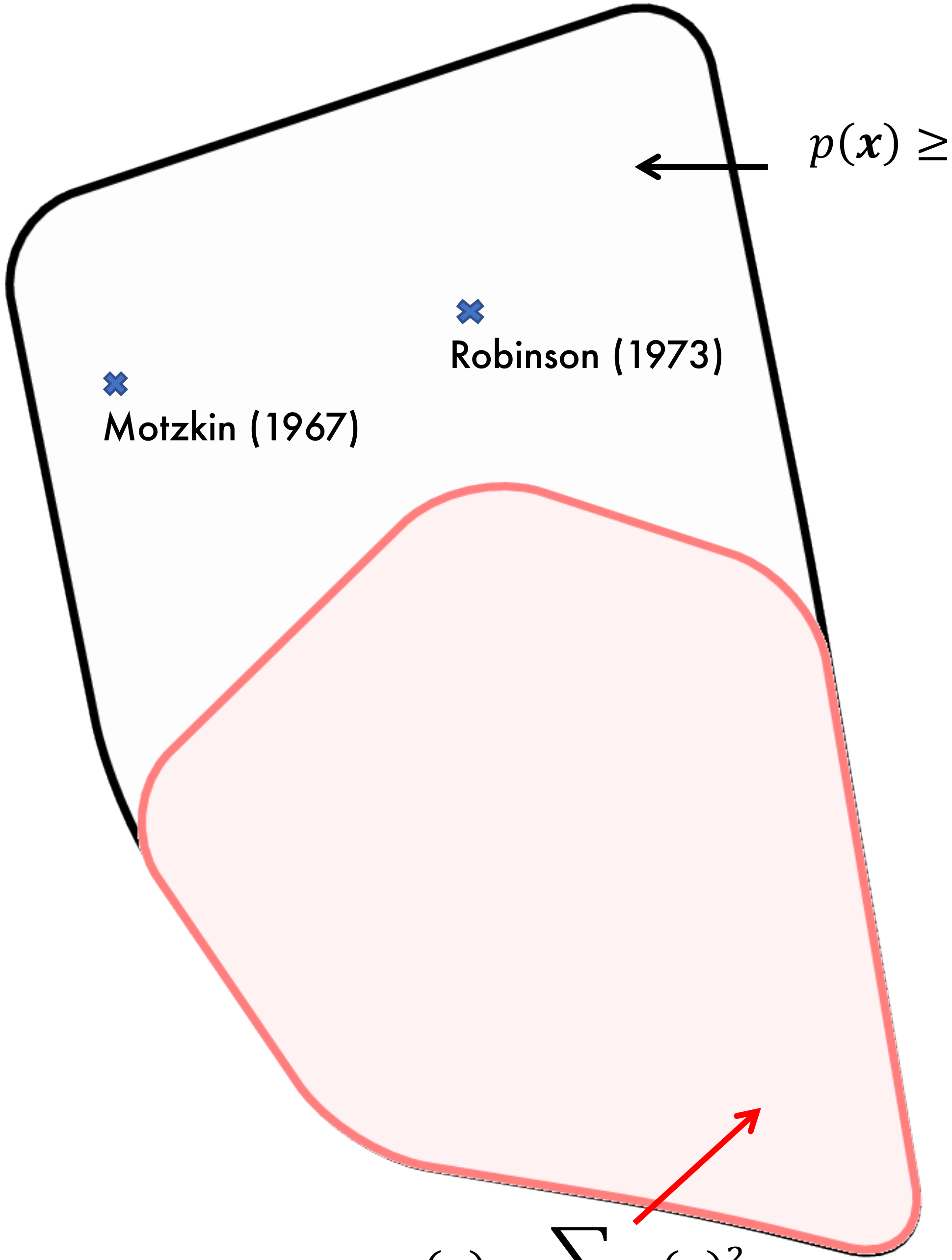
SOS vs Convex

and Generalized Cauchy-Schwarz

Bachir El Khadir

Mar 20th, 2020
MIT–Princeton Online Seminars

$p(\boldsymbol{x}) \geq 0 \ \forall \boldsymbol{x} \in \mathbb{R}^n$



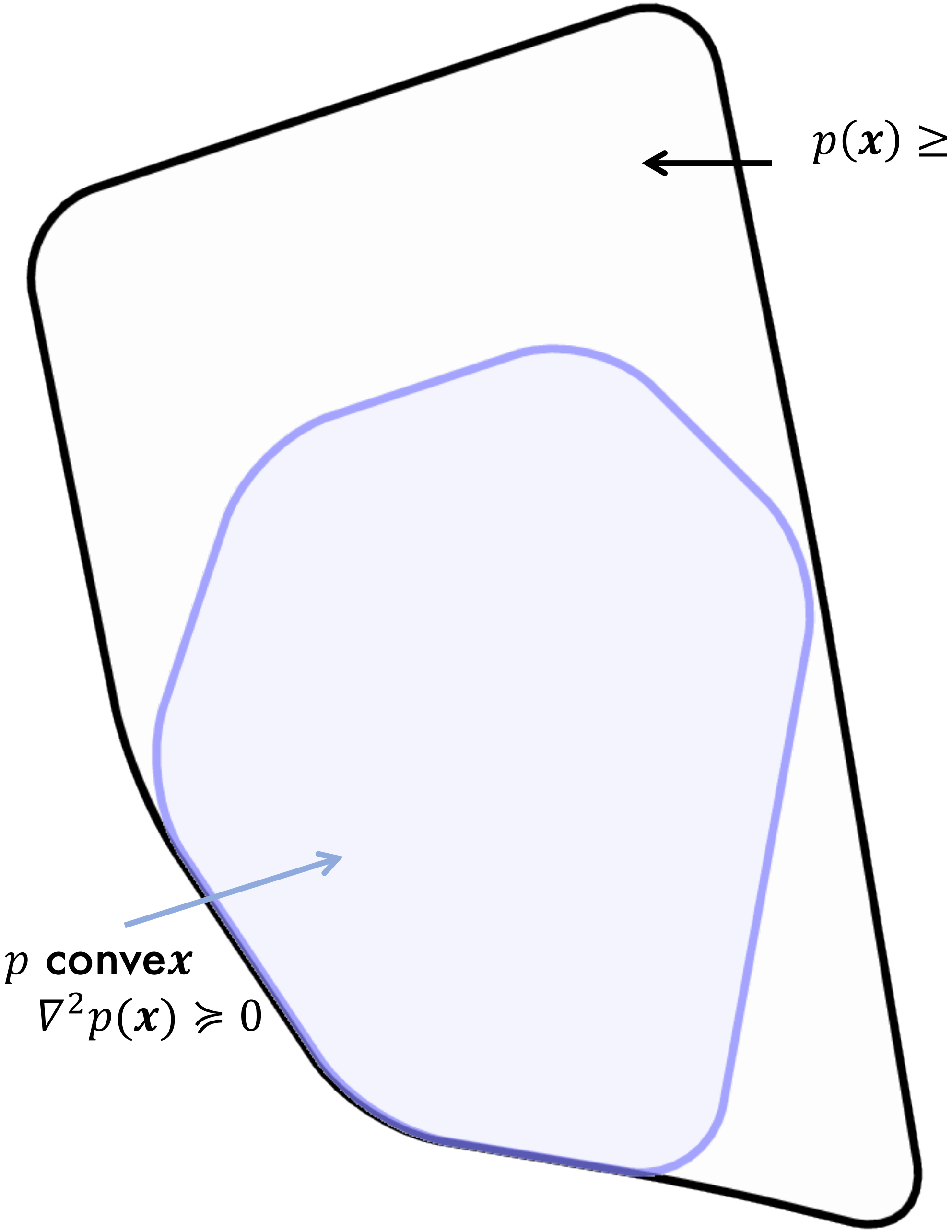
$$p(x) = \sum_i q_i(x)^2$$

