Switched Linear Systems and Infinite Products of Matrices


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Motivation (from Optimization viewpoint)

Analysis of algorithms equivalent to analysis of dynamical systems:

E.g., for minimizing a quadratic function $f(x) = \frac{1}{2}x^T Ax$:

Iterative algorithm (e.g., gradient descent) $x_{k+1} = x_k - \gamma \nabla f(x_k)$	$\Longleftrightarrow$	Dynamical system (e.g., linear recursion) $x_{k+1} = (I - \gamma A)x_k$

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For instance, one step of gradient, and one step of proximal mapping ...?

(or ADMM, or forward-backward, etc...)

Switching can be interesting

Example: Consider

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

We have

$$\lim_{k\to\infty} \|A * A * A * \ldots\| = 0, \qquad \lim_{k\to\infty} \|B * B * B * \ldots\| = 0,$$

(in fact, A and B are nilpotent, so $A^2 = B^2 = 0$), but...

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$$\lim_{k\to\infty} \|A * B * A * B * \ldots\| = \infty \qquad (!)$$

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Many, many possible variations.

Today, focus on analysis methods/tools, and a simple example.

Example

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Gradient method: $x_{k+1} = x_k - \alpha \nabla f(x_k) = (I - \alpha A)x_k$.
Optimal constant stepsize choice: $\alpha_\star = \frac{2}{m+M} = 1/3$, achieves rate $r_\star = \max_{\lambda \in [m,M]} |1 - \alpha_\star \lambda| = \frac{M-m}{M+m} = 2/3$.

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Consider now two suboptimal methods, of stepsizes $\alpha_1 = 1/5$ and $\alpha_2 = 1/2$. The corresponding rates are $r_1 = 4/5$ and $r_2 = 3/2 > 1$ (divergent!).
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Run them in "proportions" $(2/3, 1/3)$, e.g., $1 - 1 - 2 - 1 - 1 - 2 - \ldots$
The achieved rate is now $\frac{1}{\sqrt[3]{5}} \approx 0.5848$.
Better than both; actually outperforms the "optimal" constant stepsize!

Formal setting: switched linear systems

Given a finite set of matrices $\Sigma = \{A_1, \ldots, A_m\}$, consider the (switched) linear dynamical system:

$$x_{k+1} = A_{\sigma_k} x_k, \qquad \text{for } k = 0, 1, \ldots$$

where $\sigma_k \in \{1, \ldots, m\}$ and x_0 is a given initial state.

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Understanding this system is equivalent to analyzing the (left-)infinite matrix products

$$\cdots A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}$$

How does one analyze/design this kind of things?

Different setups:

- **Deterministic** (worst-case) switching
  - Arbitrary switching, perhaps constrained (e.g., by an automaton)
  - Technical tool: Joint spectral radius
- **Random** switching
  - Probabilistic setup, could be i.i.d, or other process (e.g., Markov chain)
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Also, distinguish between

- **Analysis:** quantify convergence rate of given switching scheme σ
- **Synthesis:** design a switching scheme with good properties

Joint spectral radius

Given a set of matrices $\Sigma := \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$, what is the maximum "growth rate" that can be achieved by arbitrary switching?

$$\rho(\Sigma) := \limsup_{k \to \infty} \max_{\sigma \in \{1,\ldots,m\}^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

Appeared in several different contexts: linear algebra (Rota-Strang 1960), wavelets (Daubechies-Lagarias 1992), switched linear systems, etc.

- If $m = 1$, "standard" spectral radius $\rho(A_1) = \max_i |\lambda_i|$.
- If $m \geqslant 2$, much more complicated...

Switching is hard...

Determining if $\rho(\Sigma) \leqslant 1$ is NP-hard (Tsitsiklis & Blondel 1997).

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Still, how to compute/approximate $\rho(\Sigma)$?

What approximation guarantees can we have?

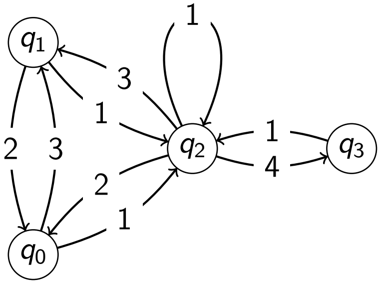
How to produce "bad" (high-growth) switching sequences?

Constrained switched systems

Switching constrained by a directed graph (automaton)

$$x_{k+1} = A_{\sigma_k} x_k$$

where $A_{\sigma_1}, A_{\sigma_2}, A_{\sigma_3}, A_{\sigma_4}, \ldots$ is a valid path.



