The Approximate Sum Capacity of the Symmetric Gaussian K-User Interference Channel

Or Ordentlich
Joint work with Uri Erez and Bobak Nazer

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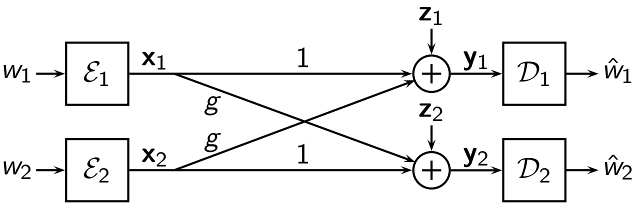
MIT, Cambridge, Massachusetts

The symmetric Gaussian 2-user IC : channel model



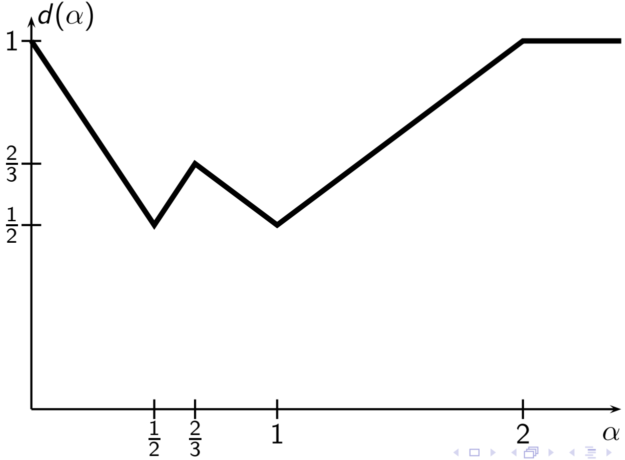
$$\mathbf{y}_k = \mathbf{x}_k + g\mathbf{x}_{\bar{k}} + \mathbf{z}_k$$

- Channel is static and real valued.
- Gaussian noises $\mathbf{z}_k$ are of zero mean and variance 1.
- All users are subject to the power constraint $\|\mathbf{x}_k\|^2 \leq n\text{SNR}$.
- Define $\text{INR} \triangleq g^2\text{SNR}$ and $\alpha \triangleq \frac{\log(\text{INR})}{\log(\text{SNR})}$.

Channel is symmetric:
sum capacity $= 2 \times$ symmetric capacity

GDoF of symmetric Gaussian 2-user IC

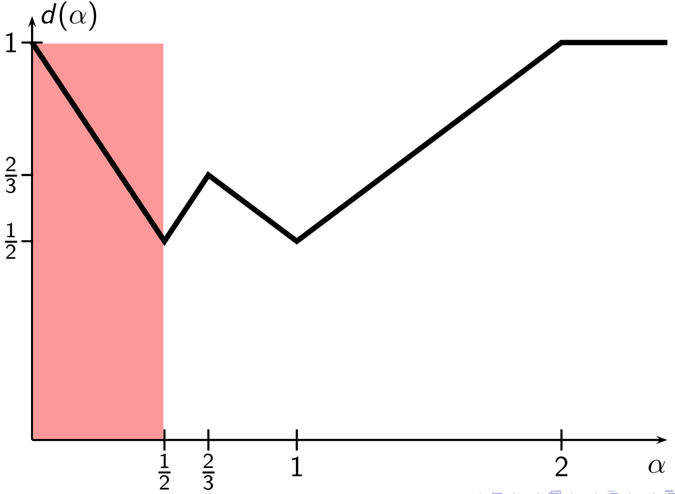
- Symmetric capacity is known to within 1/2 bit (Etkin *et al.* 08).
- DoF for each user is 1/2.
- GDoF gives more refined view