$$p(\boldsymbol{x}) = \sum_{k_1 + \ldots + k_n \leq 2d} c_k \, x_1^{k_1} \ldots x_n^{k_n}$$

Nonegative → SOS ?

n \ 2d	2	4	≥ 6
1	✓	✓	✓
2	✓	✓	✓
3	✓	✓	✗
≥ 4	✓	✗	✗

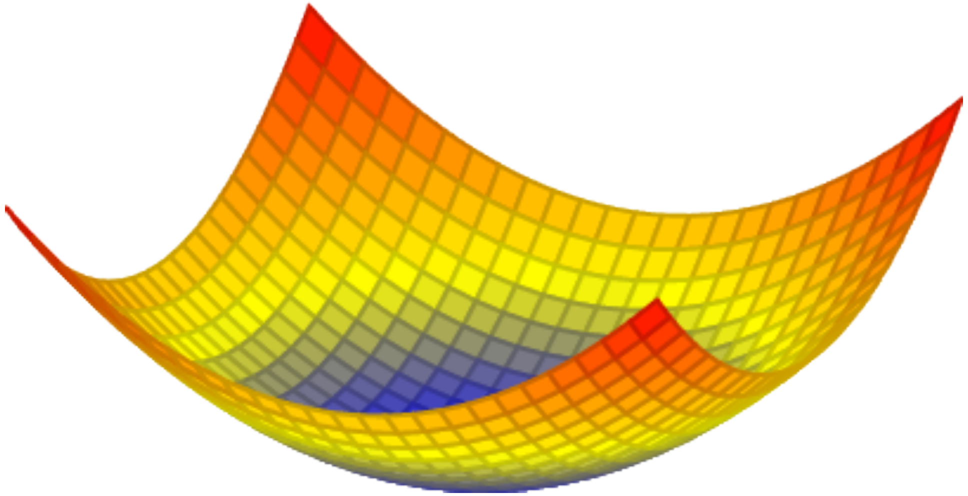
[Hilbert, 1888]

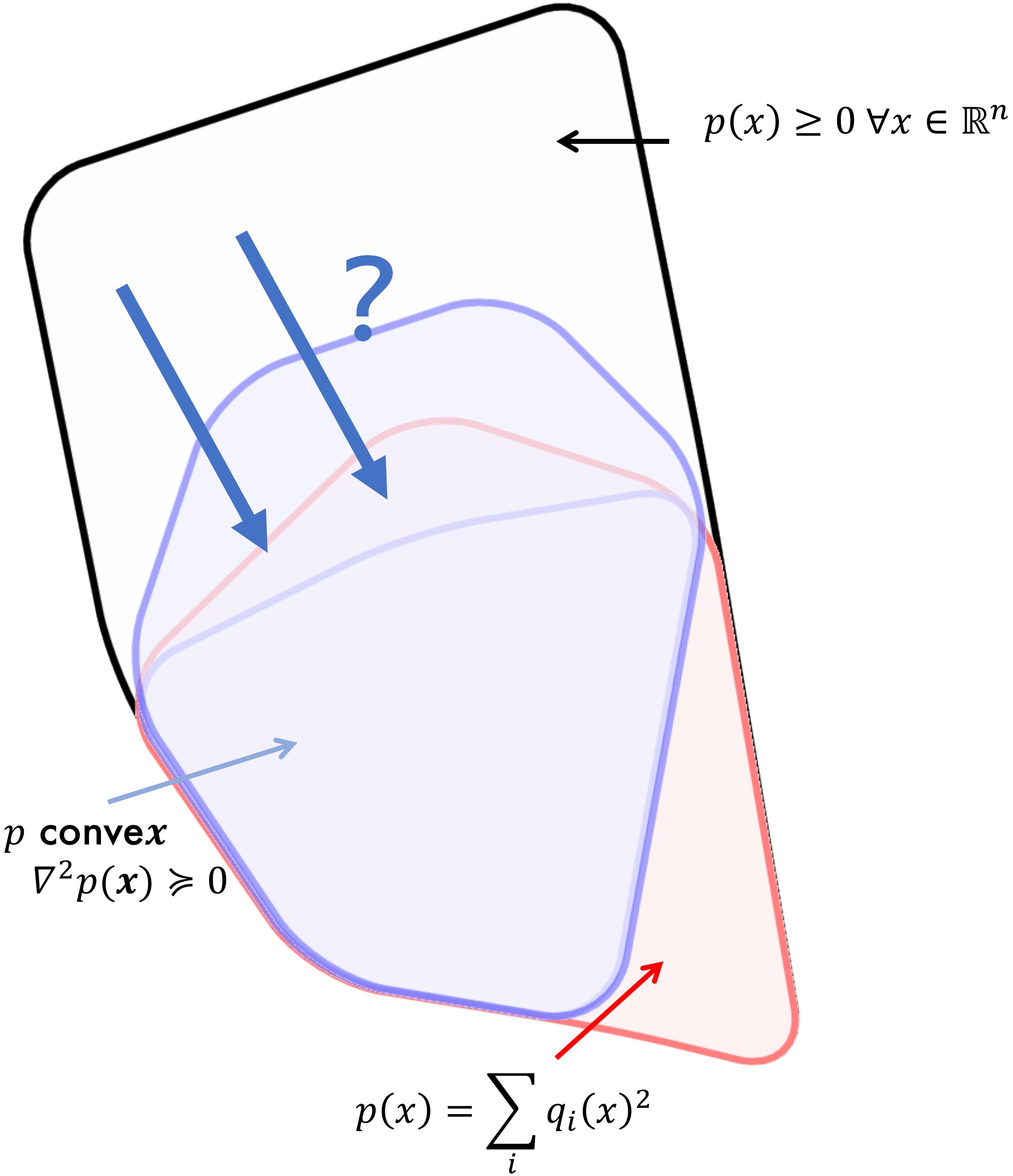


Convex ⟹ Nonnegative ?

