Moment-Sum-Of-Squares based Semidefinite Programming for Chance Constrained Optimization

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Chance Optimization

$$\underset{\text{design parameters}}{\text{maximize}} \quad \text{Probability}(\text{Success Set})$$

- **Success:** a set defined in terms of design parameters and probabilistic uncertainties.

Chance Optimization

- $x \in \mathbb{R}^n$: Design parameters
- $\omega \in \mathbb{R}^m$: Uncertain parameters with given probability distributions $\text{pr}(\omega)$
- $p_i(x, \omega), i = 1, ..., n_p$: Polynomials
- $g_i(x), i = 1, ..., n_g$: Polynomials

Chance Optimization

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{maximize}} && \text{Probability}_{\text{pr}(\omega)}\Big(\overbrace{p_i(x,\omega) \geq 0,\ i = 1, ..., n_p}^{\text{Success Set}} \Big) \\ &\text{subject to} && g_i(x) \geq 0,\ i = 1, ..., n_g \end{aligned}$$

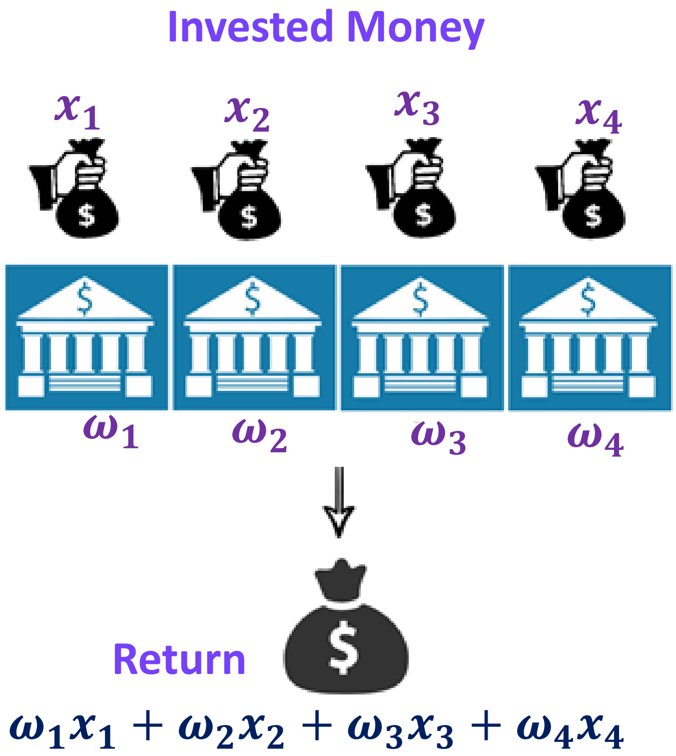
Example: Portfolio Selection Problem

- Assets with uncertain rate of return $\omega_i \sim pr_i(\omega), i = 1, \dots, 4$
- x_i invested money in asset i

- $\boldsymbol{Success}$ = Achive a return higher than "r^*"

$$\text{Return} = \{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*\}$$

- Total investment: $x_1 + x_2 + x_3 + x_4 \leq x^*$



Chance Optimization

$$\begin{aligned} \underset{x_1, x_2, x_3, x_4}{\text{maximize}} \quad & \text{Probability}(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4 \geq r^*) \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 \leq x^* \end{aligned}$$

Chance Constrained Optimization:

Chance Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}\Big(\overbrace{p_i(x,\omega) \geq 0,\ i = 1, ..., n_p}^{\text{Success Set}} \Big) \\ \text{subject to} \quad & g_i(x) \geq 0,\ i = 1, ..., n_g \end{aligned}$$

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$$\text{subject to} \quad g_i(x) \geq 0, \;\; i = 1, ..., n_g$$

Chance Constrained Optimization

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad p(x)$$
$$\text{subject to} \quad \text{Probability}_{\text{pr}(\omega)}\big(\overbrace{g_i(x,\omega) \geq 0, \;\; i = 1, ..., n_g}^{\text{Success Set}} \big) \geq 1 - \underbrace{\Delta}_{\text{Acceptable risk level}}$$

Chance (Constrained) Optimization

- These problems are, in general, **nonconvex** and **computationally hard**.

  - **e.g.,** $\text{Probability}_{\text{pr}(\omega)}(\ g(x,\omega) \geq 0\ ) = \int_{g(x,\omega)\geq 0} \text{pr}(\omega) d\omega$
    - Multivariate integral
    - In general, It does not have any analytical solution

- Existing Methods For Chance (Constrained) Optimization:

  - Sampling based Approaches
  - Particular Classes of Constraints and Uncertainties

  e.g., Linear Chance Constraints with Gaussian Uncertainties

Chance (Constrained) Optimization

- In this presentation, we develop an approach to address a **general class** of Chance (Constrained) Optimizations.

- We look for **convex relaxations** in the form of **Semidefinite Programs.**

- To obtain convex relaxations, we leverage on:
  - Theory of Measure and Moments
  - Theory of Sum-Of-Squares Polynomials
  - Duality of Conic Programs

- Ashkan Jasour, "Convex Approximation of Chance Constrained Problems: Application in Systems and Control", School of Electrical Engineering and Computer Science, The Pennsylvania State University, 2016.
- Ashkan Jasour, Necdet S. Aybat, and Constantino Lagoa, ”Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets”, SIAM Journal on Optimization, 25(3), 1411–1440, 2015.
- Ashkan Jasour, "Risk Aware and Robust Nonlinear Planning", Course Notes for MIT 16.S498, rarnop.mit.edu, 2019.

Chance (Constrained) Optimization

Chance Optimization:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad & \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i=1,\ldots,n_p\ ) \\ \text{subject to} \quad & g_i(x) \geq 0,\ \ i=1,\ldots,n_g \end{aligned}$$

Chance Constrained Optimization:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{\text{pr}(\omega)}(\ g_i(x,\omega) \geq 0,\ \ i=1,\ldots,n_g\ ) \geq 1-\Delta \end{aligned}$$

Assumptions:

- All function and constraints are **polynomial functions**.
- Sets are Compact (Archimedean[1]).

  To satisfy Archimedean, we can add the (redundant) polynomial $M - \|x\|^2$ where $M \geq 0$ such that the set $\{x: M - \|x\|^2 \geq 0\}$ contains the set.

1: For more information: rarnop.mit.edu, 2019 Lecture 3, pages 58-59

Measure and Moment based Chance Optimization - Overview

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximize}} & \text{Probability}_{\text{pr}(\omega)}\left(\, p_i(x,\omega) \geq 0, \;\; i = 1, \ldots, n_p \, \right) \\ \text{subject to} & g_i(x) \geq 0, \;\; i = 1, \ldots, n_g \end{array}$$

Infinite LP in measure space

Step 1: Infinite-dimensional LP

- We treat deign variables x as random variables.
- Instead of looking for "x", we look for its probability distribution (probability measure).

Infinite SDP in moment space

Step 2: Infinite-dimensional SDP

- Instead of looking for measures, we look for the moment sequence $(\text{i.e. } \mathrm{E}[x^\alpha], \alpha = 0, \ldots, \infty)$ of the measures.

Finite moment SDP

Step 3: Finite SDP

- Truncate matrices of **SDP of Step 2 (truncated moments)**

Step 4: We extract the solution of the original chance optimization by looking at the moments.

Measure and Moment based Chance Constrained Optimization - Overview

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x) \\ \text{subject to} & \text{Probability}_{\text{pr}(\omega)}\left(\ g_i(x,\omega) \geq 0,\ \ i=1,\ldots,n_g\ \right) \geq 1-\Delta \end{array}$$

- We look at the **dual optimization** of **moment SDP** of the chance optimization.
- We obtain Sum-Of-Squares optimization whose solution, (e.g., $p_{cc}(x)$) approximates the **feasible set** of chance constrained optimization.

$$\chi_{cc} = \{x : \text{Probability}_{\text{pr}(\omega)}\left(\ g_i(x,\omega) \geq 0,\ \ i=1,\ldots,n_g\ \right) \geq 1-\Delta\} = \{x : p_{cc}(x) \geq 1-\Delta\}$$

- We obtain inner and outer approximation of the chance constrained set.

- Deterministic Optimization:

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & p(x) \\ \text{subject to} & x \in \chi_{cc} \end{array}$$

Measure and Moment based Chance Optimization

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

$$\begin{aligned} \mathbf{P}^* = & \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(\, p_i(x,\omega) \geq 0, \ i = 1, \ldots, n_p \, \right) \\ & \text{subject to} \quad g_i(x) \geq 0, \ i = 1, \ldots, n_g \end{aligned}$$

- We treat deign variables x as random variables.
- Instead of looking for “x”, we look for its probability measure.
- Two sets of probability measures:
    - Unknown Probability measure of Design parameters: $x \sim \mu_x$
    - Given Probability measure of uncertain parameters: $\omega \sim \mu_\omega$

➢ We obtain the equivalent optimization in terms of μ_x and μ_ω.

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

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&\text{subject to} \quad g_i(x) \geq 0, \;\; i = 1, \ldots, n_g
\end{aligned}
$$

Given Success Set in (x, ω) Space:

- $\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \; p_i(x,\omega) \geq 0, \;\; i = 1, \ldots, n_p \, \}$

Given Feasible Set:

$\chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \;\; i = 1, \ldots, n_g\}$

Step 1: Infinite-dimensional LP

Reformulate **Chance Optimization** problem in terms of **measures**

$$\mathbf{P}^* = \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(p_i(x,\omega) \geq 0,\ i=1,\dots,n_p \right)$$
$$\text{subject to} \quad g_i(x) \geq 0,\ i=1,\dots,n_g$$

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Infinite dimensional Linear Program in Measures

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$ $\longrightarrow$ (volume of measure μ)

$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$ $\longrightarrow$ (Upper bound measure)

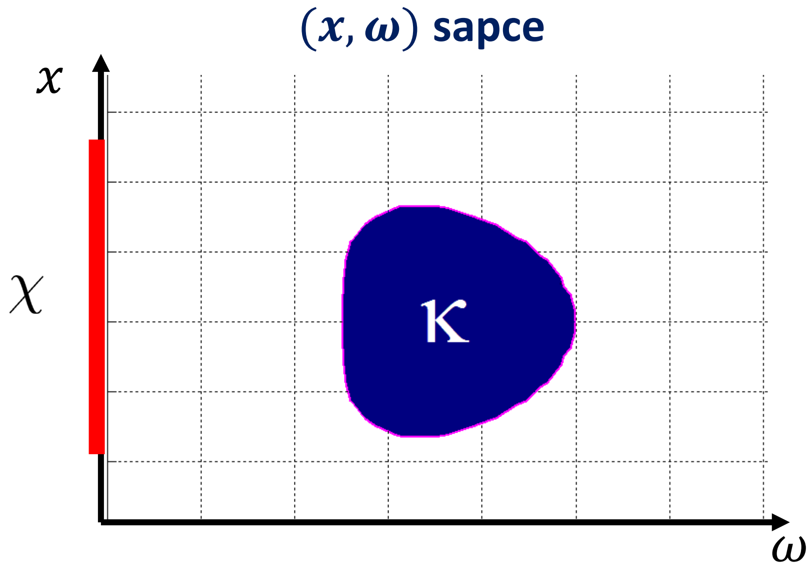
$$\mu_x \text{ is a probability measure}$$
$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

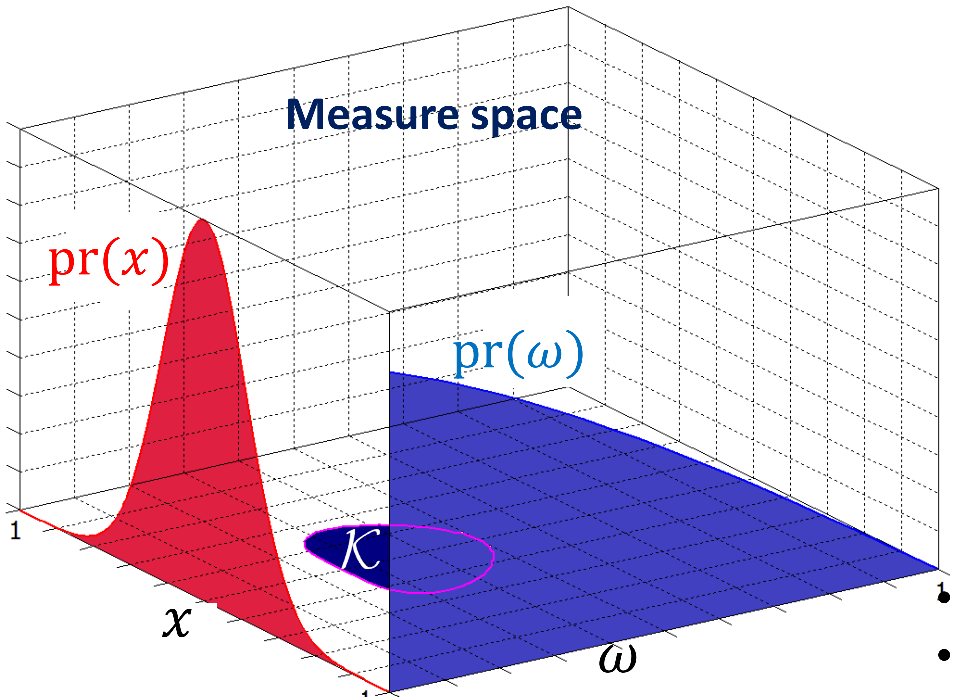
μ: Slack Measure
μ_x: Probability Measure Assigned to x
μ_ω: Known Probability Measure of ω
$\mu_x \times \mu_\omega$: joint measure of x and ω

Interpretation of Measure LP:

$$\mathbf{P}^* = \underset{x\in\mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}\left(p_i(x,\omega) \geq 0,\ i = 1, \ldots, n_p \right)$$
$$\text{subject to} \quad g_i(x) \geq 0,\ i = 1, \ldots, n_g$$

$$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m :\ p_i(x,\omega) \geq 0,\ i = 1, \ldots, n_p\ \} \qquad \chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, \ldots, n_g\}$$



