Exact semidefinite representations for genus zero curves

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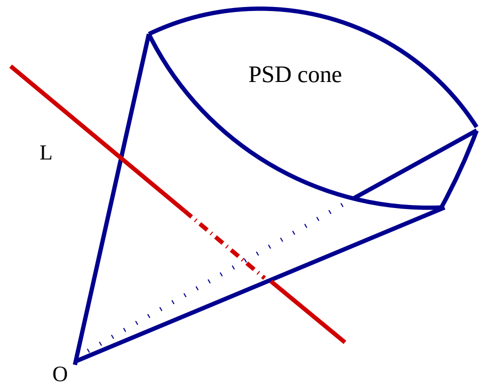
Outline

- Convex sets and semidefinite representations
- Two existing approaches
  - Rigid convexity / Lax conjecture
  - SOS construction
- A counterexample: the lemniscate
- Genus zero curves
- Construction of SDP representations
- A dual interpretation
- Extensions and conclusions

Semidefinite programming (SDP,LMIs)

A broad generalization of LP to symmetric matrices

$$\min \operatorname{Tr} CX \qquad \text{s.t.} \quad X \in \mathcal{L} \cap \mathcal{S}_+^n$$



- The intersection of an affine subspace $\mathcal{L}$ and the cone of positive semidefinite matrices.
- *Lots* of applications. A true "revolution" in computational methods for engineering applications
- Originated in control theory and combinatorial optimization. Nowadays, applied everywhere.
- Convex finite dimensional optimization. Nice duality theory.
- Essentially, solvable in polynomial time (interior point, etc.)

Semidefinite representations

A natural question in convex optimization:

What sets can be represented using semidefinite programming?

- Representability issues (e.g., closedness under projection)
  - Polytopes are closed under projection
  - Basic semialgebraic and SDP-representable sets are not

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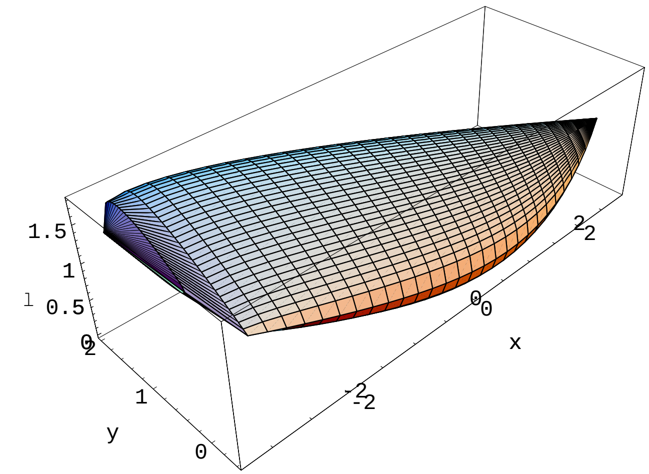
Are there “obstructions” to SDP representability?

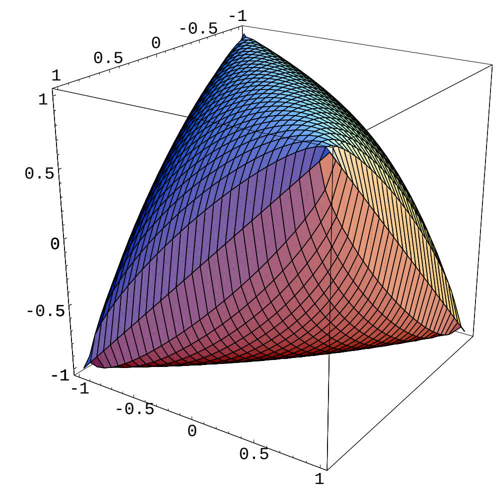
Open question: is every convex basic semialgebraic set SDP-representable? (generalized Lax conjecture)

Known SDP-representable sets

- Many interesting sets are known to be SDP-representable
- Preserved by “natural” properties: affine transformations, convex hull, polarity, etc.
- Several known structural results (e.g., facial exposedness)

Work of Nesterov-Nemirovski, Ramana, Tunçel, etc.





Existing results

Necessary conditions: $\mathcal{S}$ must be convex and semialgebraic (defined by polynomial inequalities).

Several versions of the problem:

- *Exact* vs. *approximate* representations.
- "Direct" (non-lifted) representations: no additional variables.

$$x \in \mathcal{S} \quad \Leftrightarrow \quad A_0 + \sum_i x_i A_i \succeq 0$$

- "Lifted" representations: can use extra variables (projection)

$$x \in \mathcal{S} \quad \Leftrightarrow \quad \exists y \text{ s.t. } A_0 + \sum_i x_i A_i + \sum y_j B_j \succeq 0$$

Today we focus on the "exact" version.

Direct representations

$$x \in \mathcal{S} \quad \Leftrightarrow \quad A_0 + \sum_i x_i A_i \succeq 0$$

Helton & Vinnikov (2004) fully characterized the sets $S \subset \mathbb{R}^2$ that admit a non-lifted SDP representation.

A "rigid convexity" condition: every line through the set must intersect the Zariski closure of the boundary a constant number of times (equal to the degree of the curve).

Related to hyperbolic polynomials and the Lax conjecture (Renegar, Lewis-Ramana-P. 2005)

Lax conjecture

A homogeneous polynomial $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to the direction $e \in^n$ if $t \mapsto p(x - te)$ has only real roots for all $x \in \mathbb{R}^n$.

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Ex: Let A, B, C be symmetric matrices, with $A \succ 0$. The polynomial

$$p(x, y, z) = \det(Ax + By + Cz)$$

is hyperbolic wrt $e = (1, 0, 0)$ (eigenvalues of symm. matrices are real).