Yes! Euler:

$$p(\boldsymbol{x}) = \frac{1}{2d} \boldsymbol{x}^T \nabla p(\boldsymbol{x})$$
$$= \frac{1}{2d(2d-1)} \boldsymbol{x}^T \nabla^2 p(\boldsymbol{x}) \boldsymbol{x}$$





07′, Parrilo: Convex ⟹ SOS ?

09′, Blekherman: No
For $2d \geq 4$ there are many more convex forms than sos as $n \to \infty$

Open problems:

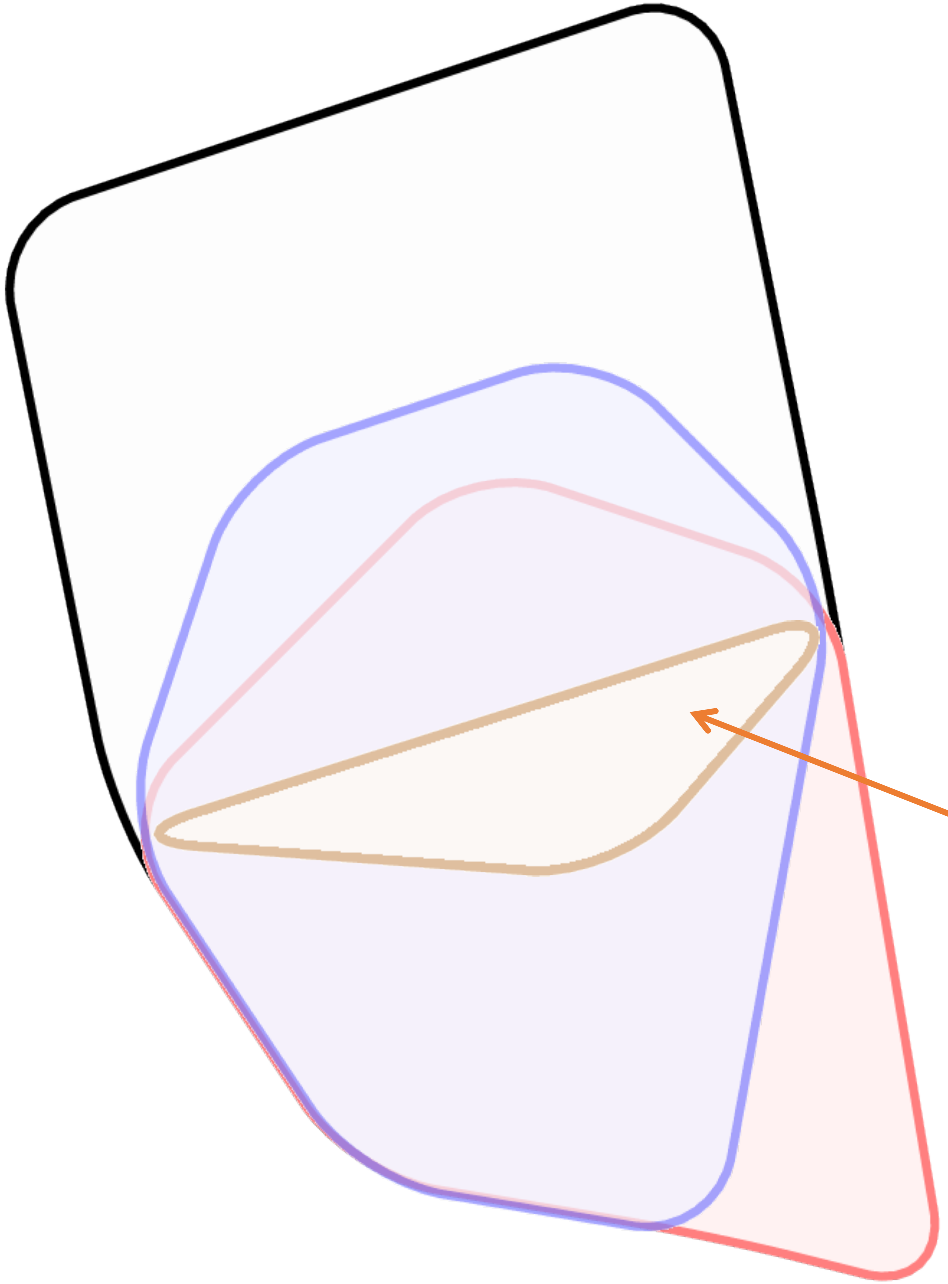
- No explicit examples are known!
- Smallest $(n, 2d)$ of such an example?

Minimal suspects:

$$(n, 2d) = (4,4), (3,6)$$

Today:

Convex quaternary quartic forms are sos.



p sos-convex
$y^T \nabla^2 p(x) y$ is sos

[Helton, Nie]

SOS-Convex ⟹ Convex

SOS-Convex ⟹ SOS

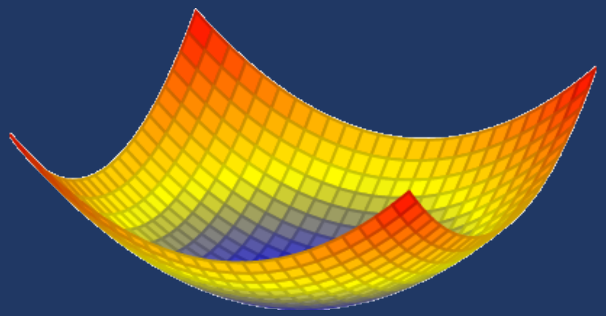
Computationally attractive!

