *(C3)* $v x(v, b) - \int_0^v x(q, b) dq \leq b\ \forall v, b$.
- *(C4)* $p(v, b) = v x(v, b) - \int_0^v x(q, b) dq\ \forall v, b$.

Definition 2. For bidder i, we call the function $x(v, b)$ a *valid allocation function* if it satisfies conditions (F0), (C1-3) of Theorem 1. We call the function $p(v, b)$ a *valid payment function* if it satisfies condition (C4) of Theorem 1 for some valid allocation $x(v, b)$.

As in Myerson [7], we can simplify the optimal program by writing the objective in terms of virtual valuations.

Definition 3. Define bidder i's *virtual valuation* function as

$$\phi_i(v, b) = v - \frac{1 - F_i(v|b)}{f_i(v|b)}$$

where $F_i(v|b) = \int_0^v f_i(q|b) dq$. (Recall that we factor the density function of the bidder's valuation/budget distribution as $f(v, b) = g_i(b) f(v|b)$).

[2]This can be applied to divisible goods case by treating fractional goods as an allocation probability.

[3]In this formulation, the (P-IC), (IR) constraints hold ex-post (regardless of other bidders' reports), so the mechanism is implementable in dominant-strategies. One can relax this by modifying these constraints so that they hold ex-interim (in expectation over other bidders' reports), in which case the mechanism becomes implementable in Bayes-Nash equilibrium.

$$\text{Maximize} \quad \sum_{i=1}^{n} \int_{\vec{t} \in T} p_i(\vec{t}) f(\vec{t}) d\vec{t} \qquad \text{(OPT P-IC)}$$

$$\begin{array}{llcll}
\text{(P-IC)} & v_i x_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) & \geq & v_i x_i(t'_i, t_{-i}) - p_i(t'_i, t_{-i}) & \forall i, t_{-i}, t_i = (v_i, b_i), t'_i = (v'_i, b'_i) \\
 & & & & \text{such that } b'_i \leq b_i \\
\text{(IR)} & v_i x_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) & \geq & 0 & \forall i, \vec{t} \\
\text{(BF)} & p_i(t_i, t_{-i}) & \leq & b_i & \forall i, \vec{t} \\
\text{(NPT)} & p_i(\vec{t}) & \geq & 0 & \forall i, \vec{t} \\
\text{(F0)} & x_i(\vec{t}) & \in & [0, s_i] & \forall i, \vec{t} \\
\text{(F1)} & \sum_i x_i(\vec{t}) & \leq & 1 & \forall \vec{t}
\end{array}$$

Figure 1: The program encoding the revenue maximizing auction (OPT P-IC)**.**

THEOREM 2 (REVENUE EQUIVALENCE). *In any valid auction, the expected revenue*

$$\sum_{i=1}^{n} \int_T p_i(\vec{t}) f(\vec{t}) d\vec{t} = \sum_{i=1}^{n} \int_T x_i(\vec{t}) \phi_i(v_i, b_i) f(\vec{t}) d\vec{t}$$

where $\phi_i(v,b)$ is bidder i's virtual valuation function.

The optimal program for the case when bidders cannot over-report budgets now becomes the program (OPT2) in Figure 2.

$$\text{Maximize} \quad \sum_{i=1}^{n} \int_t x_i(\vec{t}) \phi_i(v_i, b_i) f(\vec{t}) d\vec{t} \qquad \text{(OPT2)}$$

Such that $\forall i, t_{-i}$, the function $x(v,b) = x_i((v,b), t_{-i})$ is a valid allocation (see Definition 2) and

$$\sum_i x_i(\vec{t}) \leq 1 \qquad \forall \vec{t}$$

Figure 2: The optimal program (OPT2)**.**

4. APPROXIMATELY OPTIMAL AUCTION

4.1 LP Rounding

Currently, the feasibility constraint in (OPT2) (Figure 2) has combinatorially many constraints. To make the program tractable, we define new variables

$$\begin{array}{rcl}
x_i(v_i, b_i) & = & \int_{T_{-i}} x_i((v_i, b_i), t_{-i}) f(t_{-i}) dt_{-i} \\
p_i(v_i, b_i) & = & \int_{T_{-i}} p_i((v_i, b_i), t_{-i}) f(t_{-i}) dt_{-i}
\end{array}$$

Also we relax the feasibility constraints and only require that we allocate at most 1 *in expectation*. The relaxed program is in Figure 3.

$$\text{Maximize} \quad \sum_{i=1}^{n} \int_{v,b} x_i(v,b) \phi_i(v,b) f_i(v,b) dv db \qquad \text{(RELAXED-LP)}$$

Such that $\forall i$, the function $x_i(v,b)$ is a valid allocation (see Definition 2) and

$$\sum_i \int_{v,b} x_i(v,b) f(v,b) dv db \leq 1$$

Figure 3: The relaxed program (Relaxed-LP).

The optimal objective to (RELAXED-LP) upper-bounds the optimal revenue, because for any feasible setting $x_i(\vec{t})$ and $p_i(\vec{t})$ to the original exponentially sized LP, we can define $x_i(v,b)$ and $p_i(v,b)$ as above, and they are feasible for (RELAXED-LP).

We can solve (RELAXED-LP) efficiently using the Lagrangian formulation. The details are standard.