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Thm (Lax Conjecture): If $p(x, y, z)$ is hyperbolic wrt e, then it has such as determinantal representation.

A "polar" viewpoint

Any convex set $\mathcal{S}$ is uniquely defined by its supporting hyperplanes.

Thus, if we can optimize a *linear function* over a set using SDP, we effectively have an SDP representation.

Need to solve (or approximate)

$$\min c^T x \qquad \text{s.t. } x \in \mathcal{S}$$

If $\mathcal{S}$ is defined by polynomial equations/inequalities, can use sum of squares (SOS) techniques.

SOS background

A multivariate polynomial $p(x)$ is a sum of squares (SOS) if

$$p(x) = \sum_i q_i^2(x), \quad q_i(x) \in \mathbb{R}[x].$$

- If $p(x)$ is SOS, then clearly $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$.
- For univariate or quadratic polynomials, the converse is also true.
- Convex condition, can be reduced to SDP.

A natural SOS approach

Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0\}$. Different conditions exist to certify nonnegativity of $c^T x + d$ over $\mathcal{S}$:

- General Positivstellensatz type:

$$(1+q)(c^T x + d) \in \textbf{cone}_{k+1}(f_i), \qquad q \in \textbf{cone}_k(f_i).$$

- Schmüdgen:

$$c^T x + d \in \textbf{cone}_k(f_i)$$

- Putinar/Lasserre:

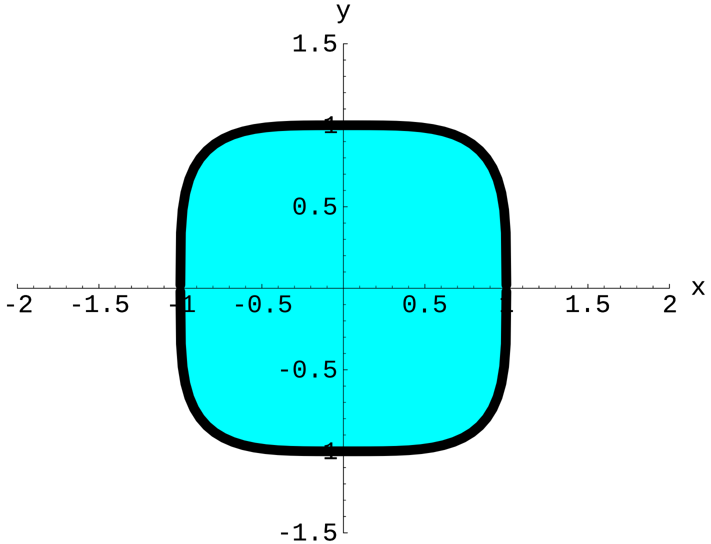
$$c^T x + d \in \textbf{preprime}_k(f_i)$$

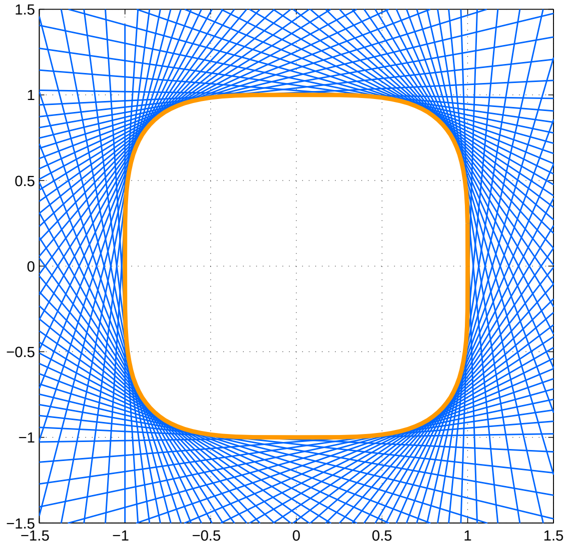
where $\textbf{preprime}_k \subseteq \textbf{cone}_k \subseteq \mathbb{R}_k[x]$. All these versions give convergent families of SDP approximations.

Concretely, for Putinar/Lasserre, if

$$c^T x + d = s_0(x) + \sum_i s_i(x) f_i(x), \qquad s_0, s_i \textbf{ are SOS}.$$

Example

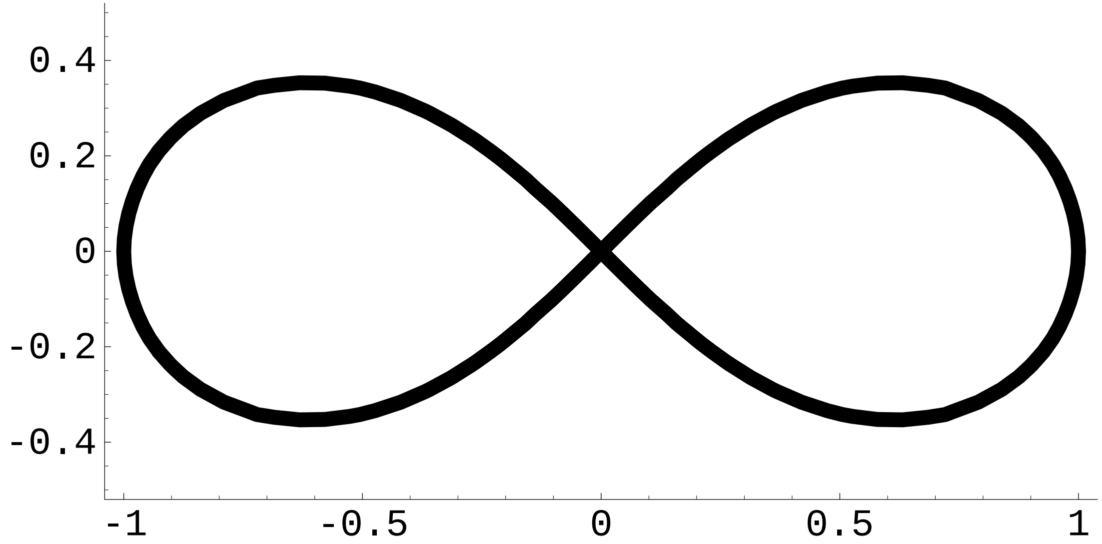




Consider the set described by $x^4 + y^4 \leq 1$

- Fails the rigid convexity condition.
- The SOS construction is exact.