Convex ⟹ SOS-Convex ?

n \ 2d	2	4	≥ 6
1	✓	✓	✓
2	✓	✓	✓
3	✓	✓	✗
≥ 4	✓	✗	✗

[Ahmadi, Blekherman, Parrilo]

$$p^* = \min_{\boldsymbol{x} \in \mathbb{R}^n} p(\boldsymbol{x})$$



p convex polynomial

Descent methods:

$$x_{k+1} = x_k - \alpha \nabla p(x_k)$$

SOS hierarchies

always convex

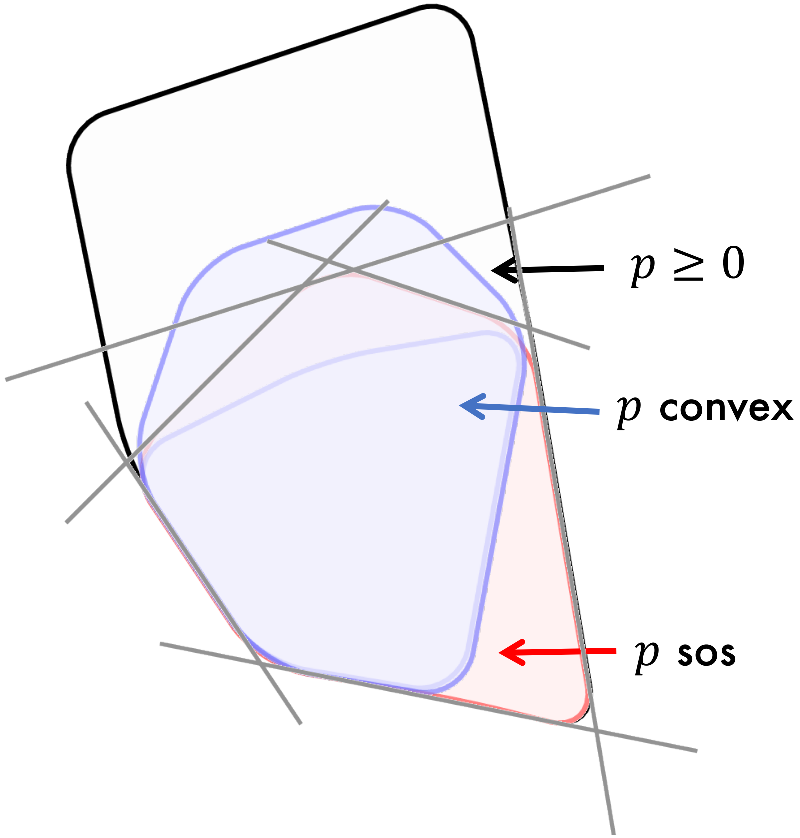
$p^{(1)} = \max \gamma$ s.t. $p - \gamma$ is sos

$\vdots$

If we knew that $\mathrm{p} - \gamma$ nonnegative $\Rightarrow \mathrm{p} - \gamma$ sos

Then $p^{(1)}$ is exact; $p^* = p^{(1)}$

General outline of the proof

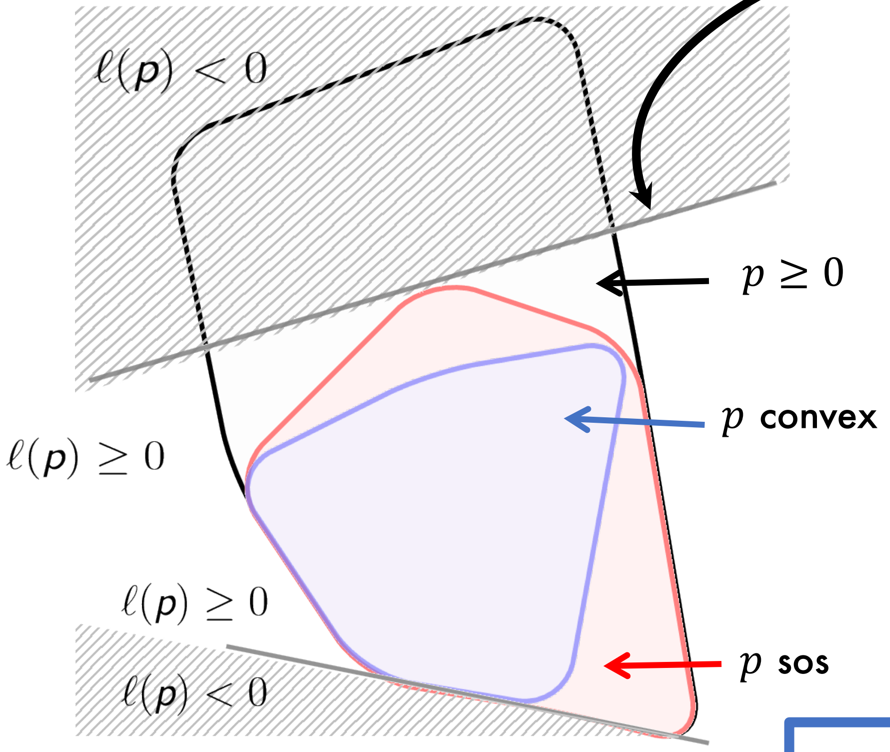


Theorem:

Convex quaternary quartic forms are sos.

1. Dual of the cone of sos forms
2. Dual of the cone of convex forms

Dual of the sos cone $(n = 4, 2d = 4)$



$$\ell : p \to \sum_{i=1,\ldots,8} \alpha_i p(z_i) \geq 0$$ [Blekherman, 2012]

$\sum_{i=1}^{8} \pm z_i z_i^T = 0$ and other conditions on α_i

[Cayley-Bacharach]

All z_i are real

Exactly two of the z_i are not real, say $z_1 = \bar{z}_2$

(I) $\sqrt{p(z_i)} \leq \sum_{j \neq i} \sqrt{p(z_j)}\ \forall i$

(II) $\sqrt{2}\sqrt{|p(z_1)| + \Re(p(z_1))} \leq \sum_{j \neq 1,2} \sqrt{p(z_j)}$

Takeaway: p sos iff it satisfies (I) and (II)

Example:

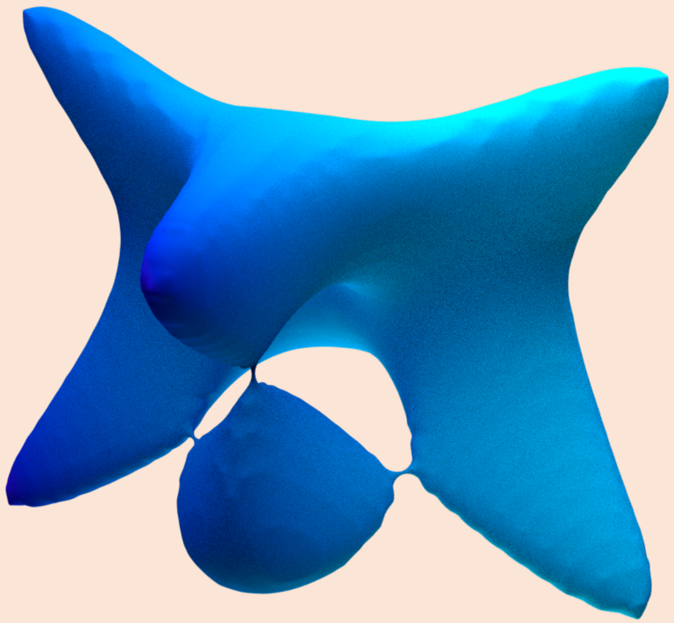
$$p(x_1, \dots, x_4) = \sum_{i=1,\dots,4} x_i^4 + \sum_{i,j,k} x_i^2 x_j x_k + 4x_1x_2x_3x_4$$ **Not SOS**

$$\boldsymbol{z_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{z_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \dots, \boldsymbol{z_8} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Cayley-Bacharach $\sum_k (-1)^k \; \boldsymbol{z_k z_k^T} = 0$

