Switched Linear Systems
and Infinite Products of Matrices

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Motivation (from Optimization viewpoint)

Analysis of algorithms equivalent to analysis of dynamical systems:
E.g., for minimizing a quadratic function $f(x) = \frac{1}{2}x^TAx$:

Iterative algorithm (e.g., gradient descent)
$x_{k+1} = x_k - \gamma \nabla f(x_k)$

Dynamical system (e.g., linear recursion)
$x_{k+1} = (I - \gamma A)x_k$
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E.g., for minimizing a quadratic function \( f(x) = \frac{1}{2}x^T Ax \):

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Q: What happens if we switch between algorithms?

For instance, one step of gradient, and one step of proximal mapping . . . ?
(or ADMM, or forward-backward, etc...)
Switching can be interesting

**Example:** Consider

\[ A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \]

We have

\[ \lim_{k \to \infty} \|A^* A^* A^* \ldots \| = 0, \quad \lim_{k \to \infty} \|B^* B^* B^* \ldots \| = 0, \]

(in fact, \(A\) and \(B\) are nilpotent, so \(A^2 = B^2 = 0\), but...
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(in fact, \(A\) and \(B\) are nilpotent, so \(A^2 = B^2 = 0\), but...)

\[
\lim_{k \to \infty} \| A \ast B \ast A \ast B \ast \ldots \| = \infty \quad (\text{!})
\]
Opportunity?

Perhaps we can also do this in reverse…?
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Take Algorithm A (slow), and Algorithm B (bad).
Can we schedule them (e.g., alternate between them, or something else) to obtain a better/faster algorithm?
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Many, many possible variations.

Today, focus on analysis methods/tools, and a simple example.
Example

Consider a convex quadratic function $f(x) = \frac{1}{2}x^T Ax$, with $m \leq \lambda_i(A) \leq M$. For concreteness, we choose $m = 1$, $M = 5$. 
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Gradient method: $x_{k+1} = x_k - \alpha \nabla f(x_k) = (I - \alpha A)x_k$.
Optimal constant stepsize choice: $\alpha_* = \frac{2}{m+M} = 1/3$, achieves rate $r_* = \max_{\lambda \in [m,M]} |1 - \alpha_* \lambda| = \frac{M-m}{M+m} = 2/3$. 

Q: Can we schedule them to make them converge? Perhaps faster, even?
Run them in "proportions" $(2/3, 1/3)$, e.g., $1 - 1 - 2 - 1 - 2 - \ldots$

The achieved rate is now $1/3 \sqrt{5} \approx 0.5848$.
Better than both; actually outperforms the "optimal" constant stepsize!
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Consider now two suboptimal methods, of stepsizes $\alpha_1 = 1/5$ and $\alpha_2 = 1/2$. The corresponding rates are $r_1 = 4/5$ and $r_2 = 3/2 > 1$ (divergent!).

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Formal setting: switched linear systems

Given a finite set of matrices $\Sigma = \{A_1, \ldots, A_m\}$, consider the (switched) linear dynamical system:

$$x_{k+1} = A_{\sigma_k} x_k, \quad \text{for } k = 0, 1, \ldots$$

where $\sigma_k \in \{1, \ldots, m\}$ and $x_0$ is a given initial state.
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Understanding this system is equivalent to analyzing the (left-)infinite matrix products

$$\cdots A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}$$
How does one analyze/design this kind of things?

Different setups:

- **Deterministic** (worst-case) switching
  - Arbitrary switching, perhaps constrained (e.g., by an automaton)
  - Technical tool: Joint spectral radius

- **Random** switching
  - Probabilistic setup, could be i.i.d, or other process (e.g., Markov chain)
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Also, distinguish between

- **Analysis**: quantify convergence rate of **given** switching scheme $\sigma$
- **Synthesis**: design a switching scheme with good properties
Joint spectral radius

Given a set of matrices $\Sigma := \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$, what is the maximum “growth rate” that can be achieved by arbitrary switching?

$$\rho(\Sigma) := \limsup_{k \to \infty} \max_{\sigma \in \{1, \ldots, m\}^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

 Appeared in several different contexts: linear algebra (Rota-Strang 1960), wavelets (Daubechies-Lagarias 1992), switched linear systems, etc.

- If $m = 1$, “standard” spectral radius $\rho(A_1) = \max_i |\lambda_i|$.
- If $m \geq 2$, much more complicated...
Switching is hard...

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Still, how to compute/approximate $\rho(\Sigma)$?

What approximation guarantees can we have?

How to produce “bad” (high-growth) switching sequences?
Constrained switched systems

Switching constrained by a directed graph (automaton)

\[
x_{k+1} = A_{\sigma_k} x_k
\]

where \( A_{\sigma_1}, A_{\sigma_2}, A_{\sigma_3}, A_{\sigma_4}, \ldots \) is a valid path.