Lemniscates

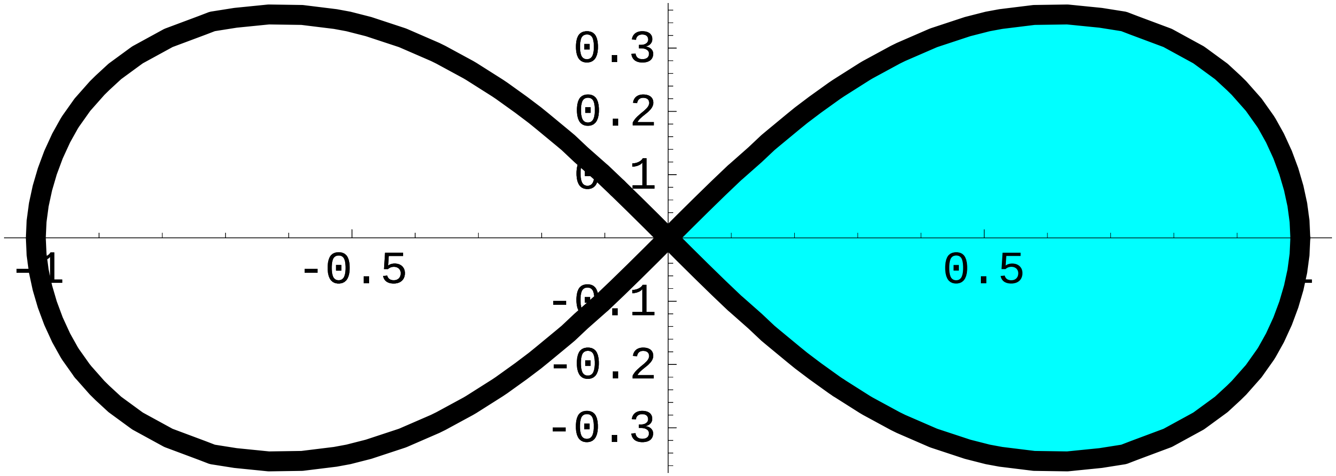


A (Bernoulli) *lemniscate* is a plane curve, defined as the set of points such that the product of the distances from two fixed points at distance $2d$ (the foci) is constant and equal to d^2.