$p(\boldsymbol{z_1}) = 32,$ $p(\boldsymbol{z_k}) = 0$ for k$\neq$ 1

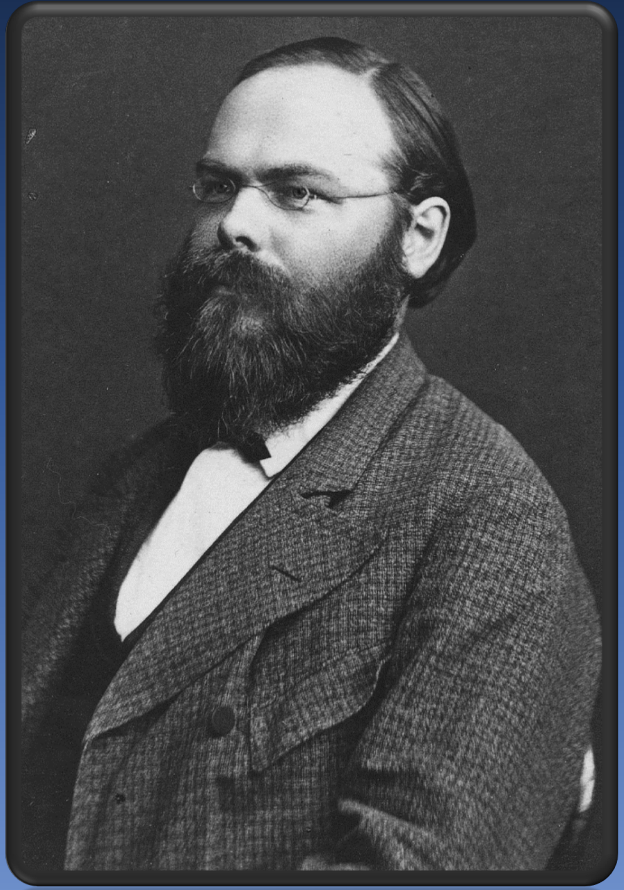
$$\sqrt{p(\boldsymbol{z_1})} \nleq \sum_{k>1} \sqrt{p(\boldsymbol{z_k})}$$



$\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, p(1, x_1, x_2, x_3) \leq 1\}$

Dual of the cone of convex forms

Generalizations of Cauchy-Schwarz



Background: Differential forms

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\boldsymbol{u} = \sum_{\boldsymbol{i}} u_i \, \boldsymbol{e_i} \in \mathbb{R}^n$$

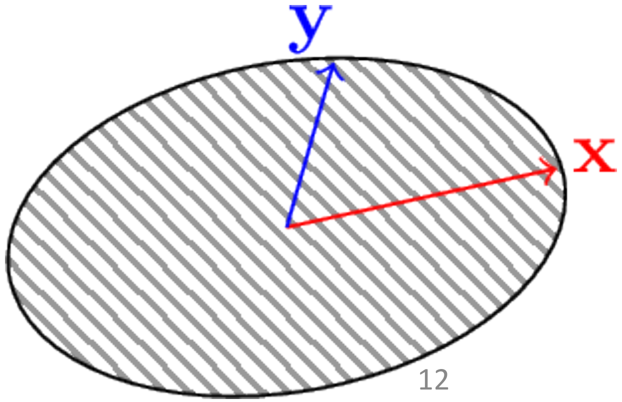
- $\partial_{\boldsymbol{u}} \coloneqq \boldsymbol{u}^T \, \nabla \; = \sum u_i \frac{\partial}{\partial x_i}$
- $\partial_{\boldsymbol{e_i}} = \frac{\partial}{\partial x_i}$
- $\mathbb{R}\left[\partial_{\boldsymbol{e_1}}, \dots, \partial_{\boldsymbol{e_n}}\right] \approx \mathbb{R}[x_1, \dots, x_n]$

p form of degree $2d$

- $\partial_{\boldsymbol{u}_1} \dots \partial_{\boldsymbol{u}_{2d}} p$ is a constant!
- Euler: $\partial_{\boldsymbol{u}} p(\boldsymbol{u}) = 2d \; p(\boldsymbol{u})$
- $\partial_{\boldsymbol{u}}^{2d} p = (2d)! \; p(\boldsymbol{u})$

$\boldsymbol{z} = \boldsymbol{x} + i \; \boldsymbol{y}$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$\partial_{\boldsymbol{z}}^d \partial_{\bar{\boldsymbol{z}}}^d p \sim \iint_{\alpha^2+\beta^2 \leq 1} p(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) \, d\alpha \, d\beta$$



Generalized Cauchy-Schwarz Inequalities

Theorem:

$\exists$ a universal constant A_d s.t.

$$\forall x, y \in \mathbb{R}^n \; \frac{1}{(2d)!} \partial_x^d \partial_y^d p \leq A_d \sqrt{p(x)} \sqrt{p(y)}$$

for every convex form p of degree $2d$.

Cauchy-Schwarz:

$$\forall x, y \in \mathbb{R}^n \; x^T Q y \leq \sqrt{x^T Q x} \sqrt{y^T Q y}$$

for every psd matrix Q.

$$\frac{1}{2} \partial_x \partial_y p \leq \sqrt{p(x)} \sqrt{p(y)}$$

for every convex quadratic form $p(x) = x^T Q x$.

$$q(\alpha, \beta) := p(\alpha x + \beta y)$$

p convex form of degree $2d$, $x, y \in \mathbb{R}^n$.

$$\frac{1}{(2\mathrm{d})!}\,\partial_{\boldsymbol{x}}^{d}\partial_{\boldsymbol{y}}^{d} p \le A\,\sqrt{p(\boldsymbol{x})}\,\sqrt{p(\boldsymbol{y})}$$

$$\frac{1}{(2\mathrm{d})!}\,\partial_{\boldsymbol{x}}^{d}\partial_{\boldsymbol{y}}^{d} p \le A\,\frac{p(\boldsymbol{x})+p(\boldsymbol{y})}{2}$$

$$\frac{1}{(2\mathrm{d})!}\,\partial_{\boldsymbol{x}}^{d}\partial_{\boldsymbol{y}}^{d} p \le \frac{A}{(2\mathrm{d})!}\,\frac{1}{2}\left(\partial_{\boldsymbol{x}}^{2d} p + \partial_{\boldsymbol{y}}^{2d} p\right)$$

$$0 \le \Big[\underbrace{A\frac{1}{2}\left(\partial_{\boldsymbol{x}}^{2d}+\partial_{\boldsymbol{y}}^{2d}\right)-\partial_{\boldsymbol{x}}^{d}\partial_{\boldsymbol{y}}^{d}}_{\ell_A}\Big]\,p$$

Cauchy-Schwarz: $x^T y \le \sqrt{x^T x}\,\sqrt{y^T y}$

Enough to show: $x^T y \le \frac{x^T x + y^T y}{2}$

Why?