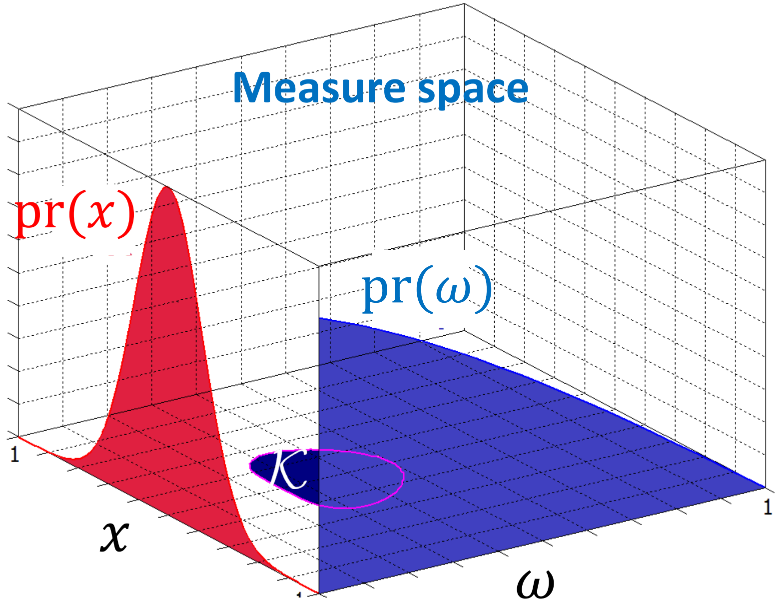


- Measure μ_x with pdf pr(x)
- Measure μ_ω with pdf pr(ω)

- **Suppose** that measure μ_x assigned to the x is **given**:

- We want to calculate the **probability of success** in terms of the probability measures μ_x and μ_ω.

$$\text{Probability}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ \ )$$



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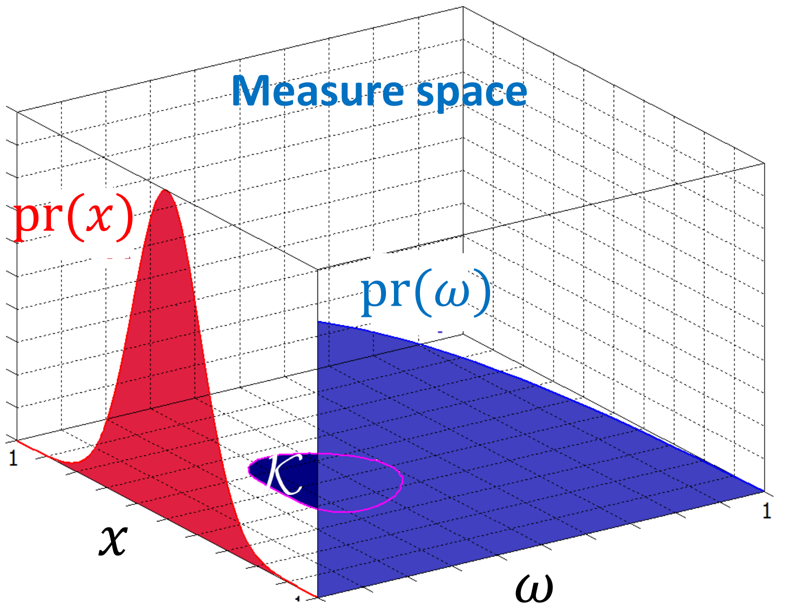
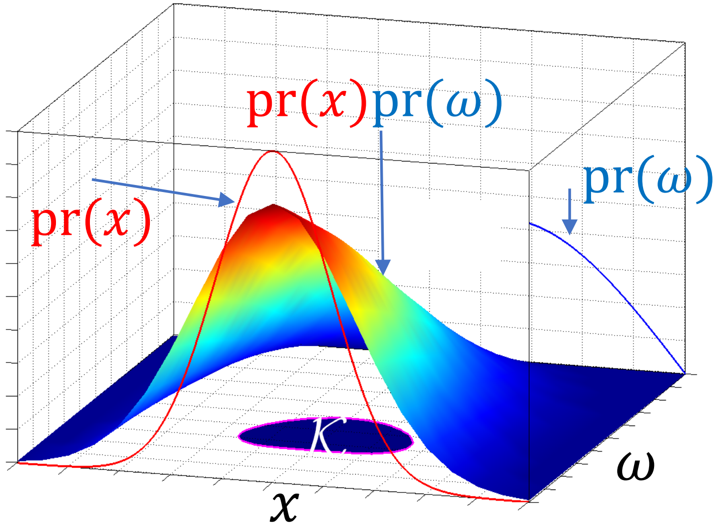
$$\text{Probability}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ \ )$$

- From the definition of the probability:

$$\text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ \ ) = \int_{\mathcal{K}=\{\ p_i(x,\omega)\geq 0,\ i=1,\ldots,n_p\ \}} d(\mu_x \times \mu_\omega)$$

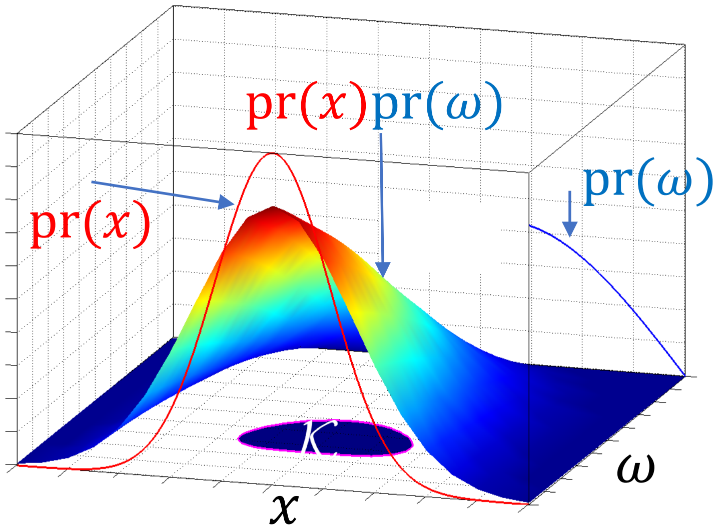
Joint probability measure of (x, ω)





- Probability of success in terms of μ_x and μ_ω

$$\text{Probability}(\ p_i(x,\omega) \geq 0,\ \ i=1,\ldots,n_p\ ) = \int_{\mathcal{K}=\{\ p_i(x,\omega)\geq 0,\ i=1,\ldots,n_p\ \}} d(\mu_x \times \mu_\omega)$$



- This is a **multivariate integral** over the (nonconvex)set $\mathcal{K}$
- Such integral, in general, does not have a **closed form solution**.

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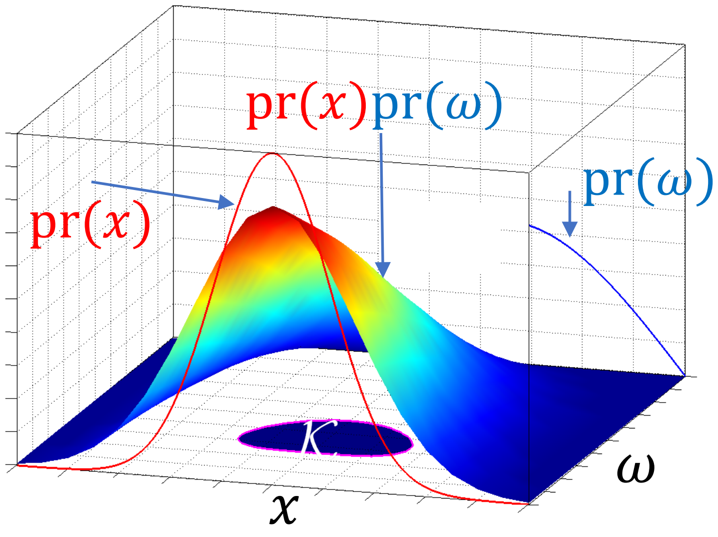

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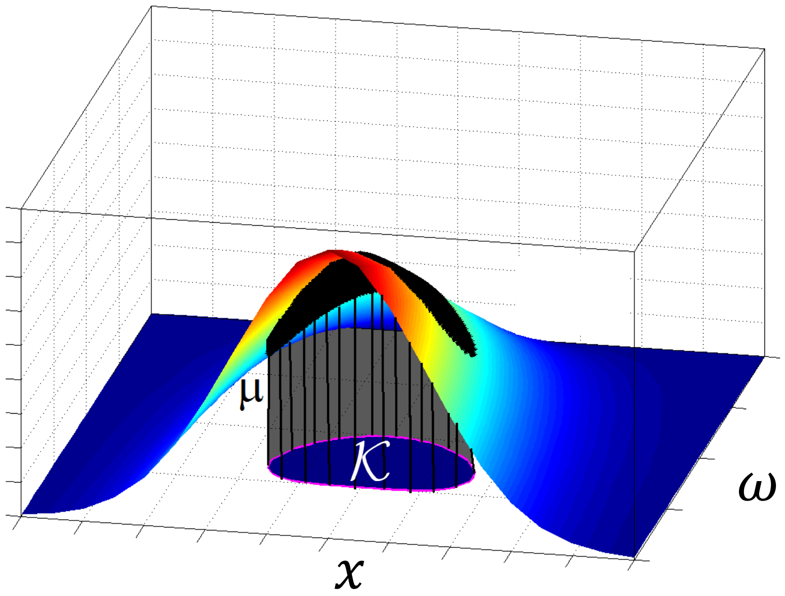
- To calculate this integral, we introduce a **slack measure** μ .
- Measure μ is supported in the set $\mathcal{K}$
- Measure μ is equal to measure $\mu_x \times \mu_\omega$ in the set $\mathcal{K}$

$$\text{Probability}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \dots, n_p\ ) = \int d\mu$$



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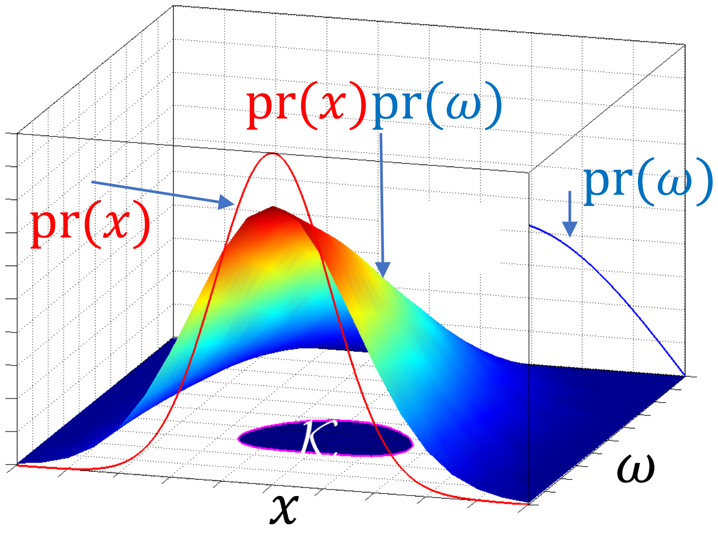

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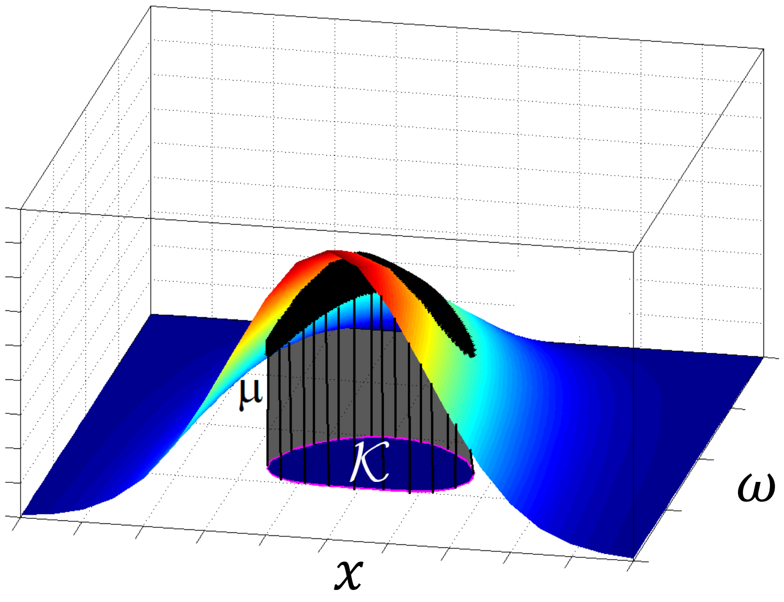
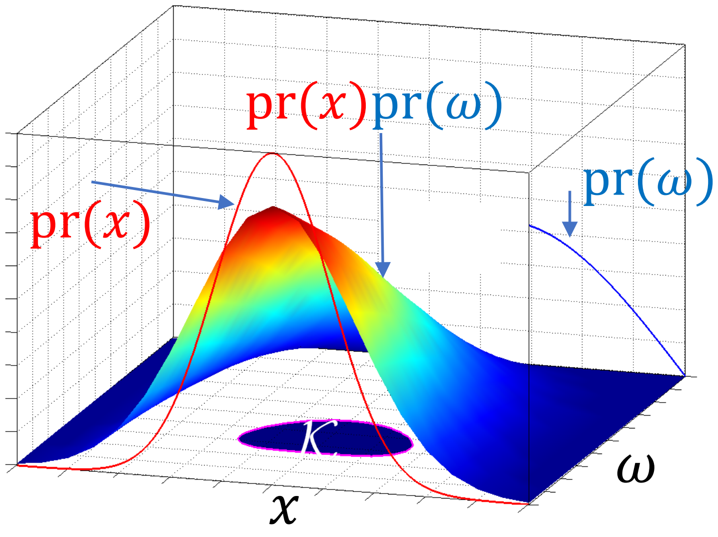
y_0 : zero-order moment of μ



