Summary of What's Known about Convergence in Concave Games

Peng Shi

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1 Concave Games

Definition 1 (Narrow Definition for single dimensionaly strategy and orthogonal strategy space)**.** *A game is called* **concave** *if each player i chooses a real quantity $x_i \in [a_i, b_i]$ to maximize utility is $u_i(\vec{x})$, where $u_i(\vec{x})$ is concave in x_i.*

Theorem 1.1 (Rosen)**.** *Concave games have (possibly multiple) Nash Equilibrium.*

2 Socially Concave Games

Definition 2 (Even-Dar et. al.)**.** *A concave game is* **socially concave** *if each utility $u_i(x_i, x_{-i})$ is convex in x_{-i}, and there exists $\lambda_i > 0$ such that $g(\vec{x}) = \sum_i \lambda_i u_i(\vec{x})$ is concave.*

Theorem 2.1 (Even-Dar et. al.)**.** *If every player in a socially concave game plays according to a procedure with external regret bound $R_i(t)$, then at time t,*

1. *The average strategy vector $\hat{x}^t$ is an ϵ^t-Nash equilibrium, where ϵ^t depends on the regret bound and minimum λ_i.*
2. *The average utility of each player i is ϵ^t close to her utility at $\hat{x}^t$(the average vector of strategies).*

3 Rosen's Framework

Definition 3 (Rosen)**.** *In a concave game, players follow the* **continuous best reply (BR) dynamic** *if there exists constants $\lambda_i > 0$ s.t. each player i adjusts x_i continuously with $\frac{dx_i}{dt} = \lambda_i \frac{\partial u_i(\vec{x})}{\partial x_i}$.*

Definition 4 (Zinkevich)**.** *In a concave game, players follow* **GIGA** *if there exists constants $\lambda_i > 0$ s.t. each player i adjusts x_i every discrete time step t with $\Delta x_i = \frac{\lambda_i}{\sqrt{t}} \frac{\partial u_i(\vec{x})}{\partial x_i}$.*

Theorem 3.1 (Rosen)**.** *Define $n \times n$ matrix function G in which $G_{ij} = \lambda_i \frac{\partial^2 u_i}{\partial x_i \partial y_i}$, for some constant choices of $\lambda_i > 0$. Then if $G + G^T$ is strictly negative definite, then the Nash equilibrium is unique, and starting at any $\vec{x_0}$ the continuous BR dynamic converges to the unique equilibrium.*

4 Incremental Results

Theorem 4.1 (Moulin)**.** *If a concave game satisfies the dominance solvability condition*

$$-\frac{\partial^2 u_i}{\partial^2 x_i} \geq \sum_j |\frac{\partial^2 u_i}{\partial x_i \partial x_j}|$$

then it satisfies Rosen's conditions in Theorem 3.1.

Definition 5. *A socially concave game is* strict *socially concave if* **any one of the following** *is satisfied:*

1. *For all i $u_i(x_i, x_{-i})$ is* strictly *concave in x_i.*
2. *There exists i, j such that $u_i(x_i, x_{-i})$ is* strictly *convex in x_{-i} and $u_j(x_j, x_{-j})$ is* strictly *convex in x_{-j}.*
3. *$f(x) = \sum_i \lambda_i u_i(x)$ is* strictly *concave in x.*

Theorem 4.2. *Strict socially concave games satisfy Rosen's condition in Theorem 3.1.*

Corollary 4.3. *In strict socially concave games, the continuous BR dynamic converge to unique equilibrium.*

Theorem 4.4. *If the conditions in 3.1 hold, and all the utility functions' first, second and third derivatives are bounded, then GIGA converges to unique equilibrium.*

Corollary 4.5. *In strict socially concave games, when players follow GIGA, not only do the statements in Theorem 2.1 hold, but the strategies converge to unique Nash equilibrium.*

5 Negative Results

Theorem 5.1. *There exists a 2 player non-strict socially concave game in which the continuous BR dynamic does not converge.*

Theorem 5.2. *Continuous better reply dynamic, in which players move in the direction of $\frac{\partial u_i(\vec{x})}{\partial x_i}$ but not necessarily with fixed proportional speed, need not converge even under Rosen's conditions in Theorem 3.1.*

Theorem 5.3. *In 2 player socially concave games, sequential best reply (player best reply in turns) need not converge.*

6 Open Questions

- What happens to Rosen's condition when negative definite is replaced with negative semi-definite? (This would include all socially concave games.) Can we make the statements in Theorem 2.1? (Does the strategies GIGA under average to ϵ-Nash equilibrium? What about the average utilities?)