Polynomial time guarantees for the Burer-Monteiro method

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Joint work with **Ankur Moitra** (MIT)
arXiv:1912.01745

Virtual Seminar on Optimization and Related Areas - 2020

Semidefinite programming

A *semidefinite program* is

$$
\text{(SDP)} \qquad \begin{aligned} \min_{X\in\mathbb{S}^n} \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \text{ for } i \in [m] \\ & X \succeq 0 \end{aligned}
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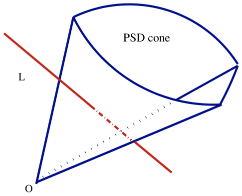

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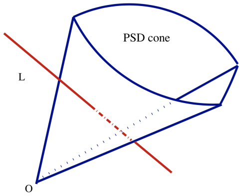

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- Polynomial time solvable with interior point methods (Nesterov-Nemirovski'87).
- Many different applications.

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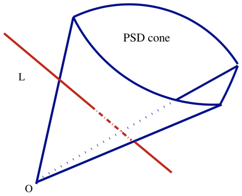

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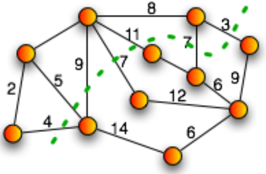
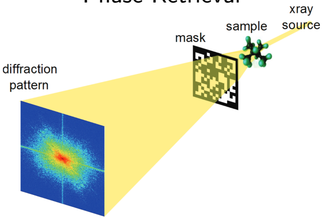
Goal: Show that SDPs can be solved in polynomial time with a more recent class of methods. The proof is remarkably geometric.

SDP applications

Max-Cut



Phase Retrieval



Matrix completion

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
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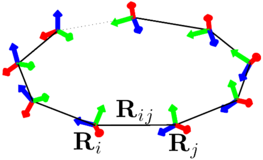
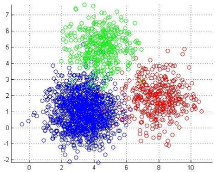
Rotation synchronization



Clustering



Solving large scale SDPs

Interior point methods intractable for large SDPs (too much memory)

Practical methods

- Low rank factorization (Burer, Monteiro)
- Frank-Wolfe / CGM (Hazan, Jaggi, Freund, Grigas, Mazumder)
- Sketching (Yurtsever, Ding, Udell, Tropp, Cevher)
- Bundle (Helmberg, Rendl, Oustry)
- Subgradients (Nesterov, Yurtsever, Tran Dinh, Cevher)

The Burer-Monteiro method is one of the most widely used in practice.

Burer-Monteiro method

$$\text{(SDP)} \qquad \min_{X \in \mathbb{S}^n} \quad C \bullet X \quad \text{s.t.} \quad A_i \bullet X = b_i \text{ for } i \in [m], \quad X \succeq 0$$

Assume the optimal solution has rank $\leq p$, and factorize

$$X = YY^T \quad \text{where } Y \in \mathbb{R}^{n \times p}$$

Use local optimization to solve the *nonconvex* problem

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How large should we choose p?

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How large should we choose p?

Theorem (Barvinok-Pataki bound)

BM is equivalent to SDP when $\binom{p+1}{2} \geq m$. So we need $p \gtrsim \sqrt{2m}$.

Known global guarantees on BM

Below Barvinok-Pataki bound
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Assume $\binom{p+1}{2} > m$.

- BM *should* work [Burer-Monteiro, Journée et al.]

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Assume compactness and $\binom{p+1}{2} > \frac{9}{2} m \log(\sigma^{-1})$ for some $\sigma > 0$.

- Randomly perturb C (magnitude σ). No spurious approximate local minima w.h.p. [Pumir-Jelassi-Boumal]

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No polynomial time guarantees known!
Obstacle: How to find a local minimum that is exactly feasible?

Our results

Assumptions:

- $\binom{p+1}{2} > (1+\eta)m$ for a fixed $\eta > 0$.
- Constraint set is smooth and compact.
- Solve BM using local method with 2nd order guarantees.

