6. Computational Complexity

- Computational models
- Turing Machines
- Time complexity
- Non-determinism, witnesses, and short proofs.
- Complexity classes: P, NP, coNP
- Polynomial-time reductions
- NP-hardness, NP-completeness
- Relativization, quantifiers, and the polynomial hierarchy
- The role of relaxations

Computation

Want to study and understand

- The power and limitations of computational methods. This requires a *formalization* of the notion of algorithm.
- What can and cannot be computed, and the resources needed to do so.
- Why polynomial problems do not have easy solutions in general.

In our context, understand relaxations from a complexity perspective.

Computational models

- Finite automata
- Pushdown automata
- Turing machines
- Church's λ-calculus
- Any modern computer programming language

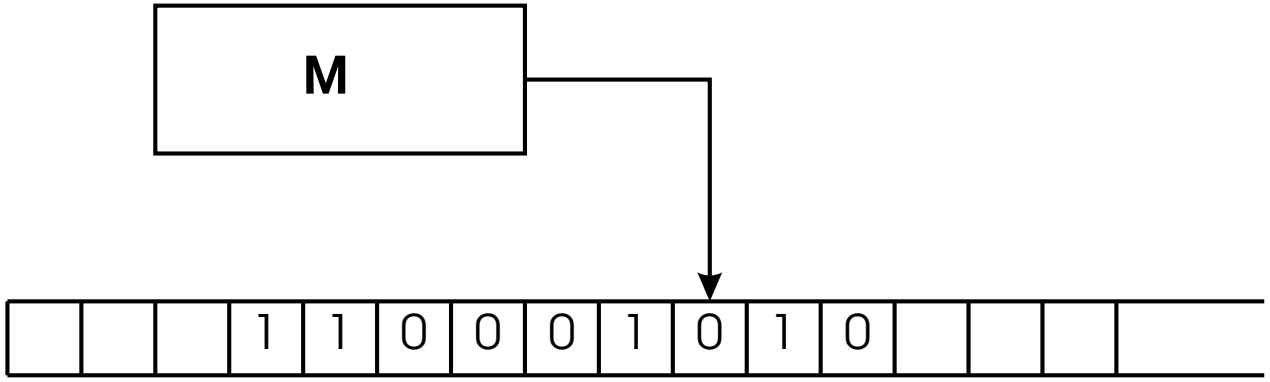
Exact choice of model is not crucial.
Last three have similar computational power.

The Church-Turing thesis:

- The intuitive notion of an algorithm is captured by a Turing machine.
- Any conceivable computational process can be performed in a Turing machine.

Turing machines

An infinite tape, with a head that moves forwards and backwards.



A finite list of rules, that describes the behavior.

Machine reads symbol in tape, moves forward or backwards, and optionally writes.

- A very idealized abstraction, of purely theoretical use
- This was *before* computers even existed! (1936)

A *formal* notion of computation allows for results on:

- Computability: what can be decided algorithmically?
  - A *qualitative* viewpoint
  - Existence of non-algorithmically decidable questions.
  - Undecidability of the Turing machine halting problem.
  - Application: quadratic integer programming is undecidable.
  - Connections with Gödel's incompleteness theorem.

- Complexity: what resources (time, memory, communication) are needed?
  - Emphasis on *quantitative* properties
  - Can we solve the problem efficiently?
  - Much more recent theory (started in 1970s)

Decision problems

Complexity classes are defined for *decision* problems, i.e., those with a *yes/no* answer.

Examples: Is this graph connected?
Is this matrix singular?
Is this polynomial nonnegative?
Is this proposition satisfiable?
Is this optimization problem feasible?

Given a problem, is the answer "Yes" or "No"?

A *language* is the set of inputs with the desired property.

So we can equivalent ask "Is this instance in the language?"
We use "problem" informally, to denote "language".

Time complexity

We are interested in the number of operations, as a function of the length of the input.