➢ Probability of success in terms of μ_x and μ_ω



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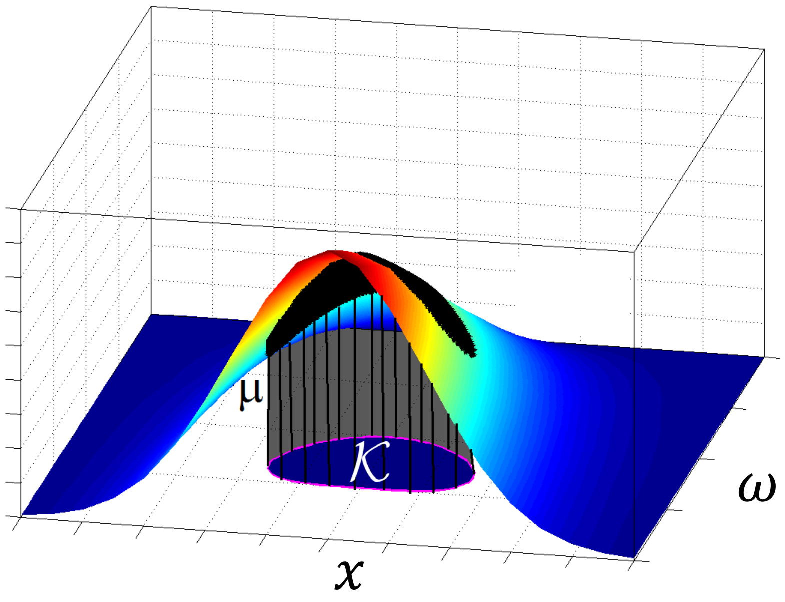

$$\text{Probability}(\ p_i(x,\omega) \geq 0,\ \ i=1,\ldots,n_p\ ) = \int d\mu$$

➢ To construct such **measure μ**, we solve the following optimization

$$\max_{\mu} \int d\mu$$

$$\mu \preccurlyeq \mu_x \times \mu_\omega \qquad supp(\mu) \in \mathcal{K}$$

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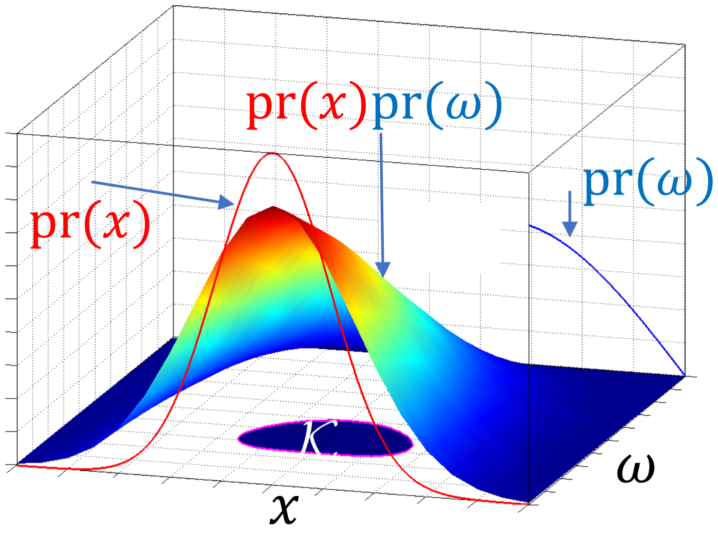

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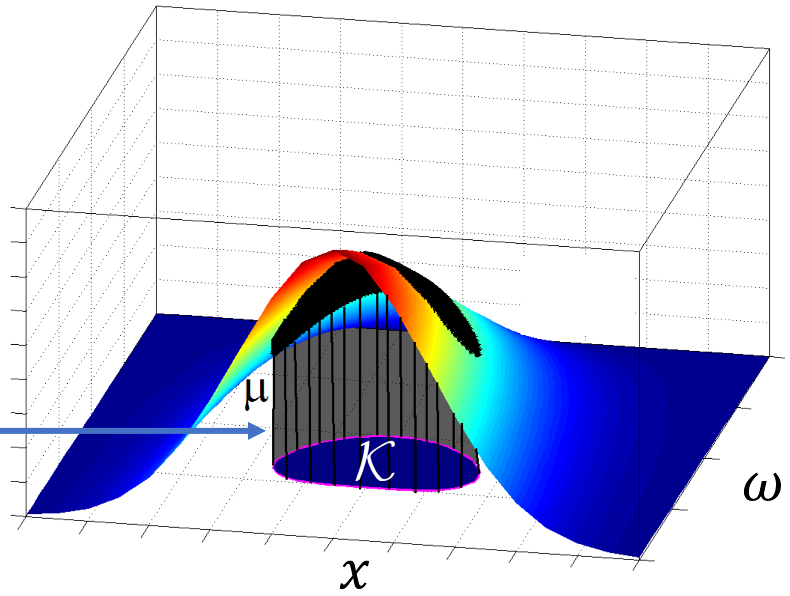

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Optimal Solution

$$\mu \preccurlyeq \mu_x \times \mu_\omega \qquad supp(\mu) \in \mathcal{K}$$

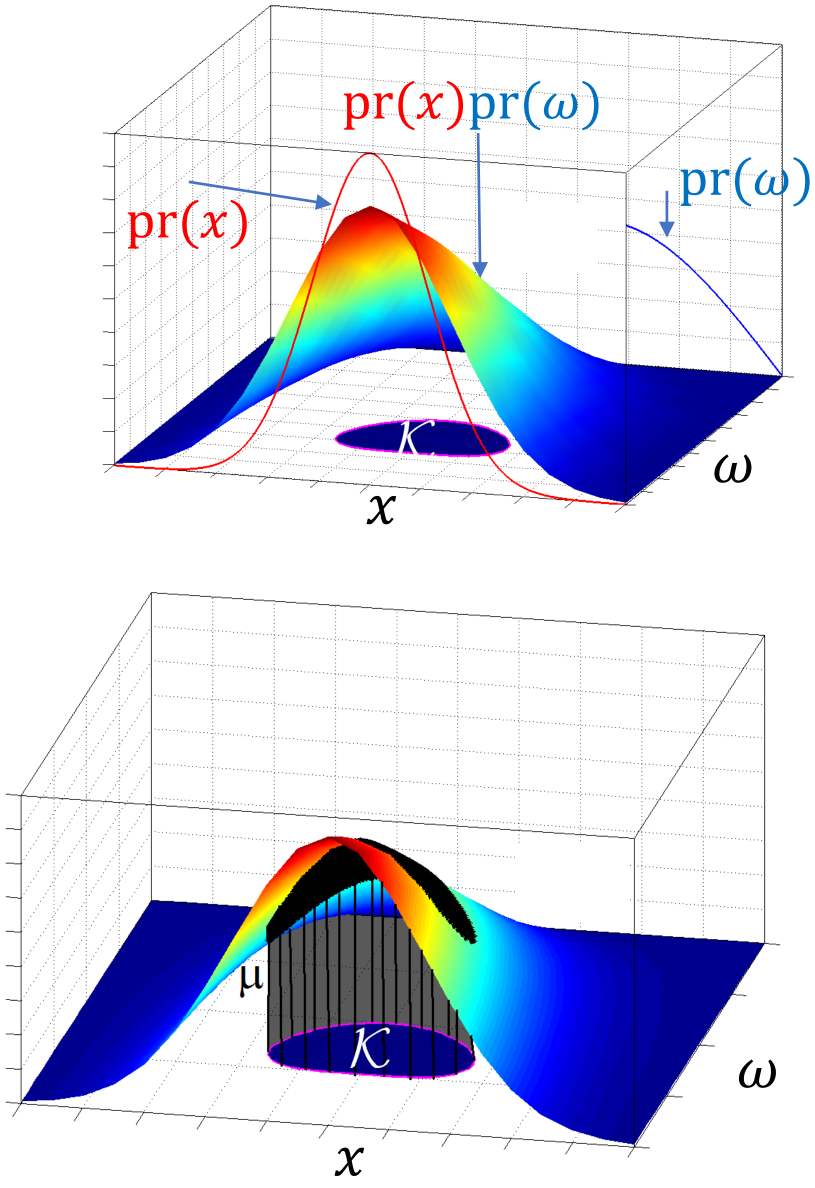
- **To maximize the probability of success ,at the same time**, we look for measure $\boldsymbol{\mu}$ and measure $\boldsymbol{\mu_x}$.

$$\text{maximize Probability}(\ p_i(x,\omega) \geq 0,\ \ i = 1, \ldots, n_p\ ) =$$

$$\max_{\mu, \boxed{\mu_x}} \int d\mu$$

$$\mu \preccurlyeq \mu_x \times \mu_\omega \quad supp(\mu) \in \mathcal{K}$$

$$\boxed{\begin{array}{l} \mu_x \text{ is a probability measure} \\ supp(\mu_x) \subset \chi \end{array}}$$



Step 1: Infinite-dimensional LP

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Infinite dimensional Linear Program in Measures

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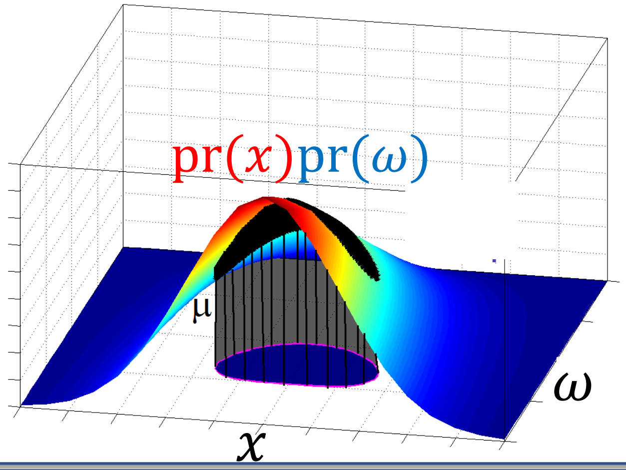

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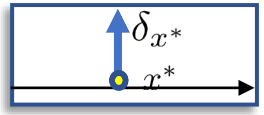
- If $\boldsymbol{\mu}^*$ is an optimal solution of measure-LP , then any $\boldsymbol{x}^* \in \boldsymbol{supp}(\boldsymbol{\mu}^*)$ is an optimal solution of the original chance optimization.

- If $x^* \in \chi$ is unique global optimal solution of the original problem,

  Then $\mu^* = \delta_{x^*}$ is unique optimal solution of measure -LP.

$\mu^* = \delta_{x^*}$



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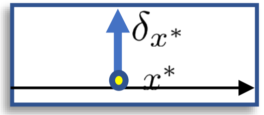
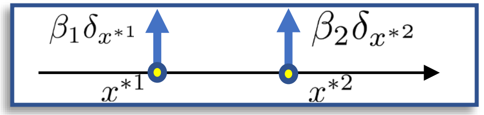


- If $x^{*i} \in \chi, \quad i = 1, \ldots, r$ are "r" global optimal solution of the original problem,

  Then $\mu^* = \sum_{i=1}^r \beta_i \delta_{x^{*i}}, \ \beta_i > 0, \sum_{i=1}^r \beta_i = 1$ is unique optimal solution of optimization in measures.

$$\mu^* = \sum_{i=1}^r \beta_i \delta_{x^{*i}}, \ \beta_i > 0, \sum_{i=1}^r \beta_i = 1$$



Theorem 3.1 : A. Jasour, N. S. Aybat, and C. Lagoa *"Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets"*, SIAM Journal on Optimization, 25(3), 1411–1440, 2015.

1 Chance Optimization

$$\mathbf{P}^* = \underset{x \in \mathbb{R}^n}{\text{maximize}} \quad \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ i = 1,\ldots,n_p\ )$$
$$\text{subject to} \quad g_i(x) \geq 0,\ i = 1,\ldots,n_g$$

2 Equivalent Problem in the measure space:

Infinite dimensional LP

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$
$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$
$$\mu_x \text{ is a probability measure}$$
$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

3 Equivalent Problem in the moment space:

- Instead of looking for measures, we look for the moment sequence (i.e. $\mathrm{E}[x^\alpha], \alpha = 0, \ldots, \infty$) of the measures.
- We need to write the objective function and constraints of the measure-LP in terms of the moments.

Necessary and Sufficient Moment Condition

Let $\mathbf{K} = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1, ..., m\}$ be a semialgebraic (Archimedean).

Sequence y has a representing measure with support contained in the set $\mathbf{K}$, if and only if, it satisfies:

$$\mathbf{M}_d(y) \succcurlyeq 0, \quad \forall d$$

Moment Matrix

$$\mathbf{M}_{d-d_{g_i}}(g_i y) \succcurlyeq 0,\ i = 1, .., m, \quad \forall d$$

Localizing Matrix

- *Chapter 3*: Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.