Theorem (Polytime optimality)

Randomly perturb C (magnitude σ). Given an initial approx feasible point, then BM computes a point that is approx feasible & approx optimal w.h.p. in $\mathrm{poly}(n, \sigma^{-1})$ *iterations.*

Smoothed analysis [Spielman-Teng '01]

- Some practical algorithms are slow in a few instances.
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BM for SDP feasibility

Goal:

$$\text{find } X \quad \text{s.t.} \quad A_i \bullet X \approx b_i \text{ for } i \in [m], \quad X \succeq 0$$

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Consider the least squares problem:

$$(\mathrm{SDP}_{ls}) \qquad \min_{X \in \mathbb{S}^n} \sum_i (A_i \bullet X - b_i)^2 \quad \text{s.t.} \quad X \succeq 0$$

The associated Burer-Monteiro problem is

$$(\mathrm{BM}_{ls}) \qquad \min_{Y \in \mathbb{S}^n} \sum_i (A_i \bullet YY^T - b_i)^2$$

Previous work on BM_{ls} relies on RIP [Bhojanapalli-Neyshabur-Srebro].

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Theorem (Polytime feasibility)

Randomly perturb $A_1, \dots, A_m$ *(magnitude σ). Then* BM_{ls} *computes an approx feasible point w.h.p. in* $\mathrm{poly}(n, \sigma^{-1})$ *iterations.*

Optimality/Criticality conditions

(SDP) $\quad \min \quad C \bullet X \quad \text{s.t.} \quad A_i \bullet X = b_i \text{ for } i \in [m], \quad X \succeq 0$

Lemma

Assuming strong duality, X is optimal for SDP iff there exists λ such that

$$A_i \bullet X = b_i, \quad X \succeq 0, \quad S(\lambda)X = 0, \quad S(\lambda) \succeq 0,$$

where $S(\lambda) := C - \sum_i \lambda_i A_i$.

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Lemma

Y is a 2-critical point of BM iff there exists λ such that

$$A_i \bullet YY^T = b_i, \quad S(\lambda)Y = 0, \quad S(\lambda) \bullet UU^T \geq 0 \;\; \forall U : A_i \bullet UY^T = 0$$

A critical point Y of BM is spurious if YY^T is not optimal for SDP.

Spurious critical points

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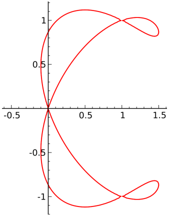
Theorem (Boumal-Voroninski-Bandeira)

Spurious critical points may only exist if

$$C \in \mathcal{M} + \mathcal{L} \subset \mathbb{S}^n$$

where $\mathcal{M} = \{S : \operatorname{rank} S \leq n - p\}$ *and* $\mathcal{L} = \operatorname{span}\{A_1, \ldots, A_m\}$

- Assume that $\binom{p+1}{2} > m$.
- $\mathcal{M} + \mathcal{L}$ is an *algebraic variety* of codimension $\binom{p+1}{2} - m$.
- Generically, no spurious points.



Approximate optimality/criticality

(SDP) X is δ-optimal if there exists λ such that

$$\|A_i \bullet X - b_i\| \leq \delta, \qquad \|S(\lambda)X\| \leq \delta, \qquad X \succeq 0, \qquad S(\lambda) \succeq -\delta I_n.$$

(BM) Y is ϵ-critical for BM if there exists λ such that

$$\|A_i \bullet YY^T - b_i\| \leq \epsilon, \qquad \|S(\lambda)Y\| \leq \epsilon^2,$$
$$S(\lambda) \bullet UU^T \geq -\epsilon \quad \forall U : \|U\| \leq 1,\ \|A_i \bullet UY^T\| \leq \epsilon.$$

Theorem (C.-Moitra)

Assume domain is smooth and compact. Given an approx feasible point, local optimization can produce an approx critical point in poly(ϵ^{-1}).

Spurious approximate critical points

An ϵ-critical point Y of BM is spurious if YY^T is not δ-optimal for SDP, with $\delta = O(\epsilon)$.

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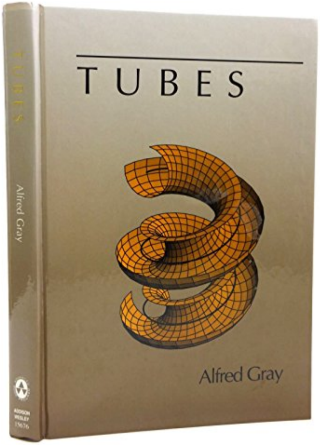


Theorem (C.-Moitra)

Spurious ϵ-critical points may only exist if

$$C \in \mathrm{tube}_\epsilon(\mathcal{M} + \mathcal{L}) \subset \mathbb{S}^n$$

where $\mathrm{tube}_\epsilon(\mathcal{W}) := \{X : \mathrm{dist}(X, \mathcal{W}) \leq \epsilon\}$