Standard $O(\cdot)$ notation: $f(n)$ is $O(g(n))$ if there exists a n_0 and $c > 0$ such that

$$n \geq n_0 \Rightarrow f(n) \leq cg(n)$$

.

Example: $\binom{n+5}{3}$ is $O(n^3)$.

For instance, we can say: a Turing machine requires $O(n^2)$ units of time, where n is the length of the input.

Complexity classes: P

P is the class of problems that can be decided in *polynomial time*, i.e., those for which the running time is $O(n^k)$, for $k \in \mathbb{N}$.

Some examples in P:

- Connectedness: Is the graph G connected?
- Nonsingularity: Is the rational matrix M singular?
- Positive semidefiniteness: Is this rational matrix PSD?
- Stability: Is this rational matrix Hurwitz stable?
- Controllability: Is this system controllable?

For all of these, there are algorithms requiring only a polynomial number of steps.

Complexity classes: P

Unambiguous and correct decision (yes or no), in polynomial time.

"Efficiently" or "quickly" are synonyms for "in polynomial time".

If a language is in P, so is its complement.

Ex: If I can efficiently decide "Is this matrix singular?", then I can also solve efficiently "Is this matrix nonsingular?"

Formally, we have that P=coP.

Complexity classes: NP

NP stands for *nondeterministic polynomial time*. What does this mean?

Given a "hint", "guess", or "certificate", can verify in polynomial time.

Examples:

- Is this logical proposition satisfiable?
  If you give me the assignment, it's easy.
- Compositeness: Is this number composite (not prime)?
  If you give me the factors, I can multiply them.
- Does this graph have a Hamiltonian path?
  Given the path, I can easily verify if it is Hamiltonian.
- Is this matrix singular?
  Give me a vector in the kernel, then I can quickly check.

A "Yes" answer is *easy to verify*.

Can use the certificate as a proof, to quickly convince someone else.

Complexity classes: NP

NP is that class of languages that:

- Have polynomial-time verifiers
- Can be decided by nondeterministic polynomial time Turing machines

"If you help me, I can *efficiently* check whether it's correct"

Verifying is easy. Deciding may be easy, or hard.

Notice that, in particular, $P \subseteq NP$

Unlike P, it is not known whether NP is closed under complement.

Complementation

Consider a logical proposition $Q = \neg((p \wedge q) \Rightarrow (q \wedge \neg r))$.

For a proposition in n variables,
the truth table has 2^n rows.

p	q	r	$p \wedge q$	$q \wedge \neg r$	Q
T	T	T	T	F	T
T	T	F	T	T	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	T	F
F	F	T	F	F	F
F	F	F	F	F	F

To verify that the proposition is *satisfiable*, it is enough to look at *one row*.
If you tell me which row, I can do that quickly.

However, if it is *unsatisfiable*, no easy shortcuts in general.
For instance, need to look at all 2^n rows.