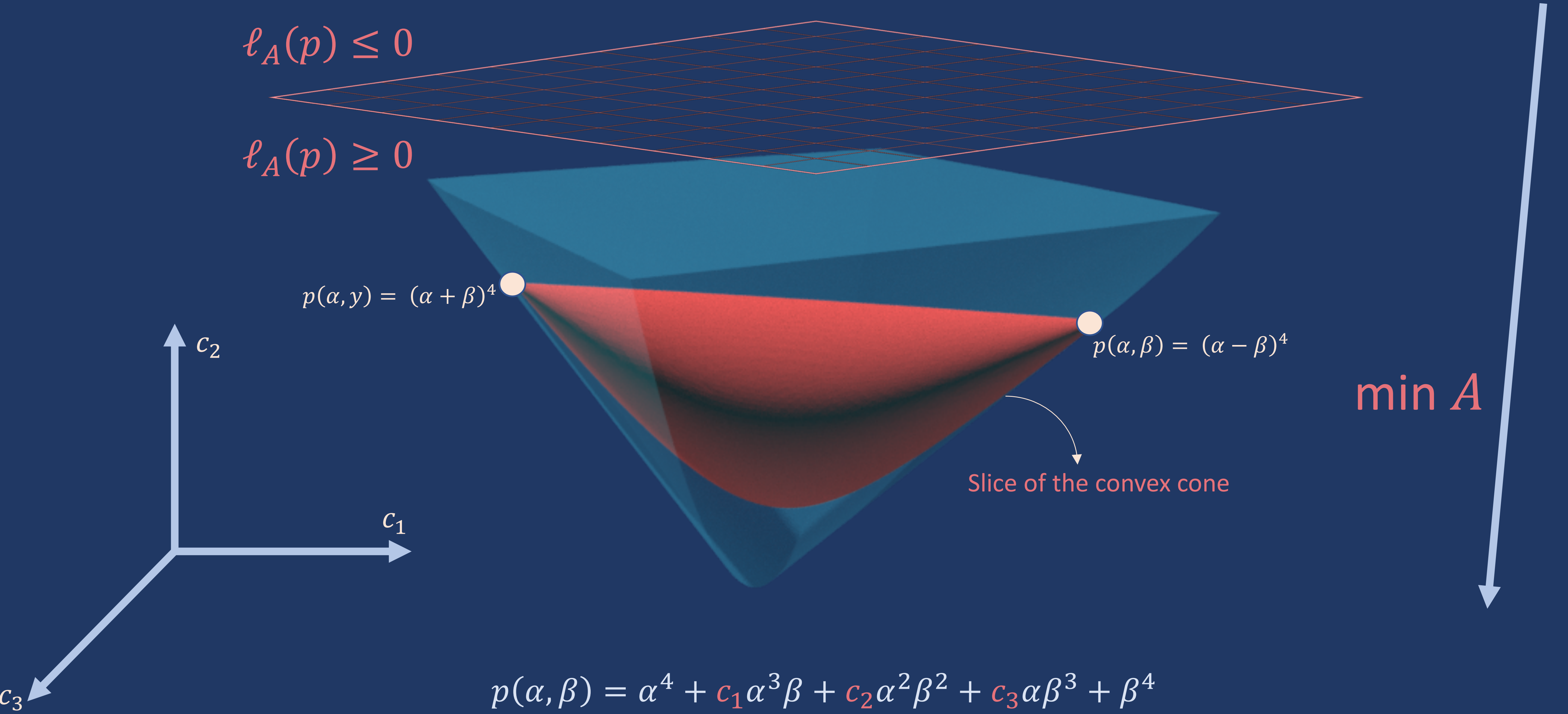
- $2\,x^T(\lambda y) \le x^T x + (\lambda y)^T(\lambda y)$
- $2\,\lambda\, x^T y \le x^T x + \lambda^2 y^T y$
- Optimize w.r.t λ

Linear Transformation: $\boldsymbol{x} \mapsto \boldsymbol{e_1}, \boldsymbol{y} \mapsto \boldsymbol{e_2}$

$$\ell_A(p) \coloneqq \left[A\frac{1}{2}\left(\partial_{\boldsymbol{e_1}}^{2d}+\partial_{\boldsymbol{e_2}}^{2d}\right)-\partial_{\boldsymbol{e_1}}^{d}\partial_{\boldsymbol{e_2}}^{d}\right]p$$

Takeaway: $\min A$ subject to $\ell_A(p) \ge 0\ \forall p$ convex

$$\min \quad A$$
$$\ell_A(p) \geq 0 \; \forall p \text{ quartic convex form in 2 vars}$$



$$p(\alpha, \beta) = \alpha^4 + c_1\alpha^3\beta + c_2\alpha^2\beta^2 + c_3\alpha\beta^3 + \beta^4$$

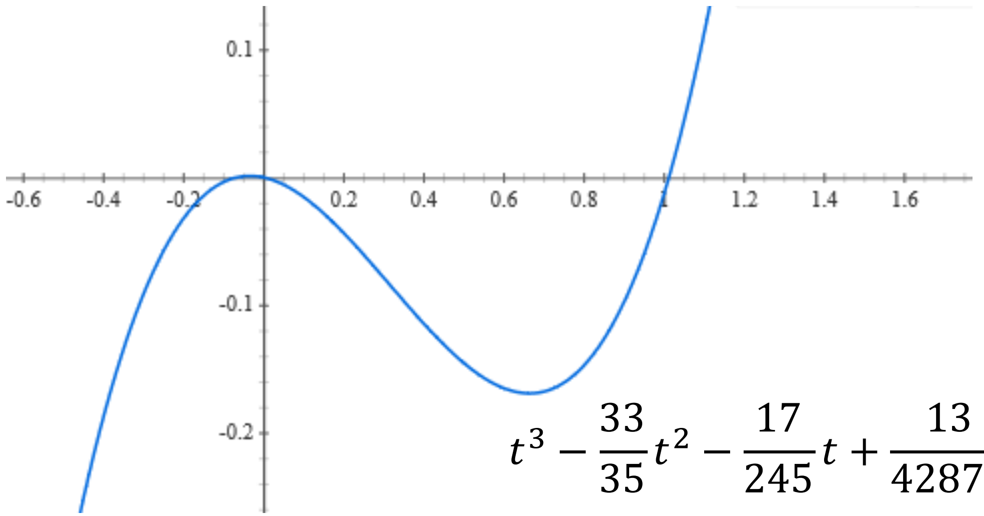
$$\frac{1}{(2d)!}\,\partial_{\boldsymbol{x}}^{d}\partial_{\boldsymbol{y}}^{d}p \le A_d \sqrt{p(\boldsymbol{x})}\sqrt{p(\boldsymbol{y})}$$