In particular, if the points are $(\pm\frac{1}{\sqrt{2}}, 0)$, then

$$x^4 + y^4 - x^2 + 2x^2y^2 + y^2 = 0.$$

Half-lemniscate

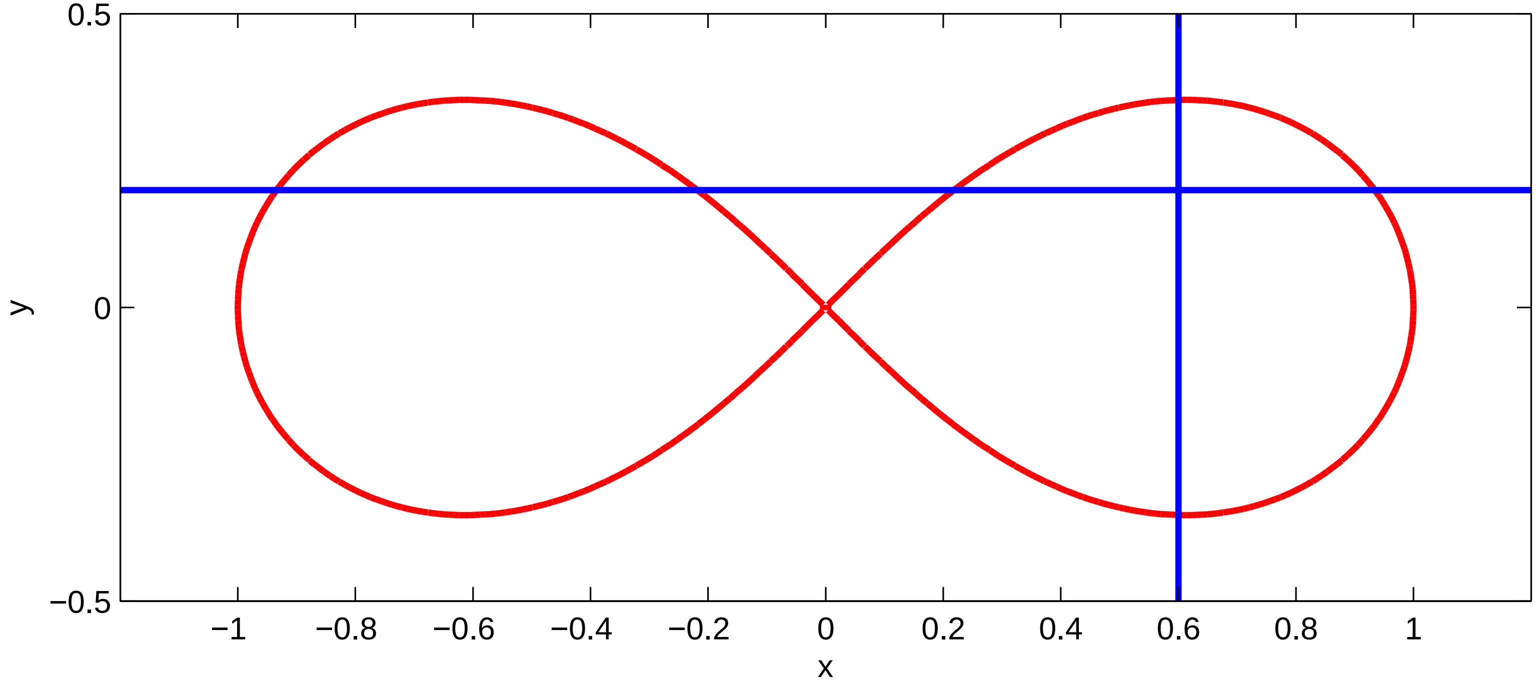


The lemniscate has two branches.

Each one is the boundary of a convex set.

Do these sets have semidefinite representations?

Rigid convexity fails



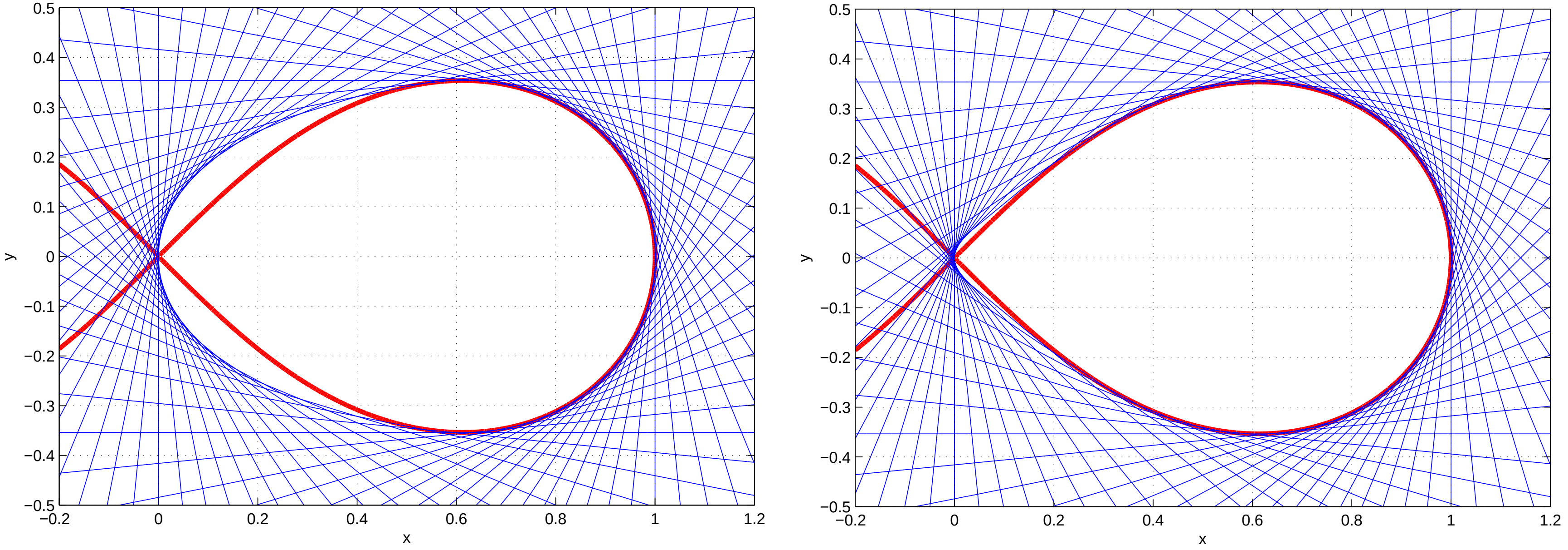
The lemniscate fails to satisfy the Helton-Vinnikov rigid convexity condition.

The number of intersections is *not* constant (sometimes 2, or 4).

Thus, no representation of the form $A_0 + A_1 x + A_2 y \succeq 0$ can exist. If an SDP description exists, it *must* use additional variables.

SOS fails

SOS schemes (Schmüdgen, Putinar/Lasserre) give outer approximations, but in this example they are *never* exact.



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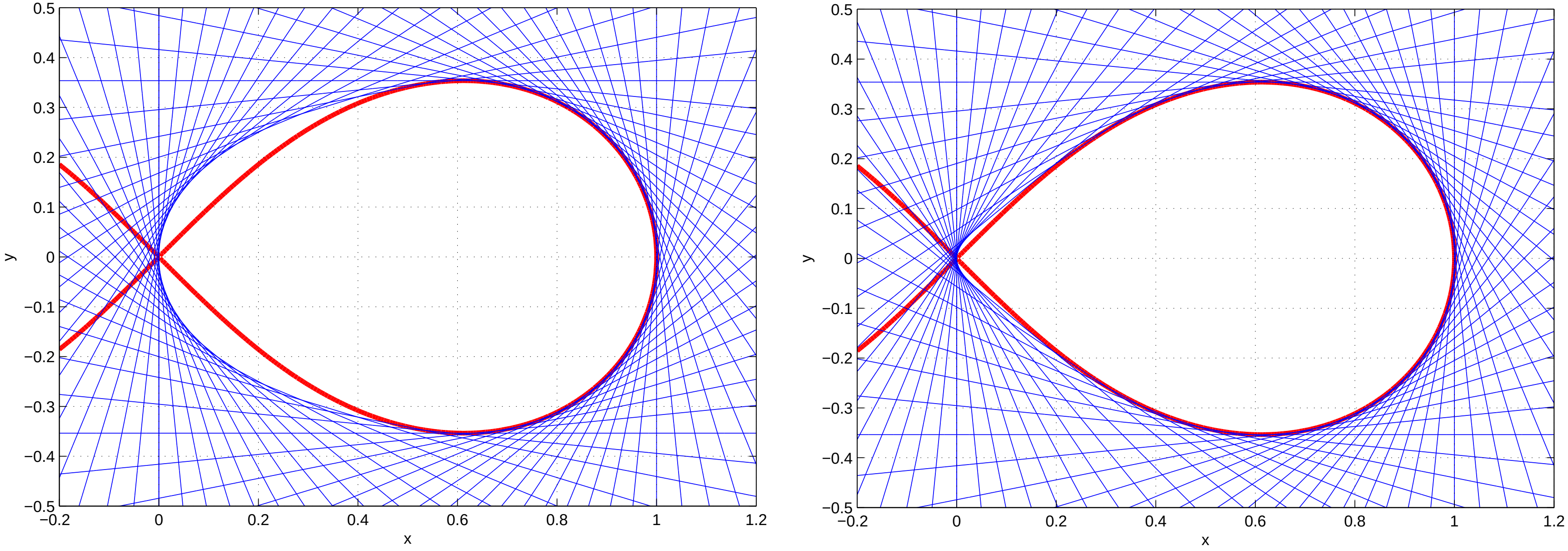