Constrained Joint Spectral Radius:

$$\rho = \limsup_{k\to\infty} \max_{\sigma} \|A_{\sigma_k} \cdots A_{\sigma_1}\|^{1/k}.$$

where $\sigma_1, \ldots, \sigma_k$ is a path.

Bounding JSR using polynomials

Theorem

Let $p(x)$ be a strictly positive homogeneous polynomial of degree $2d$, that satisfies

$$p(A_i x) \leqslant \gamma^{2d} p(x), \qquad \forall x \in \mathbb{R}^n \quad i = 1, \dots, m$$

Then, $\rho(\Sigma) \leqslant \gamma$.

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Proof: Since $p(x)$ is strictly positive, there exist $0 < \alpha \leqslant \beta$ such that

$$\alpha \|x\|^{2d} \leqslant p(x) \leqslant \beta \|x\|^{2d} \qquad \forall x \in \mathbb{R}^n.$$

and

$$\|A_{\sigma_k} \dots A_{\sigma_1}\| \leqslant \max_x \frac{\|A_{\sigma_k} \dots A_{\sigma_1} x\|}{\|x\|} \leqslant \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2d}} \frac{p(A_{\sigma_k} \dots A_{\sigma_1} x)^{\frac{1}{2d}}}{p(x)^{\frac{1}{2d}}} \leqslant \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2d}} \gamma^k.$$

The result follows by taking kth roots, and the limit $k \to \infty$.

Using sum of squares (SOS)

Want to find a strictly positive polynomial $p(x)$ that satisfies

$$p(x) \geqslant 0, \qquad \gamma^{2d} p(x) - p(A_i x) \geqslant 0, \quad i = 1, \ldots, m.$$

While this is not tractable, we can use instead

$$p(x) \text{ is SOS}, \qquad \gamma^{2d} p(x) - p(A_i x) \text{ is SOS}.$$

Then, $\rho(\Sigma) \leqslant \rho_{SOS} := \gamma$.

For fixed γ, this is an SOS program (convex optimization), can be solved using a semidefinite programming (SDP) solver. As degree $2d$ increases, better bounds.

Example

Based on (Ando and Shih 98). Consider the two matrices:

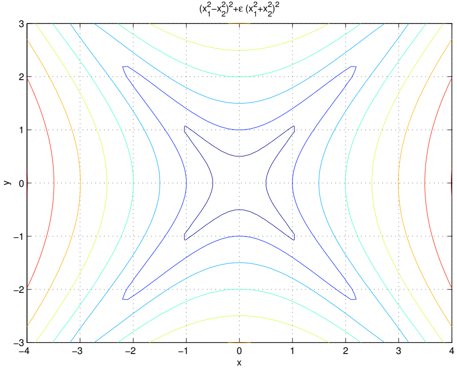
$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

- The true spectral radius is $\rho(A_1, A_2) = 1$.
- A common quadratic Lyapunov function (i.e., $d = 2$), gives $\rho(A_1, A_2) \leqslant \sqrt{2}$.

Example (cont.)

A quartic SOS Lyapunov function is enough to prove an upper bound of $1+\epsilon$ for *every* $\epsilon > 0$, since

$$p(x) = (x_1^2 - x_2^2)^2 + \epsilon(x_1^2 + x_2^2)^2$$



satisfies

$$\begin{aligned}(1+\epsilon)p(x) - p(A_1x) &= (x_2^2 - x_1^2 + \epsilon(x_1^2 + x_2^2))^2\\(1+\epsilon)p(x) - p(A_2x) &= (x_1^2 - x_2^2 + \epsilon(x_1^2 + x_2^2))^2.\end{aligned}$$

SOS-based upper bound

Find γ, $p(x)$ such that

$$p(x) \text{ is SOS}, \qquad \gamma^{2d}\, p(x) - p(A_i x) \text{ is SOS}.$$

Let $\rho_{\mathrm{SOS},2d}$ be the smallest such γ.

What (if anything) can we say about its approximation properties?

SOS Guarantees

Theorem

The degree-2d SOS upper bound for $\rho(A_1, \ldots, A_m)$ *satisfies*

$$\eta^{-\frac{1}{2d}} \rho_{\mathrm{SOS},2d} \leqslant \rho \leqslant \rho_{\mathrm{SOS},2d},$$

where $\eta = \min\{m, \binom{n+d-1}{d}\}$.