$2d$	2	4	6	8	10	12	14	…
A_d	1	1	1	1.01	1	1.06	1	…

?

Solving an SDP exactly:

1. Symmetries
2. KKT
3. Variable elimination



$$t^3 - \frac{33}{35}t^2 - \frac{17}{245}t + \frac{13}{42875}$$

```
d = 2 # 2d = 4
model = SOSModel(...)

# declare a convex polynomial
@polyvar x[1:2]
@variable model q Poly(monomials(x, 2d))
@constraint model q in SOSConvexCone()

# q(1, 0) = q(0, 1) = 1
@constraint model q(x => [1, 0]) == 1
@constraint model q(x => [0, 1]) == 1

# objective
@objective model Max coefficients(q)[d+1]

# solve
optimize!(model)
```

$$\frac{1}{70}\omega^{\frac{1}{3}} + \frac{128}{15}\omega^{\frac{1}{3}} + \frac{11}{35} \approx 1.011\,\ldots$$

with $\omega := 14336(1 + i\frac{\sqrt{3}}{9})$

Complex Cauchy-Schwarz

Real Cauchy-Schwarz: $\frac{1}{(2d)!}\,\partial_{\boldsymbol{x}}^{d}\partial_{\boldsymbol{y}}^{d}p \le A_d\,\sqrt{p(\boldsymbol{x})}\,\sqrt{p(\boldsymbol{y})}$

Theorem:

$$\forall \boldsymbol{z} \in \mathbb{C}^n \;\; \frac{1}{(2d)!}|p(\boldsymbol{z})| \;\le B_d \partial_{\boldsymbol{z}}^{d}\partial_{\bar{\boldsymbol{z}}}^{d} p$$

for every convex form p of degree $2d$.

$B_d = \frac{\binom{2(d-1)}{d-1}}{d}$ (d^{th}-Catalan number) = 1, 1, 2, 4 …

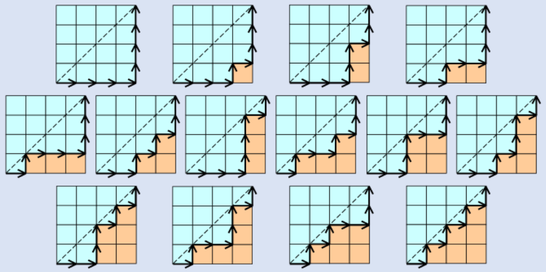
$$|p(\boldsymbol{z})| \le B'_d \iint_{\alpha^2+\beta^2\le 1} p(\alpha\,\Re(\boldsymbol{z}) + \beta\Im(\boldsymbol{z})\,)\,d\alpha\,d\beta$$

Ingredients of the proof:

- SDPs for intuition
- Basic trigonometry

Catalan numbers describe the number of:

- ways a polygon with $n+2$ sides can be cut into n triangles.
- ways to use n rectangles to tile a stair-step shape $(1,2,\ldots,n-1,n)$.
- ways in which parentheses can be placed in a sequence of numbers to be multiplied, two at a time
- planar binary trees with $n+1$ leaves
- paths of length $2n$ through an n-by- n grid that do not rise above the main diagonal

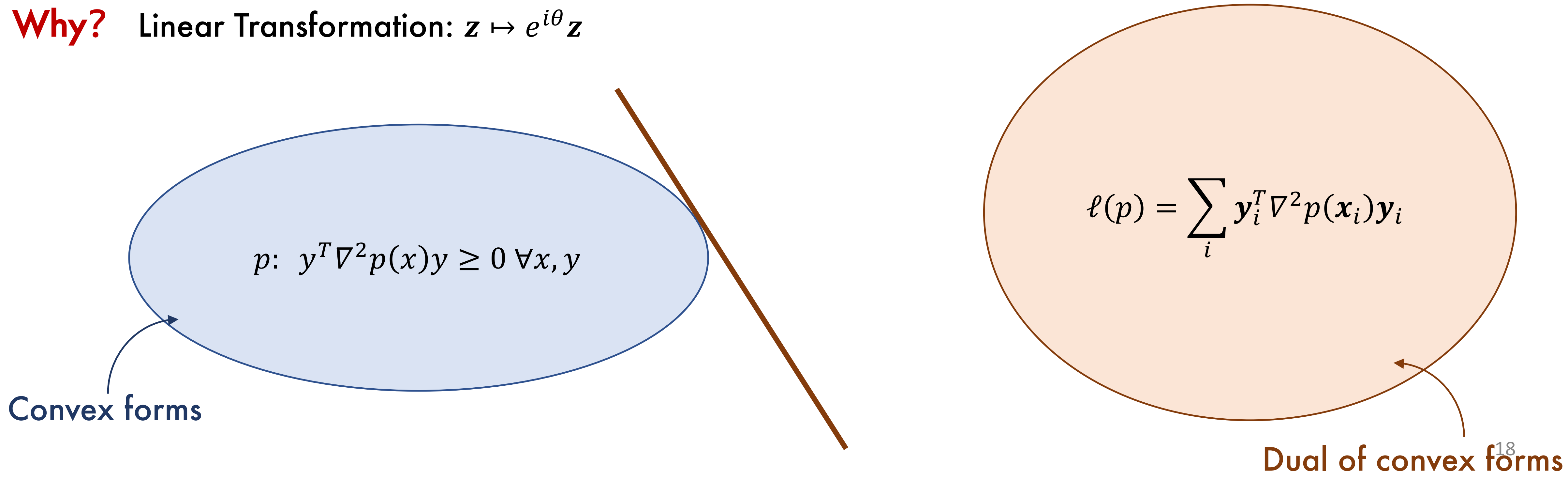


Proof of the complex Cauchy-Schwarz

Complex Cauchy-Schwarz: $B\partial_{\mathbf{z}}^{d}\partial_{\bar{\mathbf{z}}}^{d}p - \frac{1}{(2d)!}|p(\mathbf{z})| \geq 0$ (B is the d^{th}-Catalan number).