Can prove that this happens for *all* values of k.

Proof

Consider the linear functional $x + y$, which is nonnegative over S.

Its minimum over the set $\{(x, y)|p(x, y) = 0, \quad x \geq 0\}$ is zero. However, if

$$x + y = s_0(x, y) + s_1(x, y) \cdot x + t(x, y) \cdot p(x, y)$$

evaluating at $x = 0$ we have

$$y = \tilde{s}_0(y) + \tilde{t}(y)(y^2 + y^4),$$

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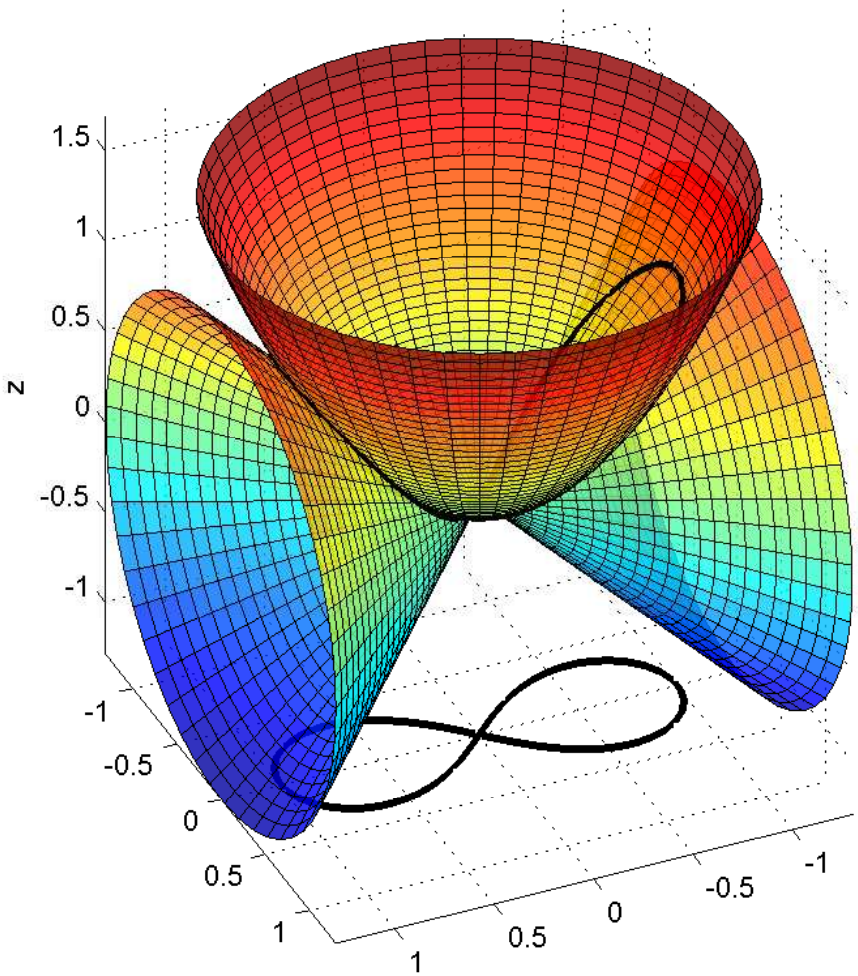
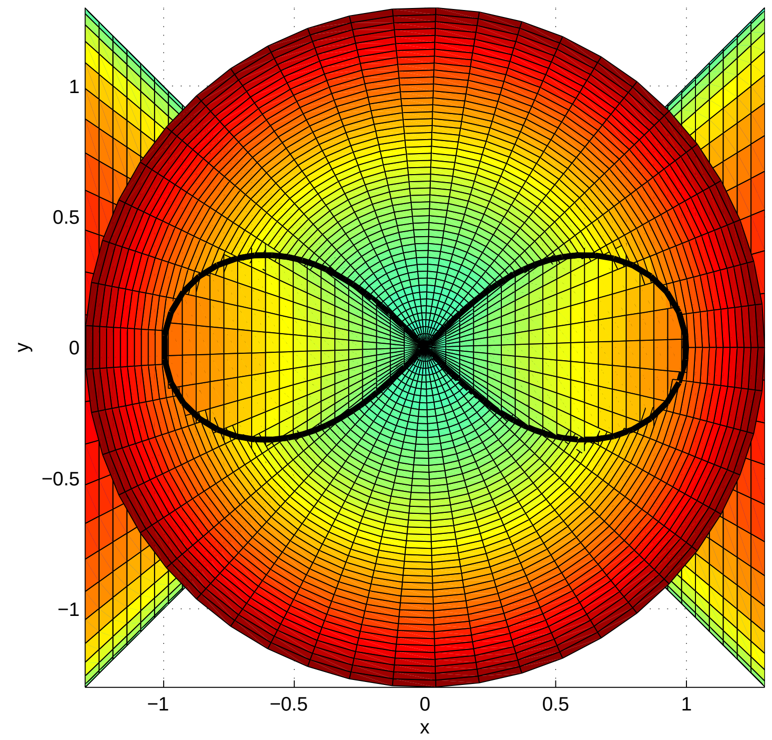
Hmmm. Perhaps the lemniscate *cannot* be represented?