4.2 The Greedy-Queuing Auction

Definition 4. [Greedy-Queuing] Define the *greedy-queuing* auction as follows: Let the optimal solution to (RELAXED-LP) (Figure 3) be $\{x_i^*(v,b)\}$. In this infeasible auction, let the expected revenue from bidder i be $R_i^* = E_{\vec{t}}[x_i^*(t_i)\phi_i(t_i)]$, the expected allocation be $a_i = E_{t_i}[x_i^*(t_i)]$, and define profitability $g_i = R_i^*/a_i$. Sort the bidders in decreasing order of profitability: $g_1 \geq g_2 \geq \cdots \geq g_n$. Allocate to bidder i,

$$x_i(\vec{t}) = (1 - \sum_{j<i} x_j(\vec{t})) x_i^*(t_i)$$

and charge $p_i(\vec{t}) = v_i x_i(\vec{t}) - \int_0^{v_i} x_i((q, b_i), t_{-i}) dq$.

THEOREM 3. *The greedy-queuing auction is valid (Definition 1), satisfying (P-IC), (IR), (NPT) and (F) ex-post.*

PROOF. This follows from the fact that the constraints encoding (P-IC), (IR), (NPT), (F) are linear, and still satisfied whenever x_i is scaled by a constant $\alpha \leq 1$. For fixed t_{-i}, this scaling is constant because $\forall j < i$, $x_j(\vec{t})$ is independent of $x_i^*(t_i)$. □

THEOREM 4. *The greedy-queuing auction achieves at least $1 - \frac{1}{e} \geq \frac{1}{1.59}$ of the optimal revenue.*

PROOF. As in Definition 4, let the optimal solution to (RELAXED-LP) be $\{x_i^*(v,b)\}$, and for bidder i, let the corresponding expected revenue and expected allocation be R_i^* and a_i respectively. Define profitability $g_i = \frac{R_i^*}{a_i}$, and assume the bidders are sorted in decreasing order of profitability.

Straightforward induction shows that under greedy-queuing, the actual allocation is

$$x_i(\vec{t}) = \prod_{j<i} (1 - x_j^*(t_j)) x_i^*(t_i)$$

By independence between agents' types, the actual expected revenue from i is

$$\begin{aligned} R_i &= E_{\vec{t}}[\prod_{j<i}(1-x_j^*(t_j))x_i^*(t_i)\phi_i(t_i)] \\ &= E_{\vec{t}}[\prod_{j<i}(1-x_j^*(t_j))]E_{t_i}[x_i^*(t_i)\phi_i(t_i)] \\ &= \prod_{j<i}(1-a_j)R_i^* \\ &= \prod_{j<i}(1-a_j)a_i g_i \end{aligned}$$

Because g_i's are non-increasing, we can use the substitution $g_i = \sum_{k=i}^n y_k$ for some $y_k \geq 0$, and write the total actual expected revenue as

$$\begin{aligned} R &= \sum_i R_i \\ &= \sum_i \prod_{j<i}(1-a_j)a_i \sum_{k=i}^n y_k \\ &= \sum_k (\sum_{i=1}^k \prod_{j<i}(1-a_j)a_i)y_k \\ &= \sum_k (1-\prod_{i=1}^k(1-a_i))y_k \end{aligned}$$

The optimal expected revenue

$$\begin{aligned} R_{opt} &\leq \sum_i R_i^* \\ &= \sum_i a_i g_i \\ &= \sum_k (\sum_{i=1}^k a_i)y_k \end{aligned}$$

Lemma 1. *Suppose $s = \sum_{i=1}^k a_i \leq 1$, where $a_i \geq 0$, then*

$$1 - \prod_{i=1}^k (1-a_i) \geq (1-e^{-1})s$$

Proof. Note that the function $f(x) = \log(1-x)$ is concave and tangent to the line $g(x) = -x$ at $x = 0$. By Jensen's inequality and using the fact that $f \leq g$,

$$\sum_{i=1}^k \log(1-a_i) \leq k\log(1-\frac{s}{k}) \leq -s$$

Hence,

$$1 - \prod_{i=1}^k (1-a_i) \geq 1 - e^{-s}$$

Now, the function $h(s) = 1 - e^{-s}$ is concave, with $h(0) = 0$ and $h(1) = 1 - e^{-1}$. By concavity h must lie above the line $l(s) = (1-e^{-1})s \; \forall 0 \leq s \leq 1$. This yields the desired bound. $\square$

Applying the lemma and using the non-negativity of y_k, we get that under greedy-queuing, the actual expected total revenue

$$R \geq \sum_k (1 - e^{-1} \sum_{i=1}^k a_i y_k \geq (1 - e^{-1} R_{opt}$$

This proves our $\frac{e}{e-1} \leq 1.59$ approximation ratio.

We now provide an example in which the integrality gap of the program (Relaxed-LP) (Figure 3) with respect to the exponential sized actual program (OPT P-IC) is exactly $\frac{e}{e-1}$, hence showing that our $\frac{e}{e-1}$-approximation is the best achievable using an algorithm based on the relaxation (Relaxed-LP).

Example 1. *Suppose that all n bidders are identical, with budget $b_i = 1$ and*

$$v_i = \begin{cases} 1 & \textit{with probability } \frac{1}{n} \\ 0 & \textit{otherwise} \end{cases}$$

In this case, the solution to (Relaxed-LP) *is $x_i^*(0,1) = 0$, $x_i^*(1,1) = 1$, with corresponding revenue bound $R^* = 1$. The optimial auction allocates to an arbitrary bidder with valuation 1, if one exists, and achieves expected revenue*

$$R_{opt} = 1 - (1 - \frac{1}{n})^n$$

When $n \to \infty$, the integrality gap

$$\frac{R^*}{R_{opt}} \to \frac{e}{e-1}$$

5. NEXT STEPS

This is only an interim report, and much more remains to be done. In particular, we want to

- We can solve for the optimal auction in small cases. We want to implement this and gain intuition.
- Can we give a PTAS for non i.i.d. cases?
- How do we recover the optimal policy in the i.i.d. case, given the interim allocation probabilities?

6. REFERENCES

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APPENDIX
A. OMITTED PROOFS