Enough to show: $B\partial_{\mathbf{z}}^{d}\partial_{\bar{\mathbf{z}}}^{d}p - \frac{1}{(2d)!}\Re(p(\mathbf{z})) \geq 0$

Why? Linear Transformation: $\mathbf{z} \mapsto e^{i\theta}\mathbf{z}$



After some algebra, it all boils down to:

$$B - \cos(2d\theta) = \frac{4^d}{2d} \sum_{k=0,\dots,d-1} \cos^2\left(\theta - \frac{\pi}{2d}k\right) \sin^{2d-2}\left(\theta - \frac{\pi}{2d}k\right) \quad \forall \theta \in \mathbb{R}$$

"The secret lives of polynomial identities"

Bruce Reznick

Theorem

If $d > r$, then for all θ,

$$\sum_{j=0}^{d-1} \left(\cos\left(\frac{j\pi}{d} + \theta\right) x + \sin\left(\frac{j\pi}{d} + \theta\right) y \right)^{2r} = \frac{d}{2^{2r}} \binom{2r}{r} (x^2 + y^2)^r \tag{5}$$

Putting everything together

Fix a convex quaternary quartic form p. Let's show it is sos!

Let $\boldsymbol{z_1}, \dots, \boldsymbol{z_8} \in \mathbb{R}^4$ such that:

$$\boldsymbol{z_1 z_1^T} = \sum_{i=2,\dots,8} \pm \boldsymbol{z_i z_i^T} \quad \Longrightarrow \quad \partial^2_{\boldsymbol{z_1}} = \sum_{i=2,\dots,8} \pm \partial^2_{\boldsymbol{z_i}}$$

$$\Longrightarrow \quad \partial^2_{\boldsymbol{z_1}} \, \partial^2_{\boldsymbol{z_1}} = \sum_{i,j=2,\dots,8} \pm \, \partial^2_{\boldsymbol{z_i}} \, \partial^2_{\boldsymbol{z_j}}$$

$$\Longrightarrow \quad \frac{1}{12} \partial^4_{z_1} p = \frac{1}{12} \sum_{i=2,\dots,8} \pm \partial^2_{z_i} \, \partial^2_{z_j} p$$

$$p(z_1) \qquad\qquad \leq \sqrt{p(z_i) p(z_j)}$$

Generalized Cauchy-Schwarz:

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^4 \;\; \frac{1}{12} \, \partial^2_{\boldsymbol{x}} \partial^2_{\boldsymbol{y}} p \leq \sqrt{p(\boldsymbol{x}) p(\boldsymbol{y})}$$

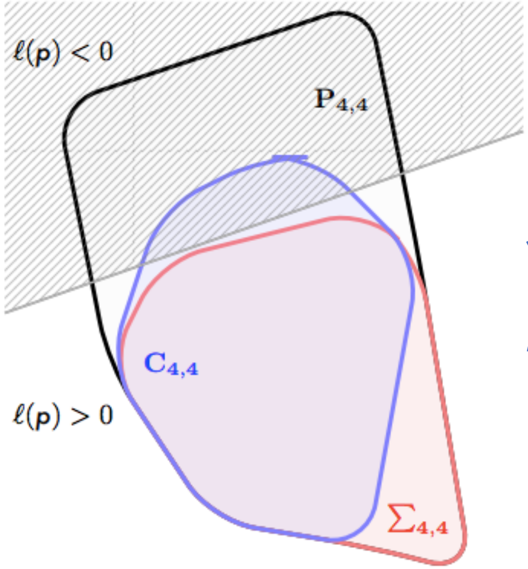
We want to show (I): $\sqrt{p(z_1)} \leq \sum_{i \geq 2} \sqrt{p(z_i)}$

What about convex ternary sextics?

$$(n = 3, 2d = 6)$$

Ternary Sextics are very similar to Quaternary Quartics!

Good understanding of
the dual of the sos cone



Type (I)

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3 \; \frac{1}{6!} \partial_{\boldsymbol{x}}^3 \partial_{\boldsymbol{y}}^3 p \leq \sqrt{p(\boldsymbol{x})} \sqrt{p(\boldsymbol{y})}$$

Type (II)

$$\forall z \in \mathbb{C}^3 \; \frac{1}{6!} |p(z)| \leq 2 \partial_z^3 \partial_{\bar{z}}^3 p$$

✗

Conjecture: Type (II) might not be needed

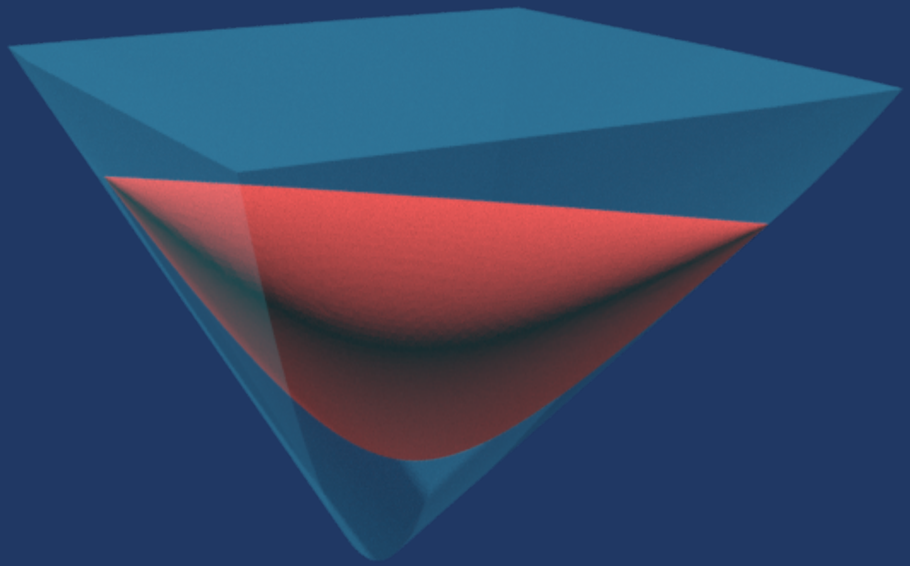
In which case, **convex** ternary sextics are also **sos**!

Conclusion and open problems

- Higher dimensions/number of variables?
- Pattern in the optimal constants in the Generalized Cauchy-Schwarz?
- Other applications of Generalized Cauchy-Schwarz?

Thanks!

arxiv/1909.07546



Want to know more?

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