SDP representation (I)

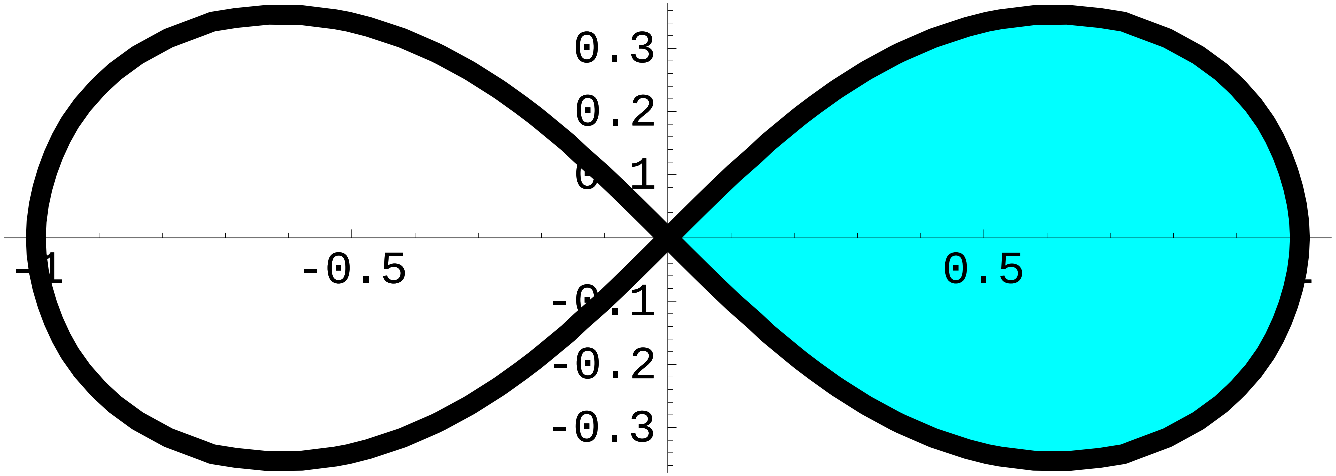
The (half) lemniscate is the intersection of a *paraboloid* and a *circular cone*:

$$S = \{x^2 + y^2 \leq z\} \cap \{y^2 + z^2 \leq x^2\}$$





SDP representation (II)



Thus, the set above can be represented as:

$$\begin{bmatrix} z & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x & y & z \\ y & x & 0 \\ z & 0 & x \end{bmatrix} \succeq 0.$$

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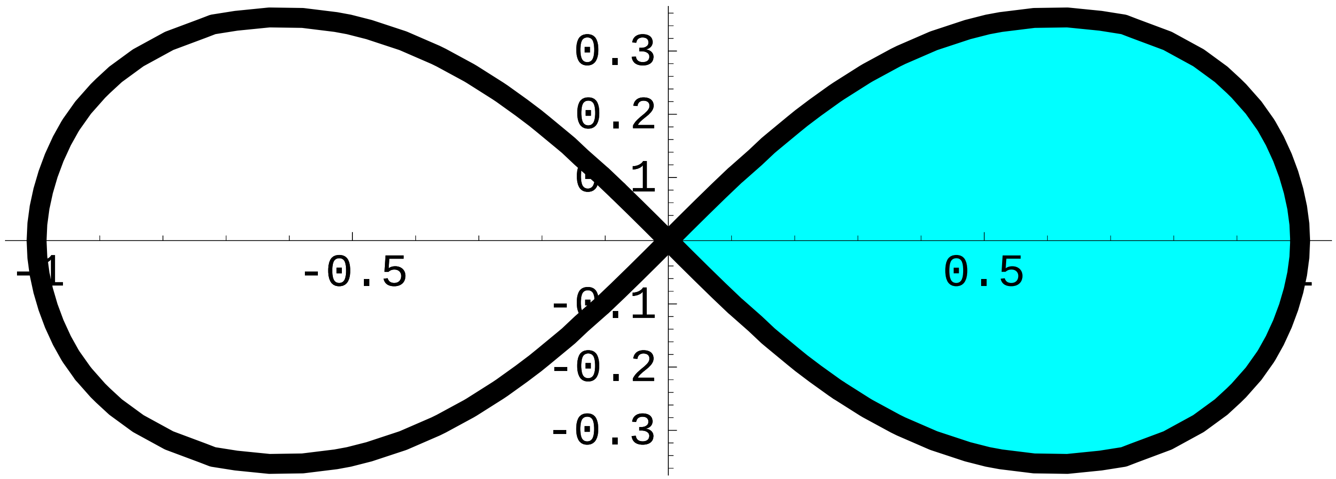

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Can we obtain this in an algorithmic way?

Algebraic curves and genus

A *plane curve* is a set in $\mathbb{R}^2$ defined by a polynomial equation $p(x, y) = 0$.