Complexity classes: coNP

coNP is the set of

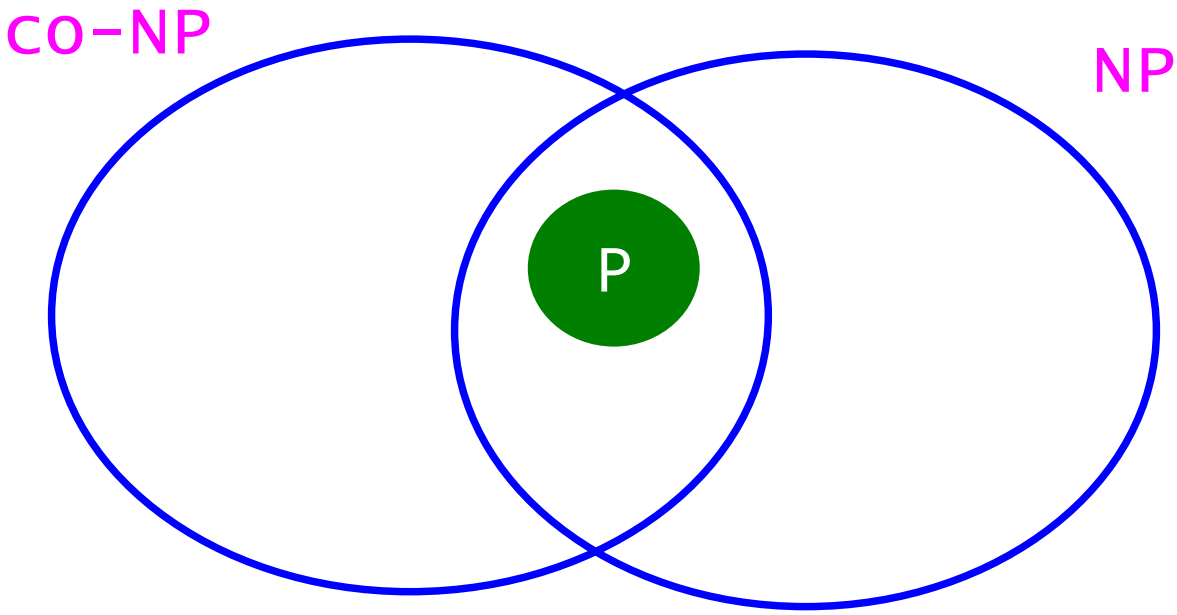
- languages for which *non-membership* can be efficiently verified, with a guess.
- problems that have polynomial-time refutations
- problems for which a "No" answer is easy to verify
- languages whose complement is in NP

Examples:

- Tautologies (propositions that are true for every possible assignment).
- Graphs without Hamiltonian circuits
- Nonnegative functions in $[0,1]^n$.

Relationships

Clearly, $P \subseteq NP$, and $P \subseteq coNP$. Therefore:



Unless they're all the same! In fact...

The big questions

There is a lot that we don't know:

- Is P=NP?
  - Does the existence of a short proof guarantee that we can find it efficiently?
- Is NP=coNP?
  - Do all tautologies have short proofs?
  - Is it equally easy to certify "yes" and "no" answers?

Have been open for more than 30 years.
They're certainly the most important open questions in theoretical computer science, and perhaps in mathematics.

A 1M USD reward from the Clay Institute for a solution.

Polynomial-time reductions

An important concept: the *reducibility* of a problem to another.

We transform a problem (language) into another one, not changing significantly the running time.

For instance, we can reduce MAXCUT to a QCQP, efficiently.

If problem A is reducible to B, we write

$$A \preceq_P B$$

The reduction implies that B is "as hard" as A. Since we could use an algorithm for problem B to solve A, it cannot be easier.

NP-hardness

A problem B is *NP-hard* if *every* problem in NP can be reduced to B.

$$A \in \text{NP} \quad \Longrightarrow \quad A \preceq_P B$$

NP-hard problems are as hard as any problem in NP

- The traveling salesperson problem
- Does this graph have a Hamiltonian path?
- Boolean quadratic optimization
- Satisfiability
- Is the structured singular value μ less than 1?

NP-hard problems are not necessarily in NP.

NP-completeness

$$\text{NP-complete} = \text{NP} \cap \text{NP-hard}$$

A problem is *NP-complete* if

- it is in NP
- *every* other problem in NP is polynomial-time reducible to it.