- As degree $d \to \infty$, approximation ratio goes to 1 (!)
- For unbounded m, related to Barvinok/John's ellipsoid.
- For fixed m, proof based on Lyapunov iteration.

P.A. Parrilo and A. Jadbabaie, Approximation of the joint spectral radius using sum of squares, *Lin. Alg. Appl.*, 428(10), pp. 2385–2402, 2008.

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But...

- How to produce "bad" (high-growth) switching sequences?

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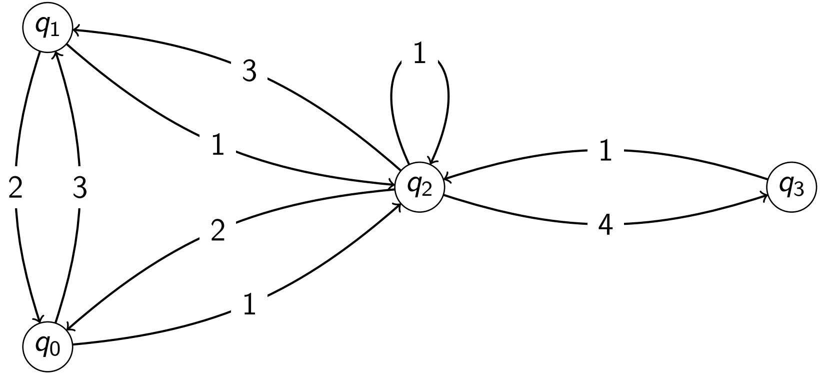
Constrained switching and upper bounds for ρ



Use *local* polynomials: $u \to_\sigma v \qquad p_v(A_\sigma x) \leqslant \gamma^{2d} p_u(x).$

SOS Program

$$
\begin{aligned}
\text{minimize } & \gamma \\
p_v(x) \text{ is SOS} & \qquad \forall v \in V \\
\gamma^{2d} p_u(x) - p_v(A_\sigma x) \text{ is SOS} & \qquad \forall (u, v, \sigma) \in E.
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\end{aligned}
$$

Q: What is the *dual*?

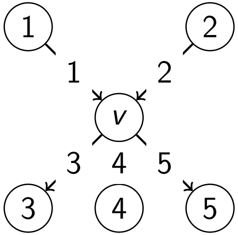
Dual: more in-flow than out-flow

Dual variables: pseudo-expectations $\widetilde{\mathbb{E}}[\cdot]$ for every *edge*, that satisfy

$$\sum_{(u,v,\sigma)\in E} \widetilde{\mathbb{E}}_{uv\sigma}[p(A_\sigma x)] \geqslant \gamma^{2d} \sum_{(v,w,\sigma)\in E} \widetilde{\mathbb{E}}_{vw\sigma}[p(x)], \qquad \forall v \in V, p(x) \text{ SOS},$$

Inequality between *incoming* and *outgoing* "probability flows":

$$\widetilde{\mathbb{E}}_{1v1}[p(A_1 x)] + \widetilde{\mathbb{E}}_{2v2}[p(A_2 x)] \geqslant \gamma^{2d}(\widetilde{\mathbb{E}}_{v33}[p(x)] + \widetilde{\mathbb{E}}_{v44}[p(x)] + \widetilde{\mathbb{E}}_{v55}[p(x)])$$



Building a high growth sequence

$$\sum_{(u,v,\sigma)\in E} \widetilde{\mathbb{E}}_{uv\sigma}[p(A_\sigma x)] \geqslant \gamma^{2d} \sum_{(v,w,\sigma)\in E} \widetilde{\mathbb{E}}_{vw\sigma}[p(x)], \qquad \forall v \in V, p(x) \text{ SOS},$$

Construct infinite sequence *backwards* using "best expectation," by lower bounding the maximum by the average.

Construction yields a guaranteed lower bound on JSR:

$$\frac{\gamma}{\left(d^{in}_{max}\right)^{\frac{1}{2d}}} \leqslant \lim_{k\to\infty} \|A_{\sigma_1} \cdots A_{\sigma_k}\|^{\frac{1}{k}}.$$

B. Legat, R. Jungers, P. Parrilo. "Generating unstable trajectories for switched systems via dual sum-of-squares techniques." HSCC2016, ACM, 2016.

The rank one case

Interesting special case: JSR of rank-one matrices ($A_i = u_i v_i^T$).