An important invariant of an algebraic curve is its *genus*. This can be defined in several ways. Classically, in terms of the degree d and the singularities σ: $g(C) := \binom{d-1}{2} - \sum_\sigma \delta_\sigma$.

Notice that in $\mathbb{C}^2$, an algebraic curve is actually a *surface* (a Riemann surface). The genus is associated with the topological genus (number of holes) of its Riemann surface.

More importantly (for us), a curve with *genus zero* is birationally equivalent to the real line.

Rational parametrizations

Every genus zero curve can be *rationally parametrized*: there exist rational functions $r(t), s(t)$, such that

$$p(r(t), s(t)) = 0.$$

Constructive procedures exist (e.g., Maple's `algcurves` package).

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For our lemniscate, for instance, we have the bijective parametrization

$$t \mapsto \left(\frac{t(1+t^2)}{1+t^4}, \frac{t(1-t^2)}{1+t^4} \right),$$

for $t \in (-\infty, \infty)$.

Examples:

Many interesting curves have genus zero:

Ellipses, Parabolas, Hyperbolas, Astroids, Cardioids, Descartes Folium, etc.

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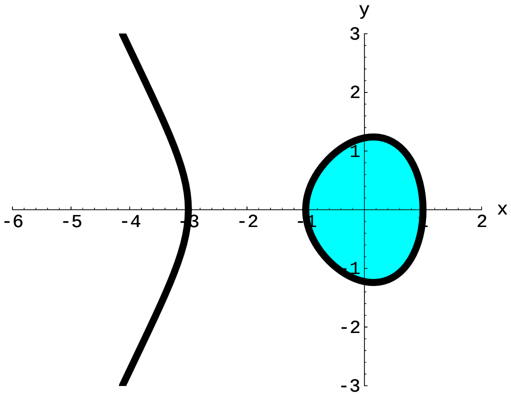
Not every plane curve has genus zero (e.g, *elliptic* curves, of genus 1).



Why is this good news?

When optimizing a linear function on the set, need to optimize

$$\min_{(x,y)\in\mathcal{C}} ax + by \qquad \Leftrightarrow \qquad \min_{t\in I} ax(t) + by(t) = \min_{t\in I} r(t),$$

where $r(t) = \frac{r_1(t)}{r_2(t)}$ is a *univariate rational function*.

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For univariate polynomials, we have PSD=SOS.

This means that, by clearing denominators, we can write an SDP relaxation that is *exact*!

$$r(t) \geq \gamma \qquad \Leftrightarrow \qquad r_1(t) - \gamma r_2(t) \geq 0 \qquad \Leftrightarrow \qquad r_1(t) - \gamma r_2(t) \text{ is SOS}$$

Same thing if t is contrained to (finite unions of) intervals.

SDP representations

Thm: (P.) Let $p(x, y)$ have genus zero. Consider the set S, defined as the convex hull of a finite collection of closed segments of the curve. Then, S has an exact SDP representation.

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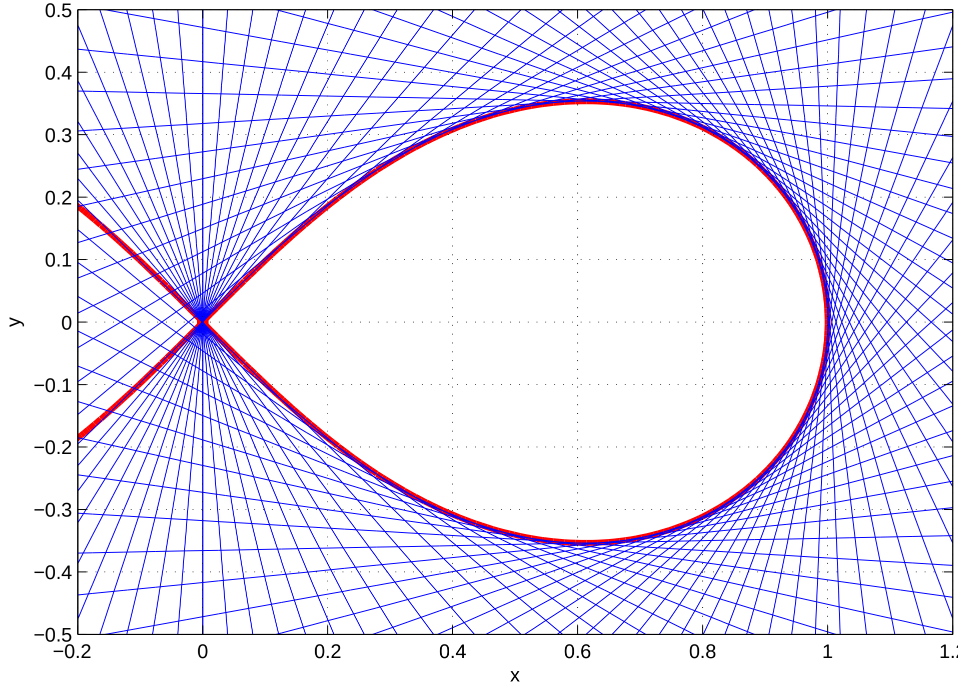

$$\max d \qquad \text{s.t.} \quad c_1 t(1+t^2) + c_2 t(1-t^2) - d(1+t^4) \text{ is SOS}$$

A dual interpretation

Writing the dual, we have that the set can be written as the pairs $(\eta_1 + \eta_3, \eta_1 - \eta_3)$, where $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4)$ satisfy

$$\begin{bmatrix} \eta_0 & \eta_1 & \eta_2 \\ \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \eta_3 & \eta_4 \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_2 & \eta_3 \end{bmatrix} \succeq 0, \qquad \eta_0 + \eta_4 = 1.$$

The variables η_i can be interpreted as "generalized moments" with respect to the weight function $1/(1+x^4)$, i.e.,

$$\eta_\alpha = \int \frac{x^\alpha}{1+x^4} d\mu,$$

The LMI constraints impose

$$\int \frac{q^2(x)}{1+x^4} d\mu \geq 0, \qquad \int \frac{x q^2(x)}{1+x^4} d\mu \geq 0.$$

Extensions to higher dimensions

Natural extensions to rational curves/surfaces in higher dimension.