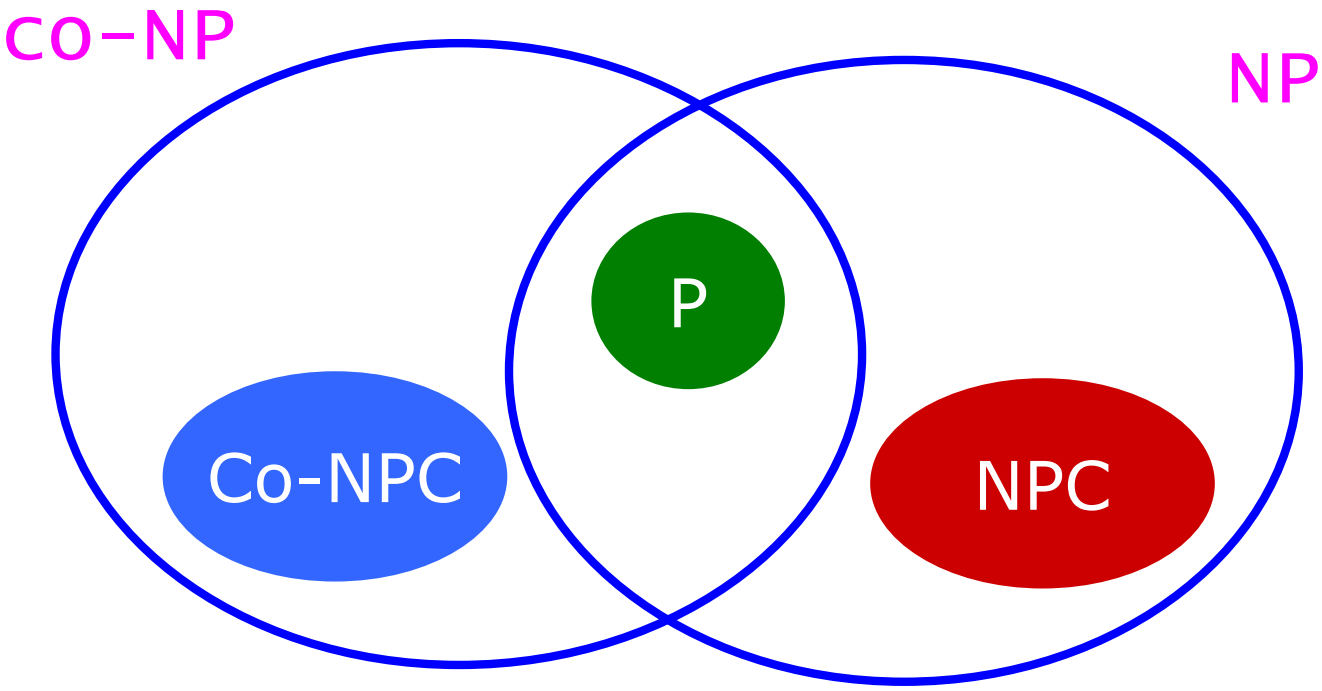
A nontrivial fact: *NP-complete* problems exist.

- Satisfiability is NP-complete (Cook-Levin).
- The proof, essentially writing down the equations for a Turing machine.

NP-complete problems are *the hardest* problems in NP.