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Moment Matrix

Localizing Matrix

Example: Moment matrix of order $d = 2$ of a measure in $\mathbb{R}$

$$\mathbf{M}_2(y) = \mathrm{E}[B_2(x) B_2^T(x)] = \mathrm{E}_\mu[\begin{bmatrix} 1 \\ x_1 \\ x_1^2 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 \end{bmatrix}] = \mathrm{E}_\mu[\begin{bmatrix} 1 & x_1 & x_1^2 \\ x_1 & x_1^2 & x_1^3 \\ x_1^2 & x_1^3 & x_1^4 \end{bmatrix}] = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix}$$

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Example: Localizing matrix for moments up to order 4 of a Measure μ in $\mathbb{R}^2$ and polynomial $g(x) = a - x_1^2 - x_2^2$

$$\mathbf{M}_1(gy) = \mathrm{E}[g(x)B_1(x)B_1^T(x)] = \begin{bmatrix} ay_{00} - y_{20} - y_{02} & ay_{10} - y_{30} - y_{12} & ay_{01} - y_{21} - y_{03} \\ ay_{10} - y_{30} - y_{12} & ay_{20} - y_{40} - y_{22} & ay_{11} - y_{31} - y_{13} \\ ay_{01} - y_{21} - y_{03} & ay_{11} - y_{31} - y_{13} & ay_{02} - y_{22} - y_{04} \end{bmatrix}$$

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Localizing Matrix

(nonnegative)measure $\mu,\ supp(\mu) \subset \mathbf{K}$ $\Longleftrightarrow$ y:moments $\mathbf{M}_d(y) \succcurlyeq 0, \ \forall d \qquad \mathbf{M}_d(g_i y) \succcurlyeq 0, \ \forall d, \ i = 1, .., m.$

2 Equivalent Problem in the measure space:

$$\mathbf{P}^*_\mu := \text{maximize}_{\mu_x,\mu} \int d\mu,$$

$$\text{s.t.} \quad \mu \preccurlyeq \mu_x \times \mu_\omega$$

$$\mu_x \text{ is a probability measure}$$

$$supp(\mu_x) \subset \chi, \quad supp(\mu) \subset \mathcal{K}$$

$$\mathcal{K} = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{R}^m : \ p_i(x,\omega) \geq 0, \ i = 1, ..., n_p \}$$

$$\chi = \{x \in \mathbb{R}^n : g_i(x) \geq 0, \ i = 1, ..., n_g\}$$

3 Moment Representation of Measures:

Measure:

$$\mu_x \qquad supp(\mu_x) \in \chi \qquad \int d\mu_x = 1$$

$\Longleftrightarrow$

Moments:

$$\mathbf{M}_\infty(y_x) \succcurlyeq 0 \qquad \mathbf{M}_\infty(g_i y_x) \succcurlyeq 0, \ |_{i=1}^{n_g} \qquad y_{x_0} = 1$$

Measure:

$$\mu \qquad supp(\mu) \in \mathcal{K}$$

$\Longleftrightarrow$

Moments:

$$\mathbf{M}_\infty(y) \succcurlyeq 0 \qquad \mathbf{M}_\infty(p_i y) \succcurlyeq 0, |_{i=1}^{n_p}$$

Measure:

$$\underbrace{\mu_x \times \mu_\omega - \mu}_{\text{(nonnegative) measure}} \succcurlyeq 0$$

$\Longleftrightarrow$

Moments:

$$\mathbf{M}_\infty(y_x \times y_\omega - y) \succcurlyeq 0$$

moments of joint measure $\mu_x \times \mu_\omega$

moments of μ

1 Chance Optimization

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Infinite dimensional SDP

$$\mathbf{P}^*_{\mathbf{mom}} := \text{maximize}_{y,y_x} \quad y_0$$
$$\text{s.t.} \quad M_\infty(\mathbf{y}) \succcurlyeq 0,\ M_\infty(p_i y) \succcurlyeq 0, \quad j = 1,\ldots,n_p$$
$$M_\infty(y_x) \succcurlyeq 0,\ M_\infty(g_i y) \succcurlyeq 0, \quad i = 1,\ldots,n_g,\ y_{x_0} = 1$$
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- Lemma 3.2 : A. Jasour, N. S. Aybat, and C. Lagoa "*Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets*", SIAM Journal on Optimization, 25(3), 1411–1440, 2015.

③ Equivalent Problem in the moment space:

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Relaxation

④ Finite SDP in moments:

Truncated moment SDP

in terms of moment up to order $2d$

$$\begin{aligned}\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := &\text{maximize}_{y,y_x} \quad y_0 \\ \text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1,\dots,n_p \\ & M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1,\dots,n_g,\ y_{x_0} = 1 \\ & M_d(y_\omega \times y_x - y) \succcurlyeq 0\end{aligned}$$

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Infinite dimensional SDP

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Relaxation

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Truncated moment SDP

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\end{aligned}
$$

Theorem:

- Optimal solution of the finite SDP converges to the optimal solution of the original SDP as $d \to \infty$
- For all $d \geq 1$ $\quad \mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \geq \mathbf{P}^*$ Upper bound of optimal objective function of chance optimization

  $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \geq \mathbf{P}^{*\mathbf{d+1}}_{\mathbf{mom}} \qquad lim_{d\to\infty}\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} = \mathbf{P}^*$ monotonically converges

- Theorem 3.3 : A. Jasour, N. S. Aybat, and C. Lagoa "*Semidefinite Programming For Chance Constrained Optimization Over Semialgebraic Sets*", SIAM Journal on Optimization, 25(3), 1411–1440, 2015.

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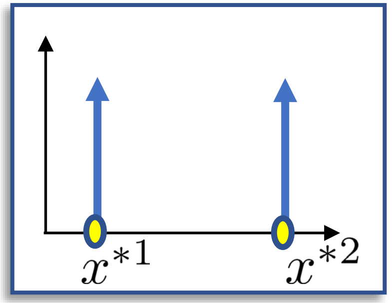
Extraction of the Solution from the moments:

- We are going to identify the **Dirac measures** by looking at the moments of x obtained by solving finite SDP.

- We extract the solution by looking at the **support** of the Dirac measures.



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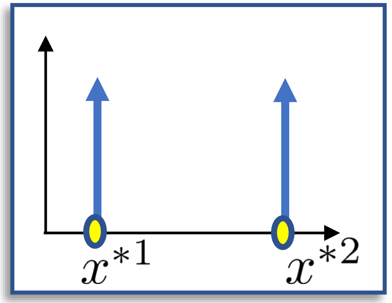
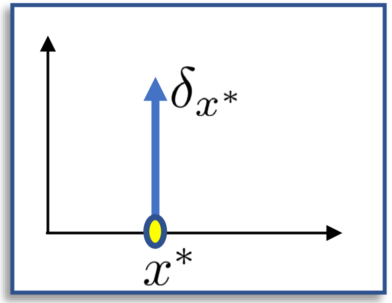


Dirac Measure:

Sequence $y = [y_\alpha, \alpha = 0, ..., 2d]$ is the moment sequence of a Dirac measure if and only if:

$$M_d(y) \succcurlyeq 0, \quad \mathrm{Rank}(M_d(y)) = 1$$

➢ To extract the support (solution): $\delta_{x^*} \longrightarrow x^* = \mathrm{E}[\delta_{x^*}] = y_1$



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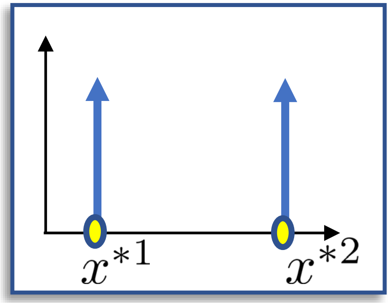


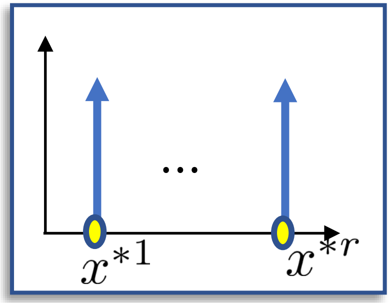
r -atomic measure:

Sequence $y = [y_\alpha, \alpha = 0, \ldots, 2d]$ is the moment sequence of linear positive combination of r Dirac measure (r -atomic representation) if and only if

$$M_d(y) \succcurlyeq 0, \quad \mathrm{Rank}(M_d(y)) = \mathrm{Rank}(M_{d-1}(y)) = r$$

(M_d is a flat extension of M_{d-1})

➢ To extract the support (solution):
we need to solve a linear algebra (relies on moment matrix factorization $\mathrm{M_d(y) = VV^T}$).[1]



1: Algorithm 4.2 : Jean Bernard Lasserre, "Moments, Positive Polynomials and Their Applications" Imperial College Press Optimization Series, V. 1, 2009.

Chance Optimization

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$$\text{subject to} \qquad g_i(x) \geq 0,\ \ i=1,\dots,n_g$$

Moment Relaxation (SDP)

$$\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} := \text{maximize}_{y,y_x} \quad y_0$$
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$$M_d(y_x) \succcurlyeq 0,\ \ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i=1,\dots,n_g,\ \ y_{x_0}=1$$
$$M_d(y_\omega \times y_x - y) \succcurlyeq 0$$

$d_{g_i} = [\frac{deg(g_i(x))}{2}]$ $d_{p_i} = [\frac{deg(p_i(x))}{2}]$

$2d \geq \max\left(deg(p_i(x)), deg(g_i(x))\right)$

- Finite SDP of order d: $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}}$ is an upper bound of P^*
- As $d \to \infty$ $\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} \downarrow \mathrm{P}^*$

➢ If obtained solution y_x^* satisfies rank condition $\text{Rank}\ M_d(y_x^*) = \text{Rank}\ M_{d-v}(y_x^*) = r \quad v = \max\{d_{g_i}\}$

$x_i^*, i=1,\dots,r,$ global solutions can be extracted by linear algebra from y_x^*

➢ Otherwise, increase d and solve new SDP.

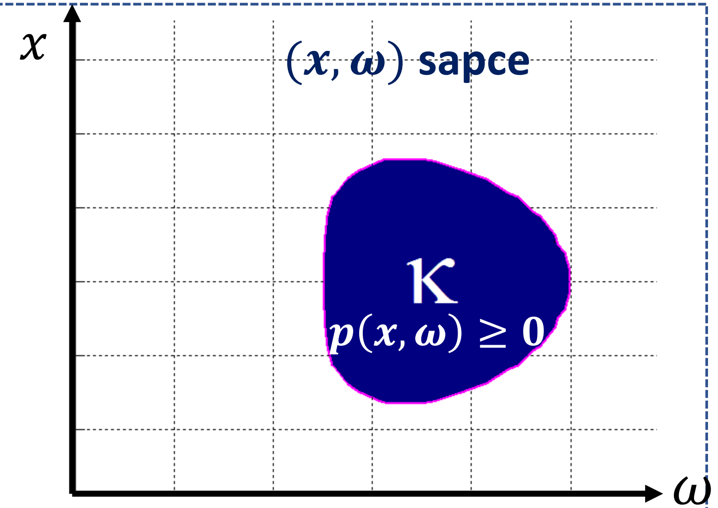
Example 1

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$

$$\text{subject to} \qquad -1 \leq x \leq 1$$

$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$

$$\omega \sim Uniform[-1,1]$$



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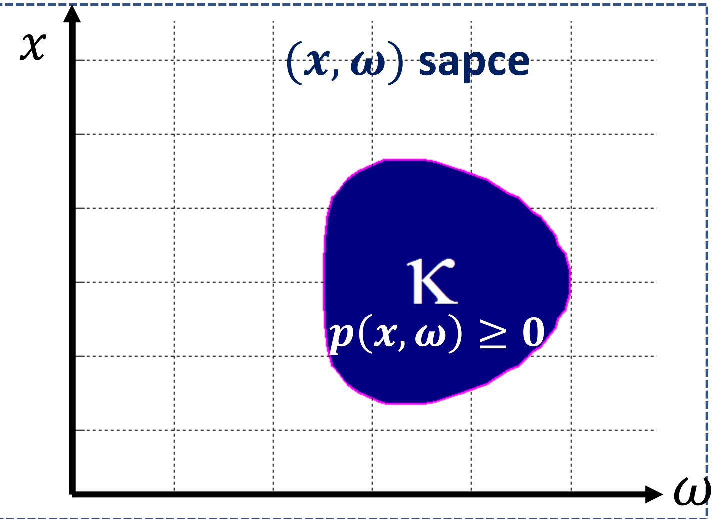


Moment SDP of Order 4:

$$\max_{y,y_x} y_{00}$$

$$M_2(y) \succeq 0 \Rightarrow \left(\begin{array}{c|cc|ccc} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right) \succeq 0$$

$$M_2(y_x y_q) - M_2(y) = \succeq 0 \Rightarrow \begin{pmatrix} 1 & y_{x1} & 0 & y_{x2} & 0 & 1/3 \\ y_{x1} & y_{x2} & 0 & y_{x3} & 0 & 1/3y_{x1} \\ 0 & 0 & 1/3 & 0 & 1/3y_{x1} & 0 \\ y_{x2} & y_{x3} & 0 & y_{x4} & 0 & 1/3y_{x2} \\ 0 & 0 & 1/3y_{x1} & 0 & 1/3y_{x2} & 0 \\ 1/3 & 1/3y_{x1} & 0 & 1/3y_{x2} & 0 & 2/5 \end{pmatrix} - M_2(y) = \succeq 0$$

$$|y_x| \leq 1$$

$$M_2(y_x) \succeq 0 \Rightarrow \begin{pmatrix} y_{x_0} & y_{x_1} & y_{x_2} \\ y_{x_1} & y_{x_2} & y_{x_3} \\ y_{x_2} & y_{x_3} & y_{x_4} \end{pmatrix} \succeq 0$$

$$y_{x_0} = 1$$

$$M_1(py) \succeq 0 \Rightarrow -y_{04} + \frac{1}{2}y_{03} - y_{22} + y_{12} - \frac{1}{4}y_{02} + \frac{1}{2}y_{21} - \frac{1}{2}y_{11} + \frac{1}{8}y_{01} - y_{40} + 2y_{30} - \frac{3}{2}y_{20} + \frac{1}{2}y_{10} - \frac{1}{16} \succeq 0$$

Obtained result:

$$y = [0.6610, 0.3305, 0.1484, 0.1687, 0.0742, 0.1022, 0.0878, 0.0387, 0.0511, 0.0406, 0.0465, 0.0210, 0.0263, 0.0203, 0.0203]$$

Upper bound of probability of success

$$y_x = [1, 0.5, 0.2558, 0.1330, 0.8521]$$

Rank Test:

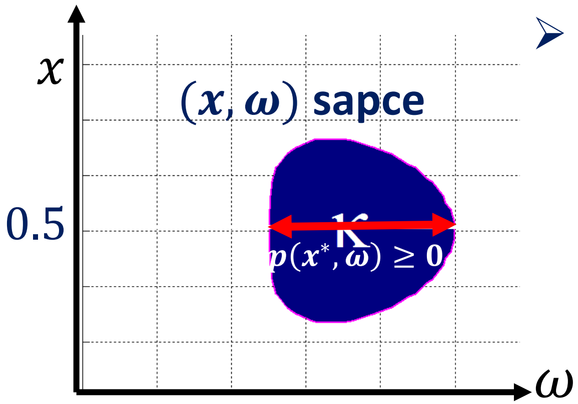
$$M_2(y_x) = \begin{pmatrix} 1.0000 & 0.5 & 0.2558 \\ 0.5000 & 0.2558 & 0.1330 \\ 0.2558 & 0.1330 & 0.8521 \end{pmatrix} \longrightarrow \text{Rank} \approx 1 \longleftarrow M_1(y_x) = \begin{pmatrix} 1.0000 & 0.5 \\ 0.5000 & 0.2558 \end{pmatrix} \Rightarrow$$