Consider $O(3)$, the group of 3×3 orthogonal matrices of determinant one. This has two connected components.

There is a well-known double-cover of $SO(3)$ from $SU(2)$ (or S^3, the four-dimensional sphere), that yields a rational parametrization of 3×3 real orthogonal matrices (the quaternions).

We can use this to provide an SDP representation of the convex hull of $SO(3)$:

$$\begin{bmatrix} Z_{11}+Z_{22}-Z_{33}-Z_{44} & 2Z_{23}-2Z_{14} & 2Z_{24}+2Z_{13} \\ 2Z_{23}+2Z_{14} & Z_{11}-Z_{22}+Z_{33}-Z_{44} & 2Z_{34}-2Z_{12} \\ 2Z_{24}-2Z_{13} & 2Z_{34}+2Z_{12} & Z_{11}-Z_{22}-Z_{33}+Z_{44} \end{bmatrix}, \quad Z \succeq 0, \quad \mathrm{Tr}\, Z = 1.$$

This is a convex set in $\mathbb{R}^9$.

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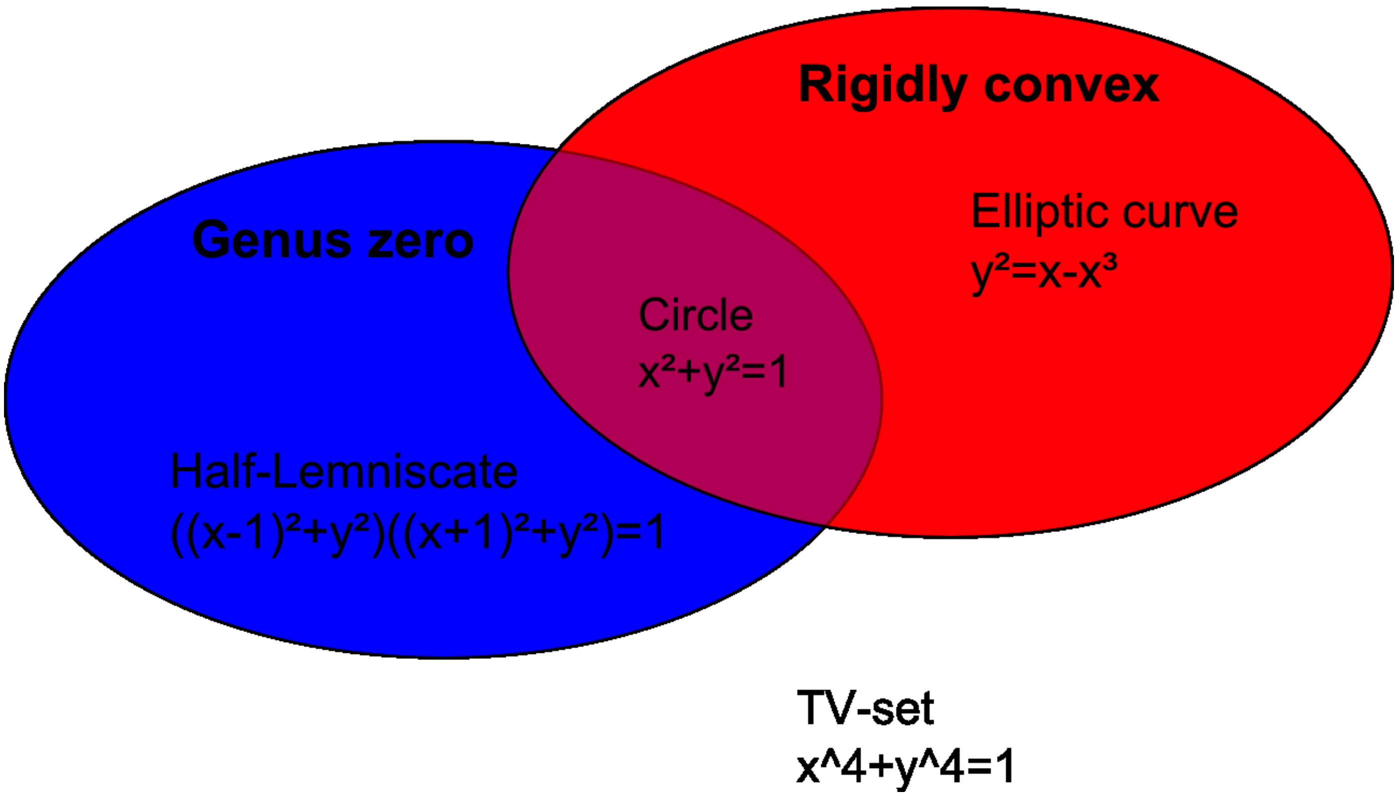
Representable sets

Two good families of SDP-representable plane curves: "rigidly convex" (Helton-Vinnikov) and *genus zero*.



Are there others? Unifications?

Relations with earlier work of Scheiderer (and others)?

Summary

- A new class of SDP-representable sets.
- A constructive procedure, interesting examples.
- Appealing interpretations.
- What is the role of singularities?
- Extensions to higher genus?
- How to obtain the “right” denominators in the Psatz?