Volumes of tubes

Theorem (Weyl 1939)

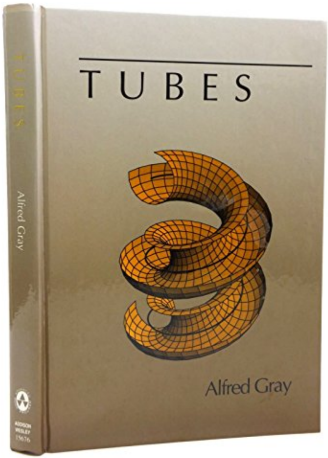
Let $V \subset \mathbb{R}^n$ manifold of codimension c. There are curvature constants $k_i(V)$ such that

$$\mathrm{Vol}[\mathrm{tube}_\epsilon(V)] = \sum_{i=c}^{n} k_i(V)\,\epsilon^i$$

Theorem (Lotz 2015)

Let $V \subset \mathbb{R}^n$ variety of codimension c defined by polynomials of degree D. Let x uniformly distributed on a ball of radius σ. Then

$$\Pr[x \in \mathrm{tube}_\epsilon(V)] \le O(nD\epsilon/\sigma)^c$$



Polytime optimality

Theorem (C.-Moitra)

Assumptions: $\binom{p+1}{2} > (1+\eta)m$, *constraint set smooth and compact, solve BM with local method with 2nd order guarantees.*

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Proof.

Method converges to an ϵ-critical point in $\mathrm{poly}(\epsilon^{-1})$. Using tubes,

$$\Pr[\text{spurious}] \;\leq\; \Pr[C \in \text{tube}_\epsilon(\mathcal{M}+\mathcal{L})] \;\leq\; \epsilon^{\binom{p+1}{2}-m} \cdot O\left(n^3/\sigma\right)^{\binom{p+1}{2}}$$

For $\binom{p+1}{2} > (1+\eta)m$ and $\epsilon = O(\sigma/n^3)^{1+1/\eta}$ the probability is tiny. □

Polytime feasibility

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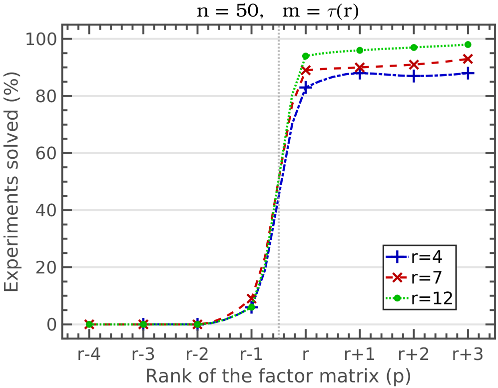
Randomly perturb $A_1, \ldots, A_m$ (magnitude σ). Then BM_{ls} computes an approximately feasible solution w.h.p. in $\mathrm{poly}(n, \sigma^{-1})$ iterations.

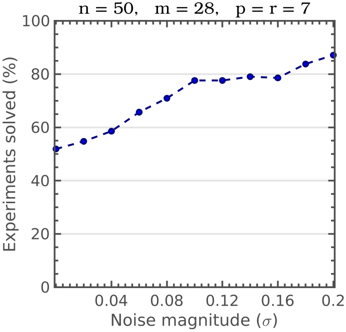
Experiments

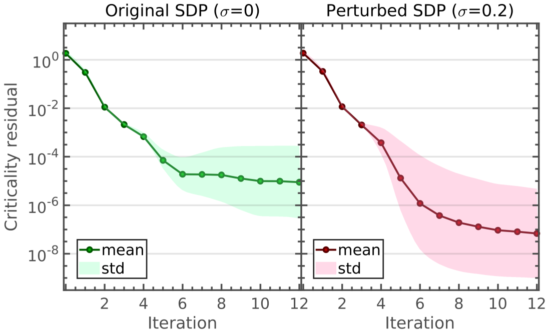
- Generate planted matrix of rank r.
- Generate SDP for which planted marix is optimal.
- Solve BM using augmented Lagrangians.



Experiments

- Fix an SDP instance for which BM behaves badly.
- Perturb the problem with small noise.
- Solve BM using augmented Lagrangians.





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- D. Cifuentes, A. Moitra, *Polynomial time guarantees for the Burer-Monteiro method*, `arXiv:1912.01745`.
- D. Cifuentes *On the Burer-Monteiro method for general semidefinite programs*, `arXiv:1904.07147`.

Thanks for your attention