Moment of Dirac probability measure $\mu_x = \delta_{x^*}$

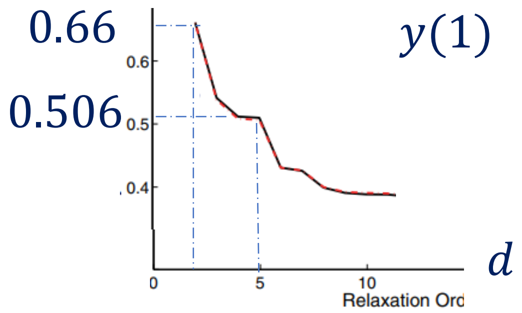
eigenvalues = 0.0046, 0.7011, 1.4022

eigenvalues =0.0046, 1.2512

$\Rightarrow$ $x^* = y_{x_1} = E[x] = 0.5$



- As relaxation order d increase $y(1)$ converges to the true probability



Example: 2

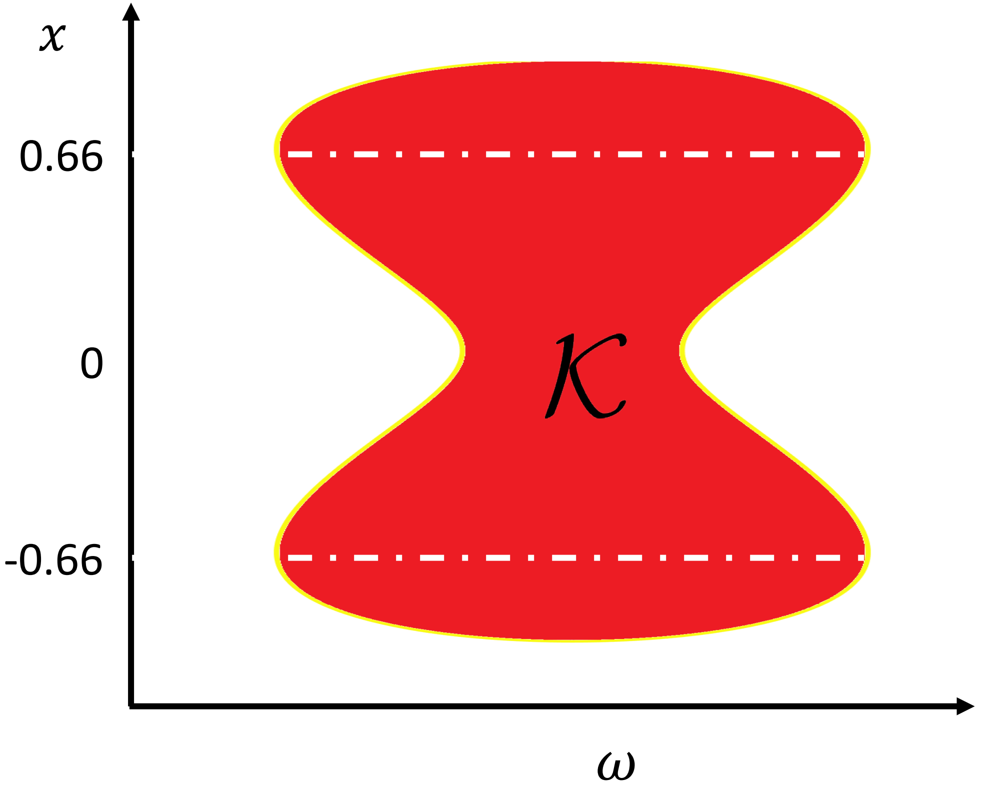
$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$

$$\text{subject to} \qquad -1 \leq x \leq 1$$

$$p(x,\omega) = \{x \in \mathbb{R}^2 : -\tfrac{1}{16}x_1^4 + \tfrac{1}{4}x_1^3 - \tfrac{1}{4}x_1^2 - \tfrac{9}{100}x_2^2 + \tfrac{29}{400} \geq 0\}$$

$$\omega \sim Uniform[-1,1]$$

d=5 $\quad y_x = [1, 0, 0.44, 0, 0.19, 00, 0.08, 0, 0.06, 0, 0.91]$



Rank Test:

$\text{Rank } M_d(y_x) = \text{Rank } M_{d-1}(y_x) \approx 2$

$\Rightarrow$ $x_2^* = -0.66, x_1^* = 0.66$

https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_2_Moment_ChanceOpt

Measure and Moment based Chance Optimization

Dual Optimization

Chance Optimization

$$\begin{array}{ll} \mathbf{P}^* = \underset{x\in\mathbb{R}^n}{\text{maximize}} & \text{Probability}_{\text{pr}(\omega)}(\ p_i(x,\omega) \geq 0,\ \ i = 1, ..., n_p\ ) \\ \text{subject to} & g_i(x) \geq 0,\ \ i = 1, ..., n_g \end{array}$$

Finite Moment SDP:

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} :=& \text{maximize}_{y,y_x} \quad y_0 \\ \text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j = 1, \ldots, n_p \\ & M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i = 1, \ldots, n_g,\ y_{x_0} = 1 \\ & M_d(y_\omega \times y_x - y) \succcurlyeq 0 \end{aligned}$$

- We find the **dual optimization** of **Moment-SDP.** (Duality of Conic Program)

Dual Optimization:

Moment SDP

$$
\begin{aligned}
\mathbf{P}^{*\mathbf{d}}_{\mathbf{mom}} :=&\text{maximize}_{y,y_x} \quad y_0 \\
\text{s.t.} \quad & M_d(\mathbf{y}) \succcurlyeq 0,\ M_{d-d_{p_i}}(p_i y) \succcurlyeq 0, \quad j=1,\dots,n_p \\
& M_d(y_x) \succcurlyeq 0,\ M_{d-d_{g_i}}(g_i y) \succcurlyeq 0, \quad i=1,\dots,n_g,\ y_{x_0}=1 \\
& M_d(y_\omega \times y_x - y) \succcurlyeq 0
\end{aligned}
$$

Dual SDP

$$
\left.
\begin{aligned}
\mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \beta \\
\text{subject to} \quad & \mathcal{W}(x,\omega)-1 \geq 0 \quad \forall (x,\omega)\in\mathcal{K} \\
& \beta - \int \mathcal{W}(x,\omega)d\mu_\omega \geq 0 \quad \forall x\in\chi \\
& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0
\end{aligned}
\right\}
$$

- Sum-Of-Squares (SOS) optimization

:SOS

:SOS

:SOS

- $\int \mathcal{W}(x,\omega)d\mu_\omega = \mathrm{E}[\mathcal{W}(x,\omega)]$

Interpretation of the dual problem

Dual optimization:

$$\begin{aligned} \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega) \geq 1 \quad \forall(x,\omega)\in\mathcal{K} \\ & \beta \geq \int \mathcal{W}(x,\omega)d\mu_\omega \quad \forall x\in\chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

- $\mathcal{W}(x,\omega) \geq 0 \quad \mathcal{W}(x,\omega) \geq 1 \;\; \forall(x,\omega)\in\mathcal{K}$
- Minimizes scaler $\beta \geq 0$
- β is upper bound of $\int \mathcal{W}(x,\omega)d\mu_\omega$
- $\int \mathcal{W}(x,\omega)d\mu_\omega = \mathrm{E}[\mathcal{W}(x,\omega)]$

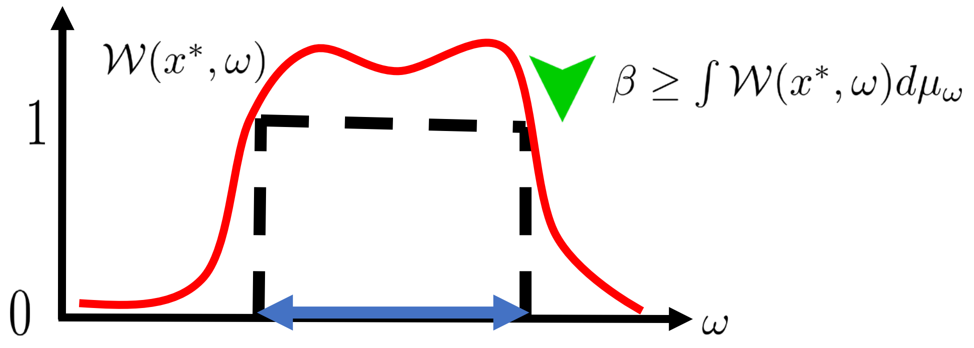
Interpretation of the dual problem

Dual optimization:

$$
\begin{aligned}
\underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad & \beta \\
\text{subject to} \quad & \mathcal{W}(x,\omega) \geq 1 \quad \forall (x,\omega) \in \mathcal{K} \\
& \beta \geq \int \mathcal{W}(x,\omega) d\mu_\omega \quad \forall x \in \chi \\
& \beta \geq 0, \mathcal{W}(x,\omega) \geq 0
\end{aligned}
$$

- $\mathcal{W}(x,\omega) \geq 0 \qquad \mathcal{W}(x,\omega) \geq 1 \ \ \forall (x,\omega) \in \mathcal{K}$
- Minimizes scaler $\beta \geq 0$
- β is upper bound of $\int \mathcal{W}(x,\omega) d\mu_\omega$
- $\int \mathcal{W}(x,\omega) d\mu_\omega = \mathrm{E}[\mathcal{W}(x,\omega)]$

- For a given design variable x^*:



$\mathcal{K} = \{(x^*,\omega) : \ p_i(x^*,\omega) \geq 0, \ i = 1, \ldots, n_p\}$

(success set for a given x^*)

- $\mathcal{W}(x^*,\omega)$:

  is an **upper bound** polynomial approximation of the indicator function of $\mathcal{K} = \{(x^*,\omega) : \ p_i(x^*,\omega) \geq 0, \ i = 1, \ldots, n_p\}$

- $\mathrm{E}[\mathcal{W}(x,\omega)] = \int \mathcal{W}(x,\omega) d\mu_\omega$:

  is an **upper bound** of probability of success for given x^*

Interpretation of the dual problem

Dual optimization:

$$\begin{aligned} \underset{\beta\in\mathbb{R},\mathcal{W}(x,\omega)\in\mathcal{C}}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega)\geq 1 \quad \forall (x,\omega)\in\mathcal{K} \\ & \beta \geq \int \mathcal{W}(x,\omega)d\mu_\omega \quad \forall x\in\chi \\ & \beta\geq 0, \mathcal{W}(x,\omega)\geq 0 \end{aligned}$$

- $\mathcal{W}(x,\omega)\geq 0 \qquad \mathcal{W}(x,\omega)\geq 1 \;\; \forall (x,\omega)\in\mathcal{K}$
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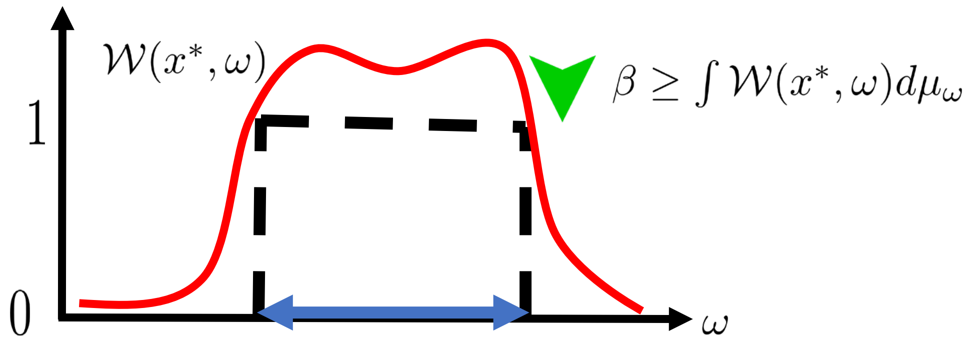

$$\mathcal{K} = \{(x^*,\omega):\; p_i(x^*,\omega)\geq 0,\; i=1,\dots,n_p\}$$

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- $\mathrm{E}[\mathcal{W}(x,\omega)] = \int \mathcal{W}(x,\omega)d\mu_\omega$:

is an **upper bound** of probability of success for given x^*

- $\mathrm{E}[\mathcal{W}(x,\omega)] = \int \mathcal{W}(x,\omega)d\mu_\omega$: Upper bound **probability of success** as a function of x
- β: Upper bound of maximum **probability of success.**

Example: Dual Optimization

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$

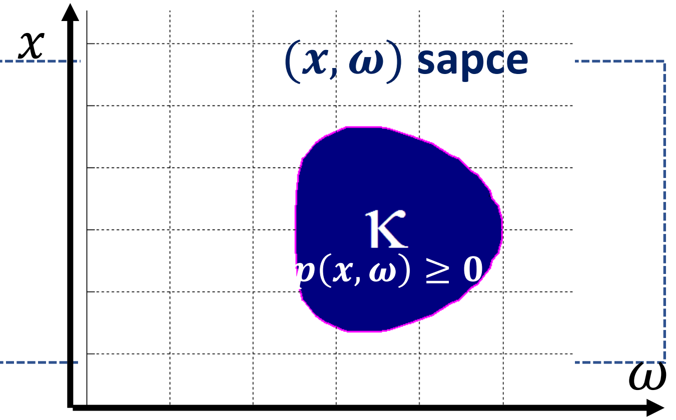
$$\text{subject to} \qquad -1 \leq x \leq 1$$

$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$

$$\omega \sim Uniform[-1,1]$$

Moment SDP:

$$x^* = y_{x_1} = 0.5$$



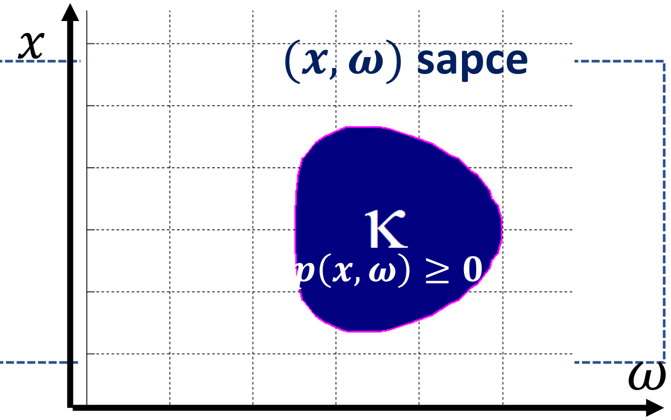
https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_1_Dual_ChanceOpt