Constrained Joint Spectral Radius:

\[
\rho = \limsup_{k \to \infty} \max_{\sigma} \| A_{\sigma_k} \cdots A_{\sigma_1} \|^1/k.
\]

where \( \sigma_1, \ldots, \sigma_k \) is a path.
Bounding JSR using polynomials

**Theorem**

Let $p(x)$ be a strictly positive homogeneous polynomial of degree $2d$, that satisfies

$$p(A_i x) \leq \gamma^{2d} p(x), \quad \forall x \in \mathbb{R}^n \quad i = 1, \ldots, m$$

Then, $\rho(\Sigma) \leq \gamma$. 
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**Proof:** Since $p(x)$ is strictly positive, there exist $0 < \alpha \leq \beta$ such that

$$\alpha \|x\|^{2d} \leq p(x) \leq \beta \|x\|^{2d} \quad \forall x \in \mathbb{R}^n.$$

and

$$\|A_{\sigma_k} \ldots A_{\sigma_1}\| \leq \max_x \frac{\|A_{\sigma_k} \ldots A_{\sigma_1} x\|}{\|x\|} \leq \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2d}} \frac{p(A_{\sigma_k} \ldots A_{\sigma_1} x)^{\frac{1}{2d}}}{p(x)^{\frac{1}{2d}}} \leq \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2d}} \gamma^k.$$

The result follows by taking $k$th roots, and the limit $k \to \infty$. 

Parrilo (MIT)
Using sum of squares (SOS)

Want to find a strictly positive polynomial $p(x)$ that satisfies

$$p(x) \geq 0, \quad \gamma^{2d} p(x) - p(A_i x) \geq 0, \quad i = 1, \ldots, m.$$  

While this is not tractable, we can use instead

$$p(x) \text{ is SOS,} \quad \gamma^{2d} p(x) - p(A_i x) \text{ is SOS.}$$

Then, $\rho(\Sigma) \leq \rho_{\text{SOS}} := \gamma$.

For fixed $\gamma$, this is an SOS program (convex optimization), can be solved using a semidefinite programming (SDP) solver. As degree $2d$ increases, better bounds.
Example

Based on (Ando and Shih 98). Consider the two matrices:

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},
\]

- The true spectral radius is \( \rho(A_1, A_2) = 1 \).
- A common quadratic Lyapunov function (i.e., \( d = 2 \)), gives \( \rho(A_1, A_2) \leq \sqrt{2} \).
A quartic SOS Lyapunov function is enough to prove an upper bound of $1 + \epsilon$ for every $\epsilon > 0$, since

$$p(x) = (x_1^2 - x_2^2)^2 + \epsilon(x_1^2 + x_2^2)^2$$

satisfies

$$(1 + \epsilon)p(x) - p(A_1x) = (x_2^2 - x_1^2 + \epsilon(x_1^2 + x_2^2))^2$$

$$(1 + \epsilon)p(x) - p(A_2x) = (x_1^2 - x_2^2 + \epsilon(x_1^2 + x_2^2))^2.$$
SOS-based upper bound

Find $\gamma$, $p(x)$ such that

\[
p(x) \text{ is SOS, } \gamma^{2d} p(x) - p(A_i x) \text{ is SOS.}
\]

Let $\rho_{SOS,2d}$ be the smallest such $\gamma$.

What (if anything) can we say about its approximation properties?
SOS Guarantees

Theorem

The degree-2d SOS upper bound for $\rho(A_1, \ldots, A_m)$ satisfies

$$\eta^{-\frac{1}{2d}} \rho_{\text{SOS},2d} \leq \rho \leq \rho_{\text{SOS},2d},$$

where $\eta = \min\{m, \left(\frac{n+d-1}{d}\right)\}$.

- As degree $d \to \infty$, approximation ratio goes to 1 (!)
- For unbounded $m$, related to Barvinok/John’s ellipsoid.
- For fixed $m$, proof based on Lyapunov iteration.

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But...

- How to produce “bad” (high-growth) switching sequences?

Joint spectral radius (deterministic)

High-growth sequences and duality

Constrained switching and upper bounds for $\rho$

Use *local* polynomials: $u \rightarrow_{\sigma} v$ \[ p_v(A_{\sigma}x) \leq \gamma^{2d} p_u(x). \]
SOS Program

minimize $\gamma$

$p_v(x)$ is SOS

$\gamma^{2d} p_u(x) - p_v(A_\sigma x)$ is SOS

$\forall v \in V$

$\forall (u, v, \sigma) \in E.$
SOS Program

minimize $\gamma$

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$\gamma^{2d} p_u(x) - p_v(A_\sigma x)$ is SOS $\forall (u, v, \sigma) \in E.$

**Q:** What is the dual?
Dual: more in-flow than out-flow

Dual variables: pseudo-expectations $\tilde{E}[:,]$ for every edge, that satisfy

$$\sum_{(u,v,\sigma) \in E} \tilde{E}_{uv\sigma}[p(A_\sigma x)] \geq \gamma^{2d} \sum_{(v,w,\sigma) \in E} \tilde{E}_{vw\sigma}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},$$