Remarks

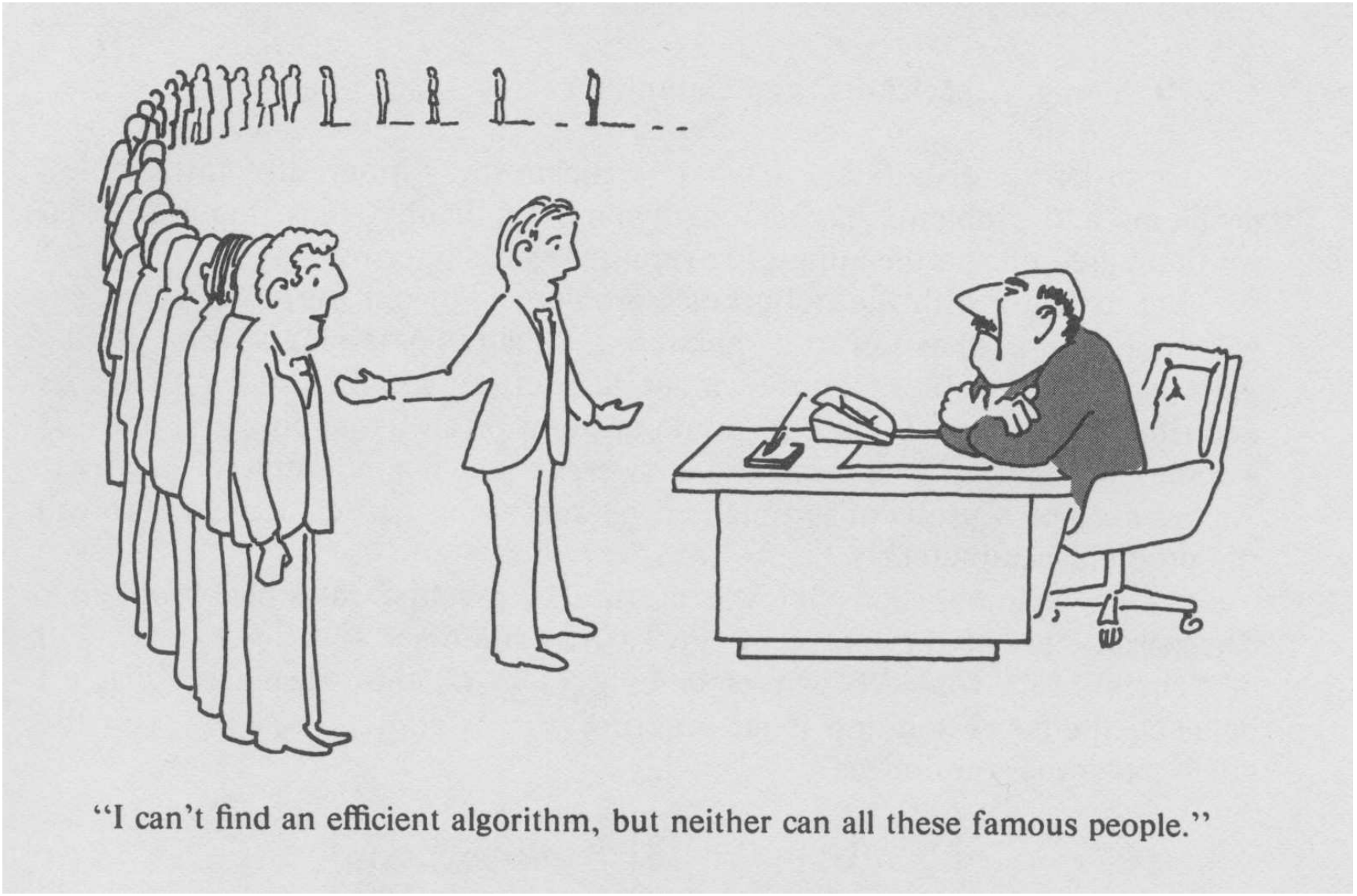
The updated picture:



A statement "Problem X is in NP" means that it is easy in some sense.

Problems can be *much* harder than NP-complete. No efficient way of verifying either positive or negative solutions!

The famous cartoon



From Garey & Johnson (1979).

Polynomial feasibility is NP-hard

Sketch: 0-1 linear programming is NP-hard, by reduction from 3SAT.

Using the encoding $T \leftrightarrow 1, F \leftrightarrow 0$, we have:

$$q_i \vee q_j \vee q_k \text{ is true } \iff x_i + x_j + x_k \geq 1.$$

In other words, we encode logic with arithmetic.

0-1 LP is clearly a special case of polynomial feasibility, since we can write the 0-1 constraints as $x_i^2 - 1 = 0$.

All the required transformations are polynomial time.

So the following holds:

$$A \in \text{NP} \quad \Longrightarrow \quad A \preceq_P \text{3SAT} \preceq_P \text{01LP} \preceq_P \text{FEAS}$$

which implies that deciding polynomial feasibility is NP-hard.