Example: Dual Optimization

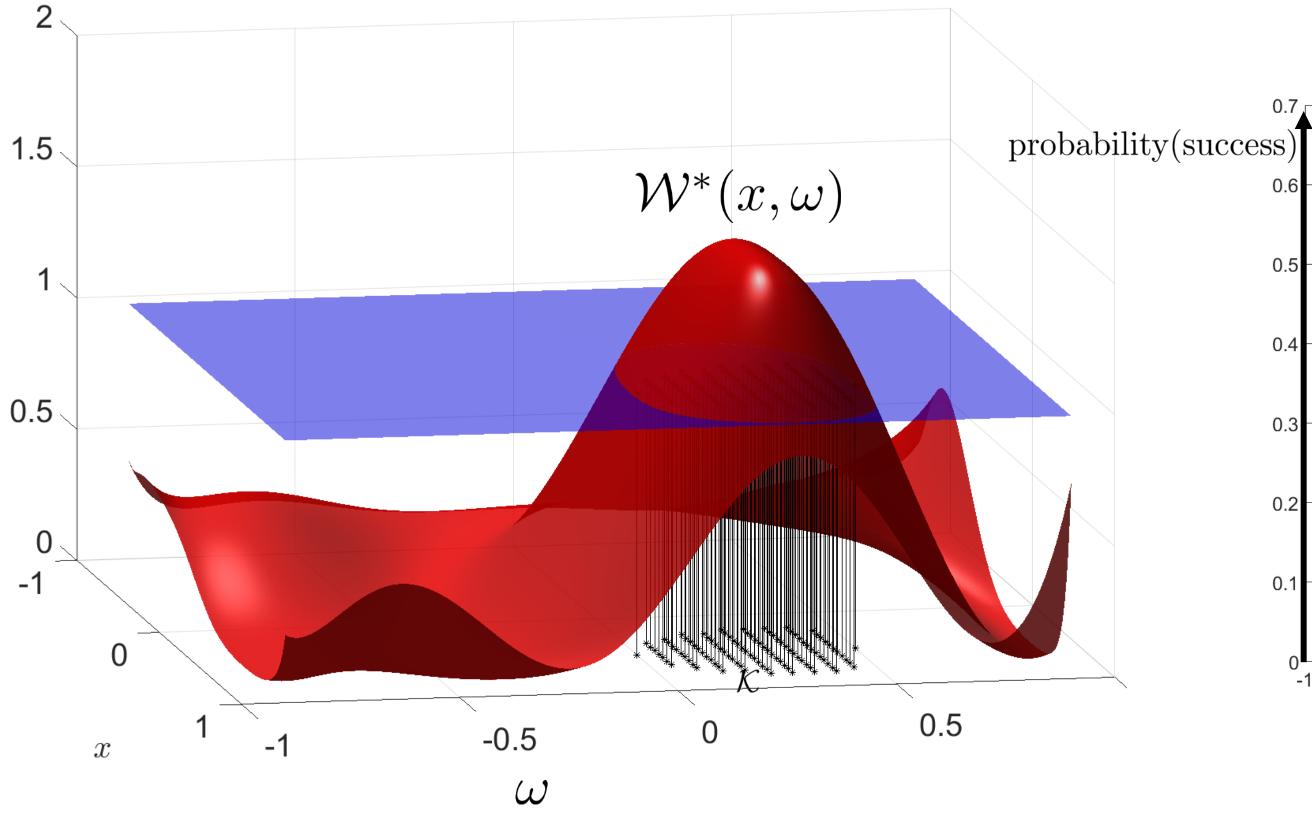
$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
$$\text{subject to} \qquad -1 \leq x \leq 1$$
$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$
$$\omega \sim Uniform[-1,1]$$

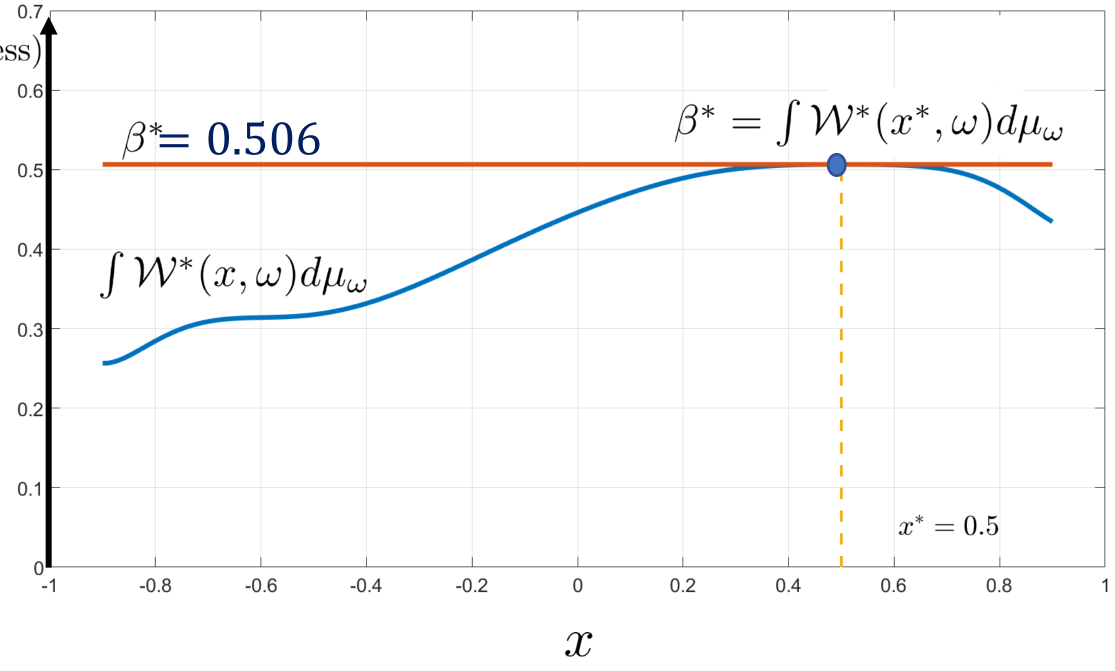
$\Rightarrow$ Moment SDP:

$x^* = y_{x_1} = 0.5$



Dual Optimization (SOS) $d = 5$:





https://github.com/jasour/rarnop19/tree/master/Lecture7_ChanceOptimization/Example_1_Dual_ChanceOpt

Example: Dual Optimization

$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$

$$\text{subject to} \qquad -1 \leq x \leq 1$$

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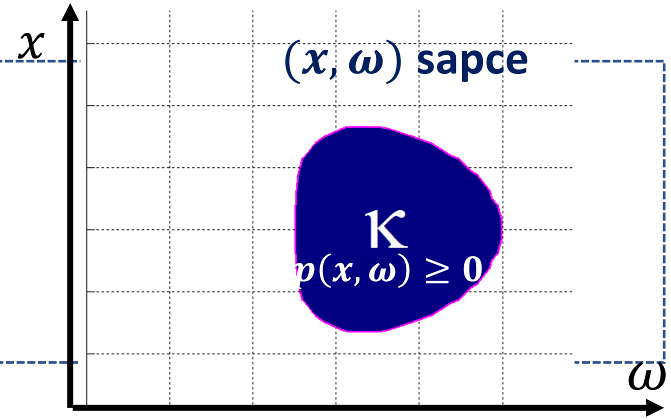
$$\omega \sim Uniform[-1,1]$$

Moment SDP:

$$x^* = y_{x_1} = 0.5$$



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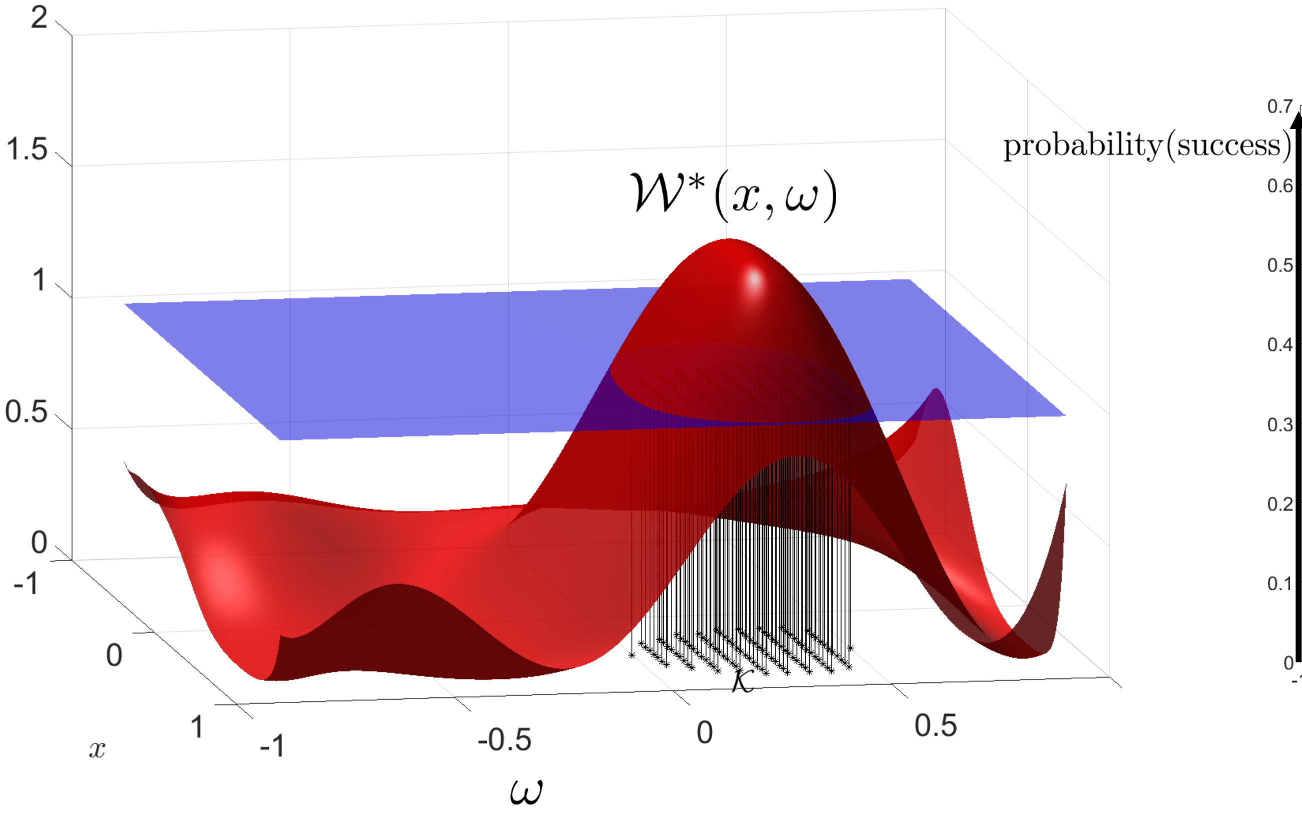
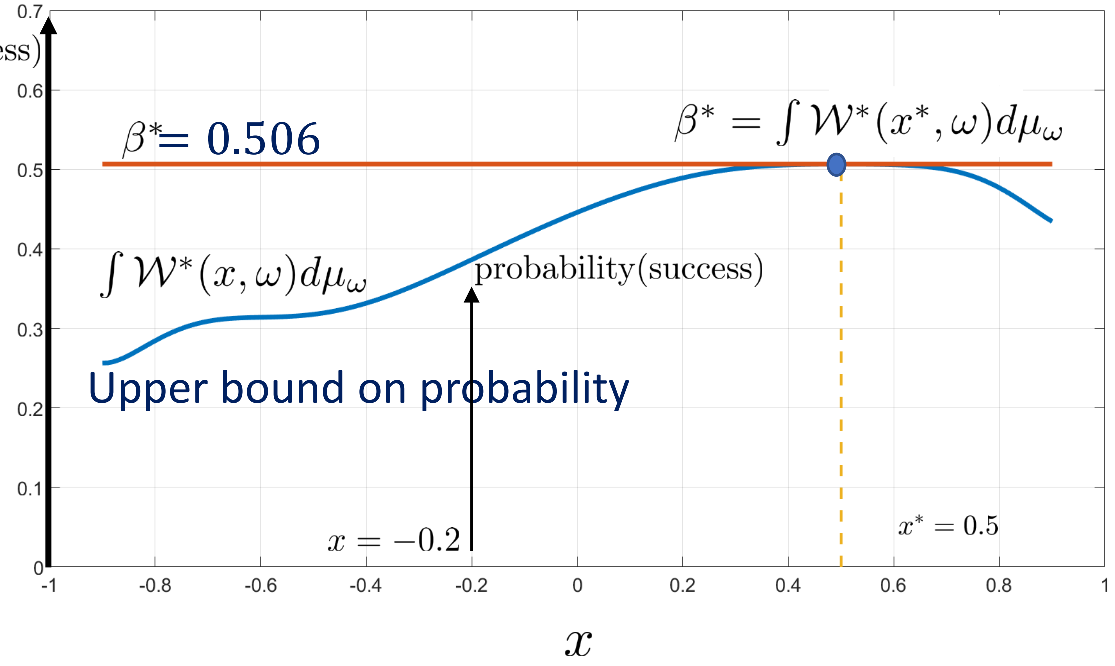