Inequality between _incoming_ and _outgoing_ “probability flows”:

$$\tilde{E}_{1v1}[p(A_1 x)] + \tilde{E}_{2v2}[p(A_2 x)] \geq \gamma^{2d}(\tilde{E}_{v33}[p(x)] + \tilde{E}_{v44}[p(x)] + \tilde{E}_{v55}[p(x)])$$
Building a high growth sequence

\[
\sum_{(u,v,\sigma) \in E} \tilde{E}_{uv\sigma}[p(A_{\sigma}x)] \geq \gamma^{2d} \sum_{(v,w,\sigma) \in E} \tilde{E}_{vw\sigma}[p(x)], \quad \forall v \in V, p(x) \text{ SOS},
\]

Construct infinite sequence \textit{backwards} using “best expectation,” by lower bounding the maximum by the average.

Construction yields a guaranteed \textit{lower} bound on JSR:

\[
\frac{\gamma}{(d_{\text{in, max}}^{\text{in}})^{\frac{1}{2d}}} \leq \lim_{k \to \infty} \left\| A_{\sigma_1} \cdots A_{\sigma_k} \right\|^{\frac{1}{k}}.
\]

The rank one case

Interesting special case: JSR of rank-one matrices \((A_i = u_i v_i^T)\).
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Geometric picture: for simplicity, take \(A_i\) symmetric \((A_i = u_i u_i^T)\).

Then the switched system

\[
x_{k+1} = A_{\sigma_k} x_k \\
= u_{\sigma_k} (u_{\sigma_k}^T x_k)
\]

corresponds to projecting \(x_k\) onto lines \(u_{\sigma_k}\).
JSR of rank one matrices as a graph problem

Given matrices $A_i = u_i v_i^T$ define a weighted directed graph $G = (E, V)$, where $E = [m]$ and $w_{ij} = \log |v_i^T u_j|$.

Optimal sequences can be obtained by solving the maximum cycle mean (MCM) problem for $G$, i.e., finding a directed simple cycle of largest average weight.

MCM can be efficiently solved using dynamic programming (Karp 1978).

Rank-one JSR is nice: efficiently solvable, optimal sequence always periodic (Ahmadi-P., also Gurvits-Samorodnitski).

Random switching: Lyapunov exponent

Given a set of matrices $\Sigma := \{A_1, \ldots, A_m\}$ and a probability distribution $p \in \Delta_m$, what is the “growth rate” of random product of i.i.d. matrices?

$$R_p(\Sigma) := \lim_{k \to \infty} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

- (Furstenberg-Kesten 1960) $R_p(\Sigma) = e^{\lambda_p(\Sigma)}$ a.s., where

$$\lambda_p(\Sigma) := \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \left[ \log \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\| \right]$$

- Characterization in terms of an invariant measure (hides complexity...)

Applications: ergodic theory, dynamical systems, fractals, stochastic linear systems, etc.
Random switching is hard...

**Analysis**: Given \((\Sigma, p)\), **compute** convergence rate \(R_p(\Sigma)\).
- Deciding **stability** (i.e. if \(R_p(\Sigma) < 1\)) is **undecidable** (Tsitsiklis & Blondel 1997).
- Still: how to compute/approximate? Special cases?
- Recently, nice convex bound (Sutter-Fawzi-Renner, arXiv:1905.03270)

**Design**: Given \(\Sigma\), **optimize** convergence rate \(\min_{p \in \Delta_m} R_p(\Sigma)\).
- Deciding **stabilizability** (i.e. if \(\min_{p \in \Delta_m} R_p(\Sigma) < 1\)) is **NP-hard** (Altschuler-P. 2019).
- Hard even for “simple” case of rank-one matrices, in contrast to analogous optimization for JSR!
Rank one case

Consider symmetric, rank-one matrices $\Sigma = \{A_i = u_i u_i^T\}_{i=1}^m$.

**Analysis:** Simple formula: $\lambda_p(\Sigma) = \sum_{ij=1}^m p_i p_j \log |u_i^T u_j|$.
- Ergodic formula: average time spent on edges of weighted graph
- Quadratic form on the simplex
- Computable, and in polynomial time.

**Design:** NP-hard to decide if $\min_{p \in \Delta_m} \lambda_p(\Sigma) < 0$. (i.e., if $R_p(\Sigma) < 1$).
- Reduction from Motzkin-Straus formulation of Independent Set, and use its hardness of approximation.

Summary

- Switching is a powerful algorithmic tool, sometimes counterintuitive
- Key techniques: JSR and Lyapunov exponents
- SOS-based approximation for joint spectral radius
- Rank one case more tractable, but design problem still hard
- When is JSR/Lyapunov an algebraic function of data?