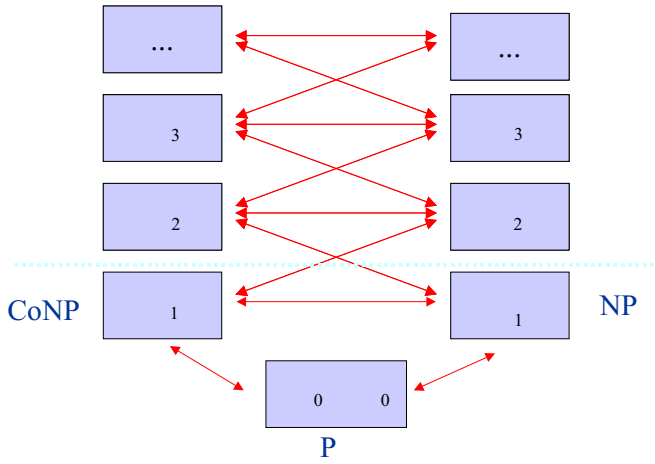
Relativization and the polynomial time hierarchy

What happens if we allow "subroutines" (*oracles*)?

For instance, if deciding NP-complete problems is "free"?

Define $\Sigma_0 P = \Pi_0 P =$ P, and recursively:

$$\Sigma_{k+1} P = \text{NP with } \Sigma_k P \text{ oracle},$$
$$\Pi_{k+1} P = \text{coNP with } \Sigma_k P \text{ oracle}$$



We have $\Sigma_1 P =$ NP, $\Pi_1 P =$ coNP. Higher classes harder.

There are problems that are complete for PH: alternating quantifiers.

Ex: $(\exists x)(\forall y)(\exists z) Q(x, y, z)$ is in $\Sigma_3 P$.

These appear in games, minimax, robust control, etc.

Hard to certify solutions, either in the positive or the negative.

If NP=coNP then the polynomial time hierarchy collapses.

Caveats

Complexity theory admits several variations, according to the specific choices:

Weak vs. strong	Computational model	Class
Exact problem	Turing	P
ϵ-approximation	Arithmetic	NP, coNP
		Σ_k, Π_k

For instance, for linear programming feasibility:

- Exact problem, rational data and Turing model, it is in P.
- Exact problem, real arithmetic model, complexity is open.
- Polynomial time in $\log \frac{1}{\epsilon}$ for approximate solutions.

Assessing difficulty

Why do we think of some optimization problems as easy, and others as hard?

"Good" optimization problems:

- LP is polynomial in the Turing model
- Weak LP/SDP are polynomial in the arithmetic real model

"Bad" problems: general polynomial optimization

- NP-hard in the Turing model
- NP-complete in real arithmetic model

Regardless of some important open questions, this classification agrees well with practical experience.

Some positive thoughts

- For practical-sized instances, we may be able to solve problems in an acceptable time.
- Approximate solutions may be easy
  For instance, naive coin flipping gives a $\frac{1}{2}$ algorithm for MAXCUT.
- But, NP-complete problems differ wildly in approximation properties.
  MAXCUT is quite nice, CLIQUE cannot be approximated within *any* constant factor.
- Problems may be easy on average, for a given probability distribution.
  In general, results depend critically on the chosen ensemble
- Classes of *instances* may admit short proofs.
  The main idea behind *relaxations*.

Relaxations and short proofs

In general, problems in coNP don't have short proofs (unless NP=coNP).

For a given problem, some instances are easier than others. For any concrete instance, short proofs may exist, and be easy to find.

Example:

- We do not know how to decide or certify efficiently that a graph does not contain a Hamiltonian cycle. *If* the graph has a bridge, then it is easy.
- Proving $\mu(M) < 1$ is hard. But, *if* the SDP upper bound satisfies $\bar{\mu}(M) < 1$, we are done.

If NP$\neq$coNP, there exist true propositions for which we cannot concisely explain why they're true!

A complexity zoo

Different computational models give rise to different complexity classes:

- Working space (memory): PSPACE, EXPSPACE, ...
- Parallel: NC, PT/WK, ...
- Randomized versions: PP, BPP, RP, ZPP, ...
- Real and complex computation: $\mathrm{NP}_{\mathbb{R}}$, $\mathrm{NP}_{\mathbb{C}}$,...
- Approximation and optimization: APX, PTAS, ...
- Quantum: BQP, ...
- Enumeration: #P, ...
- Interactive: IP, MIP, AM, PCP, ...