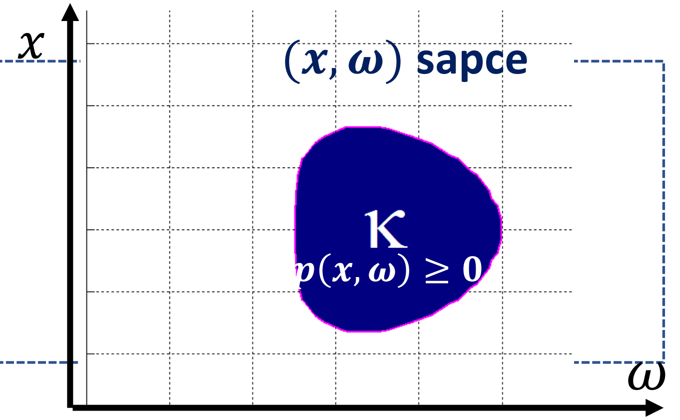


Example: Dual Optimization

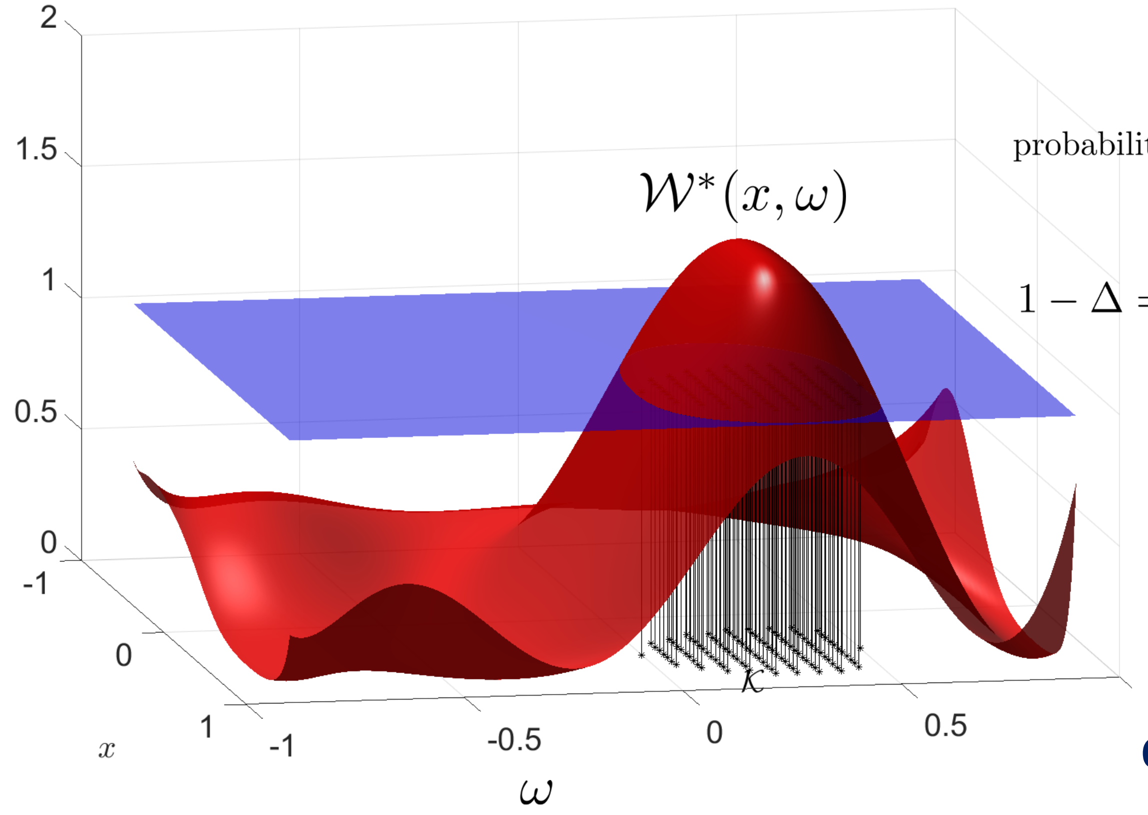
$$\mathbf{P}^* = \underset{x}{\text{maximize}} \quad \text{Probability}(\ p(x,\omega) \geq 0\ )$$
$$\text{subject to} \quad -1 \leq x \leq 1$$
$$p(x,\omega) = 0.5\omega\left(\omega^2 + (x-0.5)^2\right) - \left(\omega^4 + \omega^2(x-0.5)^2 + (x-0.5)^4\right)$$
$$\omega \sim Uniform[-1,1]$$

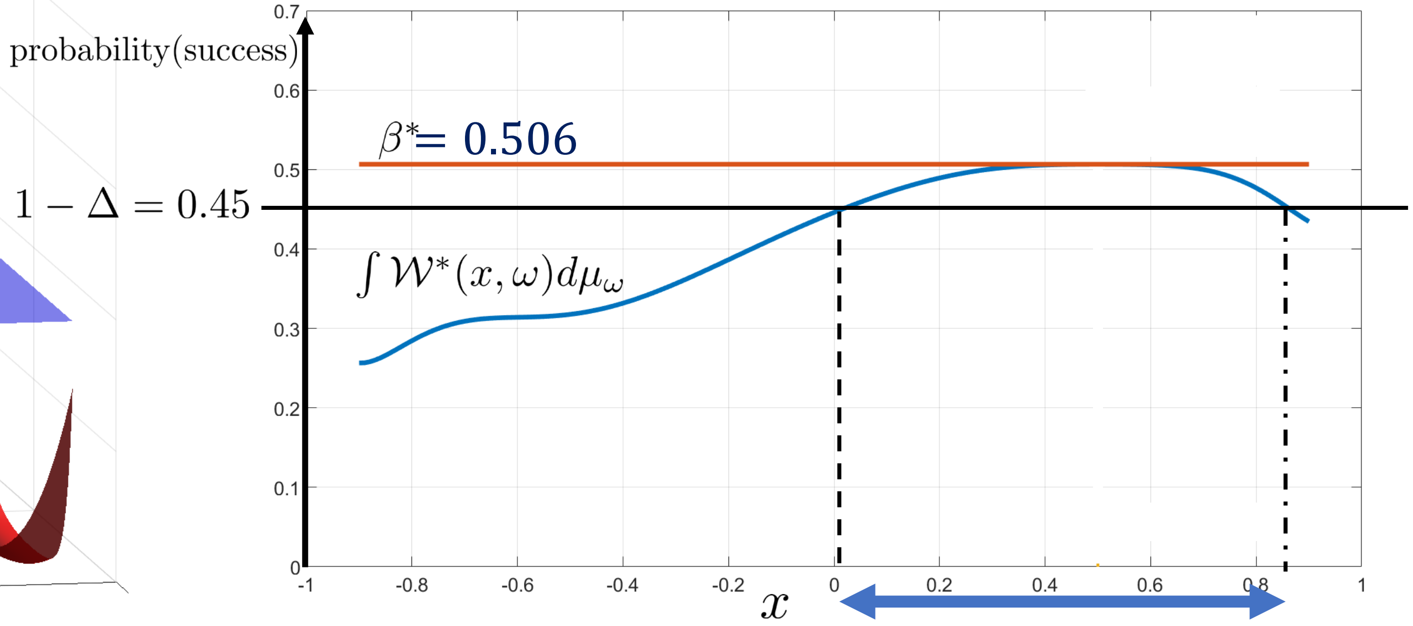
Moment SDP:

$$x^* = y_{x_1} = 0.5$$



Dual Optimization (SOS) $d = 5$:





Outer approximation of $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

$$\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}^*(x,\omega)\ d\mu_\omega \geq 1 - \Delta\}$$

Chance Constrained Set: $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta\in\mathbb{R}, \mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \beta \\ \text{subject to} \quad & \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \beta - \int \mathcal{W}(x,\omega) d\mu_\omega \geq 0 \quad \forall x \in \chi \\ & \beta \geq 0, \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

Outer Approximation :

$$\chi_{cc} = \left\{ x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1 - \Delta \right\}$$

Modified Formulation:

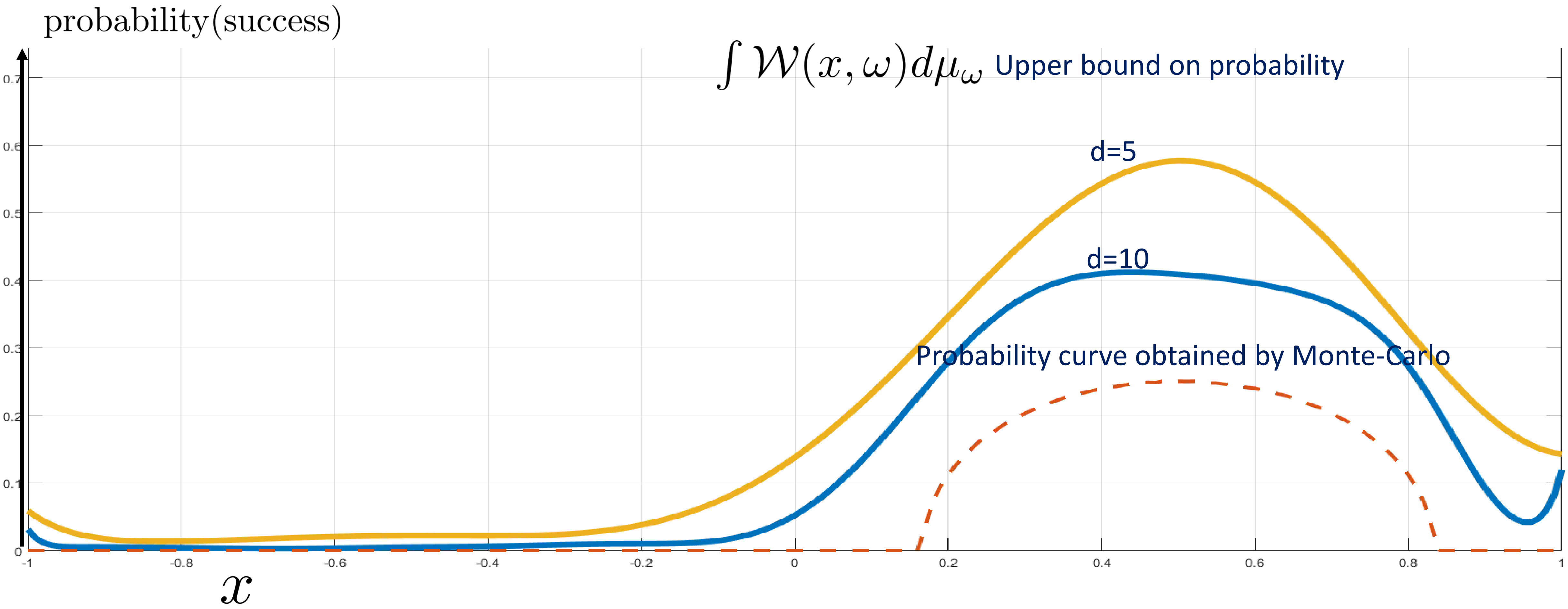
$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\mathcal{W}(x,\omega)\in\mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \mathcal{W}(x,\omega) d\mu_\omega dx \\ \text{subject to} \quad & \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

Outer Approximation :

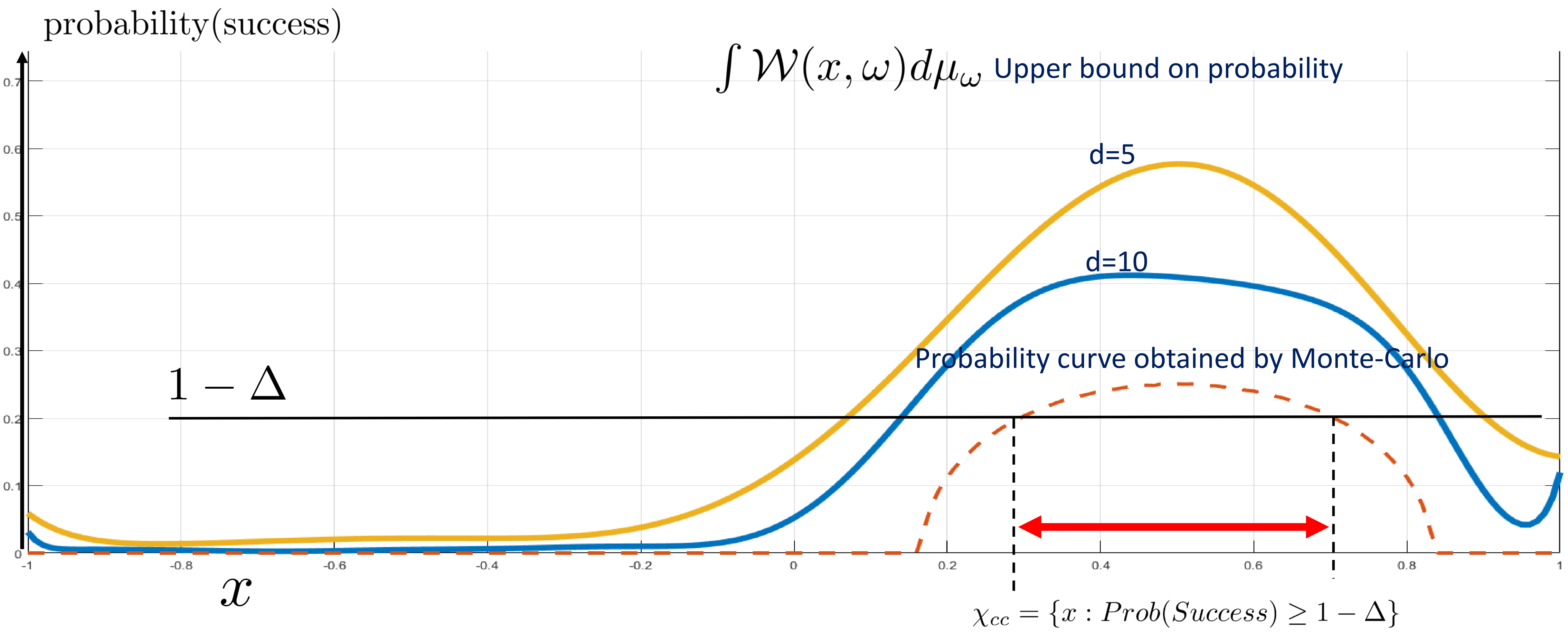
$$\chi_{cc} = \left\{ x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1 - \Delta \right\}$$

- J. B. Lasserre, Representation of Chance-Constraints With Strong Asymptotic Guarantees, IEEE Control Systems Letters , Volume: 1 , Issue: 1, 2017.