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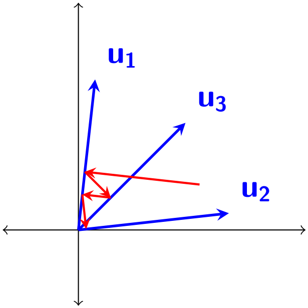
Geometric picture: for simplicity, take A_i symmetric ($A_i = u_i u_i^T$).

Then the switched system

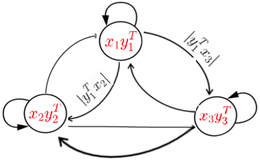
$$\begin{aligned} x_{k+1} &= A_{\sigma_k} x_k \\ &= u_{\sigma_k}(u_{\sigma_k}^T x_k) \end{aligned}$$

corresponds to projecting x_k onto lines u_{σ_k}.



JSR of rank one matrices as a graph problem

Given matrices $A_i = u_i v_i^T$ define a weighted directed graph $G = (E, V)$, where $E = [m]$ and $w_{ij} = \log |v_i^T u_j|$.



Optimal sequences can be obtained by solving the maximum cycle mean (MCM) problem for G, i.e., finding a directed simple cycle of largest average weight.

MCM can be efficiently solved using dynamic programming (Karp 1978).

Rank-one JSR is nice: efficiently solvable, optimal sequence always periodic (Ahmadi-P., also Gurvits-Samorodnitski).

A. Ahmadi, P. Parrilo. "Joint spectral radius of rank one matrices and the maximum cycle mean problem," *IEEE CDC*, 2012.

Random switching: Lyapunov exponent

Given a set of matrices $\Sigma := \{A_1, \ldots, A_m\}$ and a probability distribution $p \in \Delta_m$, what is the "growth rate" of random product of i.i.d. matrices?

$$R_p(\Sigma) := \lim_{k\to\infty} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

- (Furstenberg-Kesten 1960) $R_p(\Sigma) = e^{\lambda_p(\Sigma)}$ a.s., where

$$\lambda_p(\Sigma) := \lim_{k\to\infty} \frac{1}{k} \mathbb{E}\left[\log \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|\right]$$

- Characterization in terms of an invariant measure (hides complexity...)

Applications: ergodic theory, dynamical systems, fractals, stochastic linear systems, etc.

Random switching is hard...

Analysis: Given (Σ, p), **compute** convergence rate $R_p(\Sigma)$.

- Deciding stability (i.e. if $R_p(\Sigma) < 1$) is undecidable (Tsitsiklis & Blondel 1997).
- Still: how to compute/approximate? Special cases?
- Recently, nice convex bound (Sutter-Fawzi-Renner, `arXiv:1905.03270`)

Design: Given Σ, **optimize** convergence rate $\min_{p\in\Delta_m} R_p(\Sigma)$.

- Deciding stabilizability (i.e. if $\min_{p\in\Delta_m} R_p(\Sigma) < 1$) is NP-hard (Altschuler-P. 2019).
- Hard even for "simple" case of rank-one matrices, in contrast to analogous optimization for JSR!

Rank one case

Consider symmetric, rank-one matrices $\Sigma = \{A_i = u_i u_i^T\}_{i=1}^m$.

Analysis: Simple formula: $\lambda_p(\Sigma) = \sum_{ij=1}^m p_i p_j \log |u_i^T u_j|$.

- Ergodic formula: average time spent on edges of weighted graph
- Quadratic form on the simplex
- Computable, and in polynomial time.

Design: NP-hard to decide if $\min_{p\in\Delta_m} \lambda_p(\Sigma) < 0$. (i.e., if $R_p(\Sigma) < 1$).

- Reduction from Motzkin-Straus formulation of Independent Set, and use its hardness of approximation.

J. Altschuler, P. Parrilo. "Lyapunov Exponent of Rank One Matrices: Ergodic Formula and Inapproximability of the Optimal Distribution," arXiv:1905.07531.

Summary

- Switching is a powerful algorithmic tool, sometimes counterintuitive
- Key techniques: JSR and Lyapunov exponents
- SOS-based approximation for joint spectral radius
- Rank one case more tractable, but design problem still hard
- When is JSR/Lyapunov an algebraic function of data?

A. Ahmadi, P. Parrilo. "Joint spectral radius of rank one matrices and the maximum cycle mean problem," *IEEE CDC*, 2012.

A. Ahmadi, R. Jungers, P. Parrilo, M. Roozbehani. "Joint spectral radius and path-complete graph Lyapunov functions," *SIAM J. on Control and Optimization*, 52(1), 687–717, 2014.

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