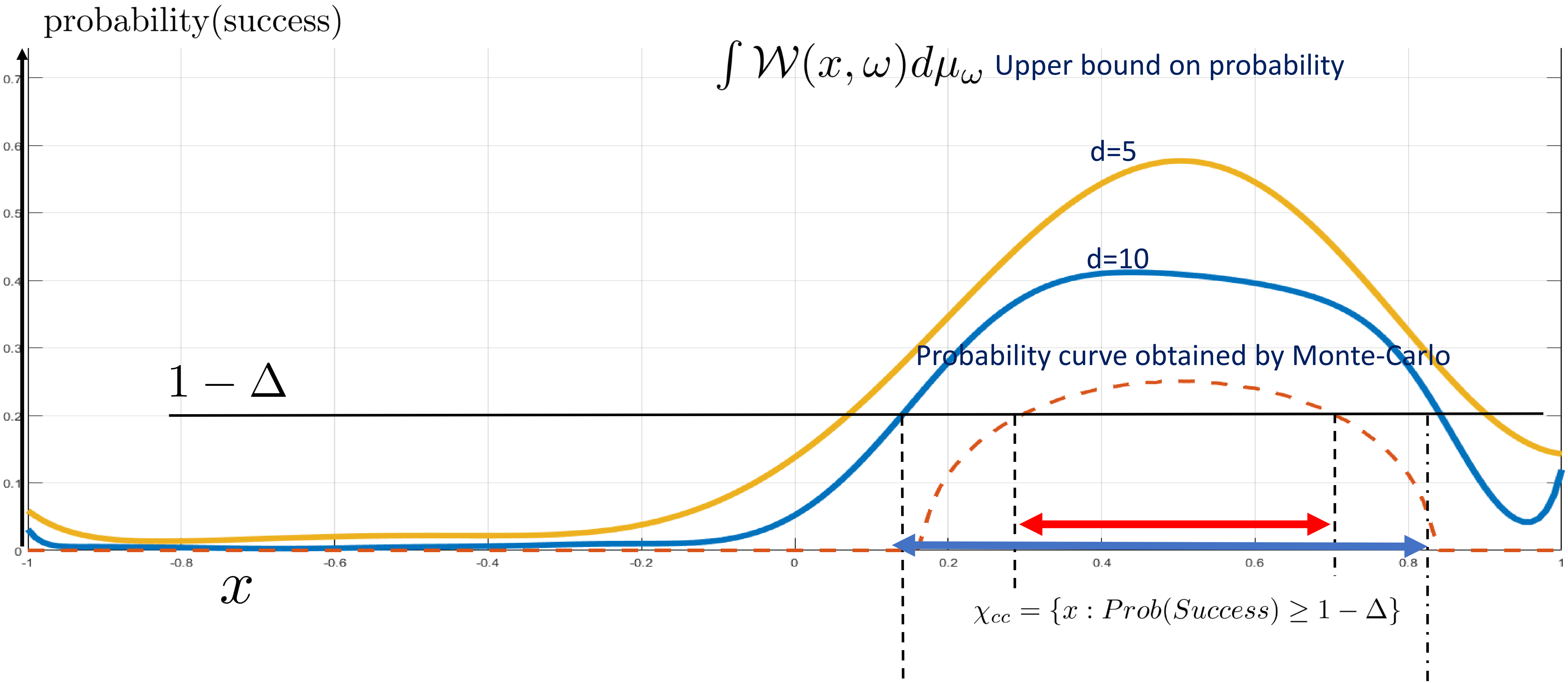
Example 1: Modified SOS Program for Chance Constrained Set



Example: Modified SOS Program for Chance Constrained Set



Example: Modified SOS Program for Chance Constrained Set



Outer approximation : $\chi_{cc} = \left\{x : \int \mathcal{W}(x,\omega)\ d\mu_\omega \geq 1-\Delta\right\}$

Chance Constrained Set: Inner Approximation

- We can apply same methodology to the Complement Set **(failure Set)**

  - Chance Constrained Set:

    $$\{x : \mathbb{R}^n : Prob(failure) \leq \Delta\}$$

  - Outer approximation:

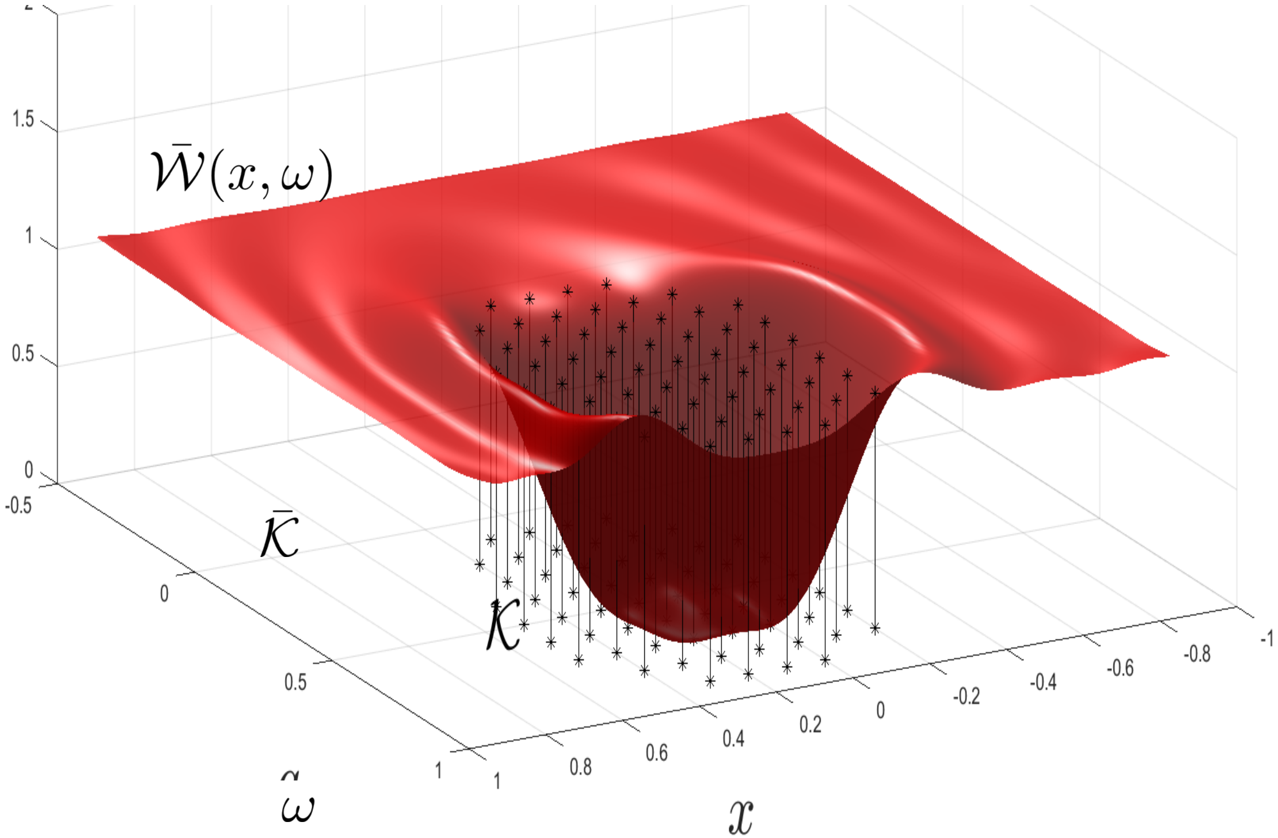
    $$\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x, \omega)\ d\mu_\omega \leq \Delta\}$$

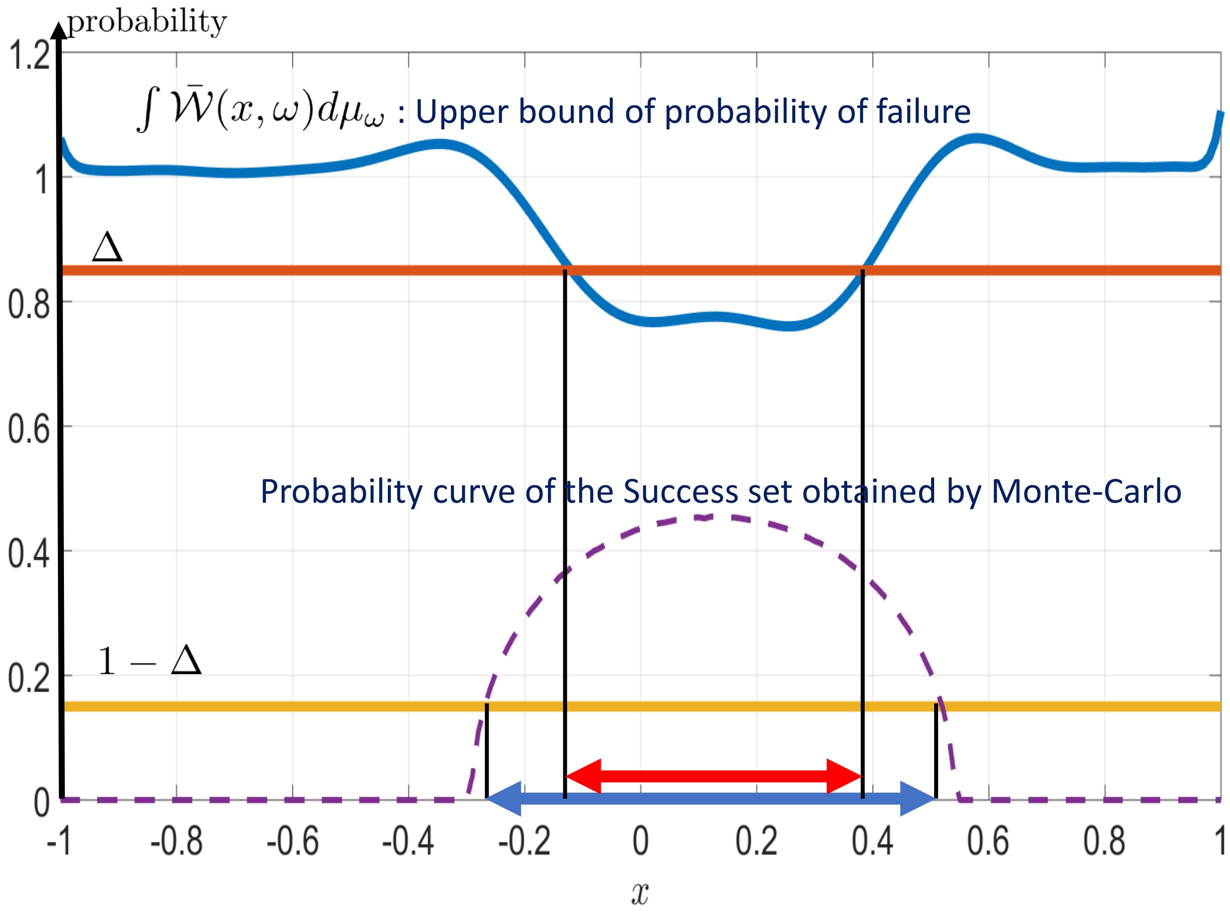
  - Inner approximation of $\{x \in \mathbb{R}^n : Prob(Success) \geq 1 - \Delta\}$

Example: Inner approximation

$$\mathcal{K} = \{(x,\omega) : p(x,\omega) = (0.8x-0.1)^4 - 0.27\omega((0.8x-0.1)^2 + 0.3\omega^2) + 0.3\omega^2(0.8x-0.1)^2 + +0.09\omega^4 \geq 0\}$$

$\omega \sim Uniform[-1,1]$ $\{x : Prob(Success) \geq 1-\Delta\}$





Inner approximation of $\{x : Prob(Success) \geq 1-\Delta\}$: $\chi_{cc} = \{x \in \mathbb{R}^n : \int \bar{\mathcal{W}}(x,\omega)\, d\mu_\omega \leq \Delta\}$

Chance Constrained Optimization

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & \text{Probability}_{\text{pr}(\omega)}\left(g_i(x,\omega) \geq 0, \; i = 1, ..., n_g \right) \geq 1 - \Delta \end{aligned}$$

Deterministic Opt

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & p(x) \\ \text{subject to} \quad & x \in \chi_{cc} \end{aligned}$$

SOS Programing for Chance Constrained Set

$$\begin{aligned} \mathbf{P}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \mathcal{W}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \mathcal{W}(x,\omega) d\mu_\omega dx \\ \text{subject to} \quad & \mathcal{W}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \mathcal{K} \\ & \mathcal{W}(x,\omega) \geq 0 \end{aligned}$$

$$\begin{aligned} \bar{\mathbf{P}}^{*\mathbf{d}}_{\mathbf{sos}} = \underset{\beta \in \mathbb{R}, \bar{\mathcal{W}}(x,\omega) \in \mathbb{R}_d[x,\omega]}{\text{minimize}} \quad & \int \bar{\mathcal{W}}(x,\omega) d\mu_\omega dx \\ \text{subject to} \quad & \bar{\mathcal{W}}(x,\omega) - 1 \geq 0 \quad \forall (x,\omega) \in \bar{\mathcal{K}} \\ & \bar{\mathcal{W}}(x,\omega) \geq 0 \end{aligned}$$

complement set

Chance Constrained Set $\{x \in \mathbb{R}^n : \text{Probability}(Success) \geq 1 - \Delta\}$

- **Outer approximation :**

$$\chi_{cc} = \{x \in \mathbb{R}^n : \int \mathcal{W}(x,\omega) \; d\mu_\omega \geq 1 - \Delta\}$$

- As polynomial order $d \to \infty$

χ_{cc} Convergences to the true chance constrained set

- **Inner approximation :**

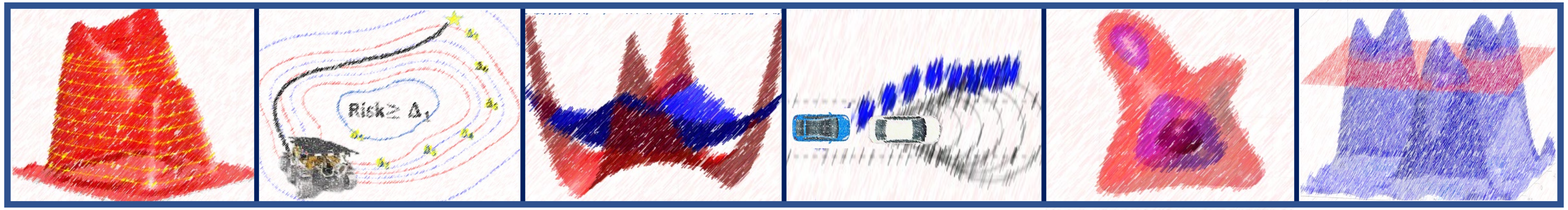
$$\bar{\chi}_{cc} = \{x \in \mathbb{R}^n : \int \bar{\mathcal{W}}(x,\omega) \; d\mu_\omega \leq \Delta\}$$

- As polynomial order $d \to \infty$

$\bar{\chi}_{cc}$ Convergences to the true chance constrained set

- For more **details, examples, codes, applications** in uncertain dynamical systems and robotics:

rarnop.mit.edu,
*Ashkan Jasour, "**Risk Aware** and **Robust Nonlinear Planning**(rarnop)",*
Course Notes for MIT 16.S498, Fall, 2019.



- Related Publications: jasour.mit.edu/publications