Symmetric Gaussian 2-user IC

Noisy interference regime

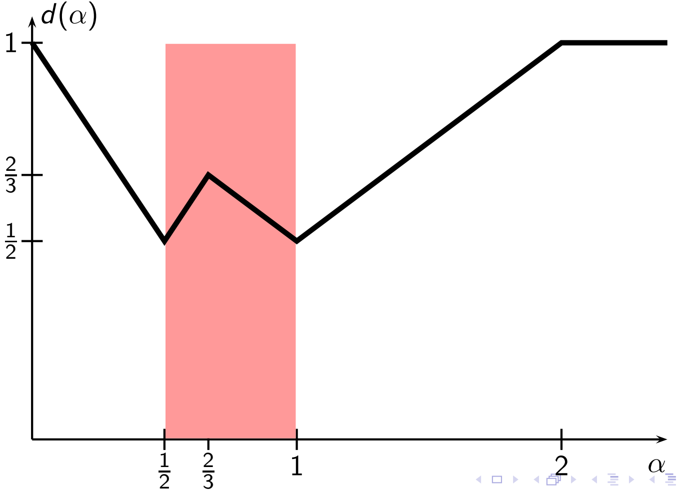
- Treat interference as noise



Symmetric Gaussian 2-user IC

Weak interference regime

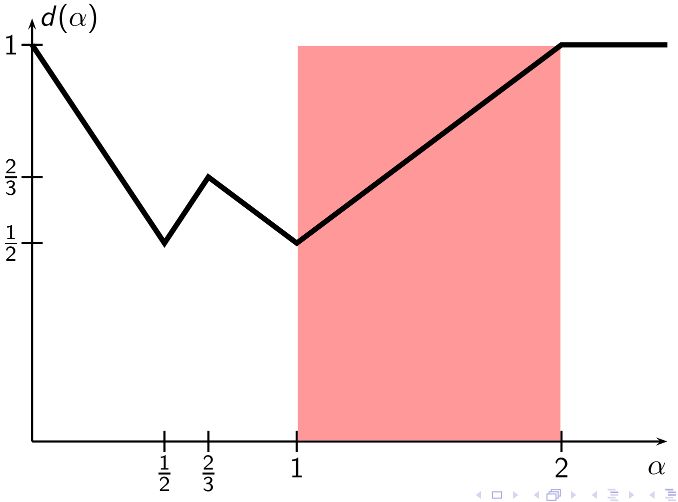
- Jointly decode intended message and part of interference (Han-Kobayashi).



Symmetric Gaussian 2-user IC

Strong interference regime

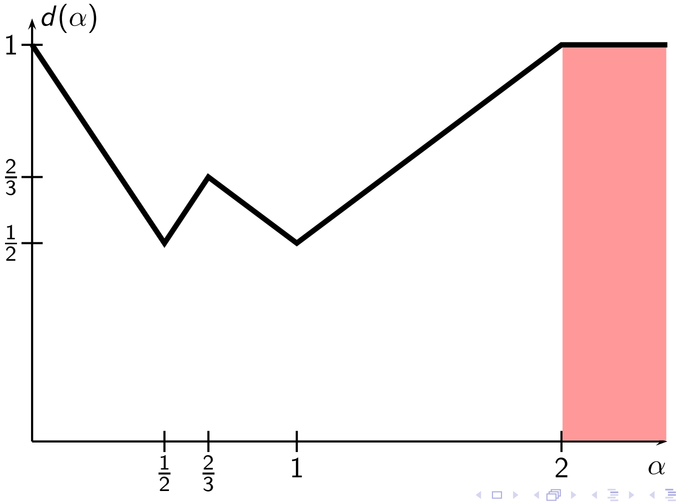
- Jointly decode intended message and interference



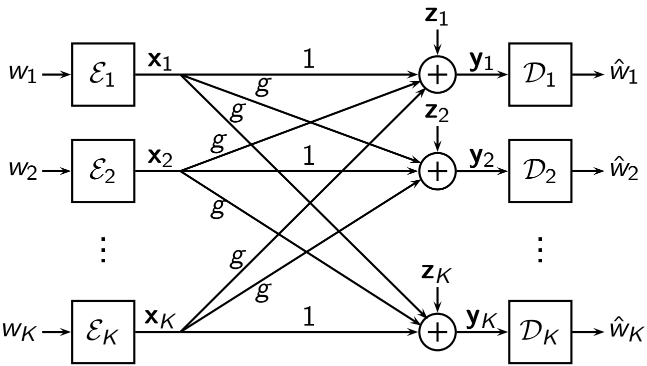
Symmetric Gaussian 2-user IC

Very strong interference regime

- Decode interference and then successively decode intended message



The symmetric Gaussian K-user IC : channel model



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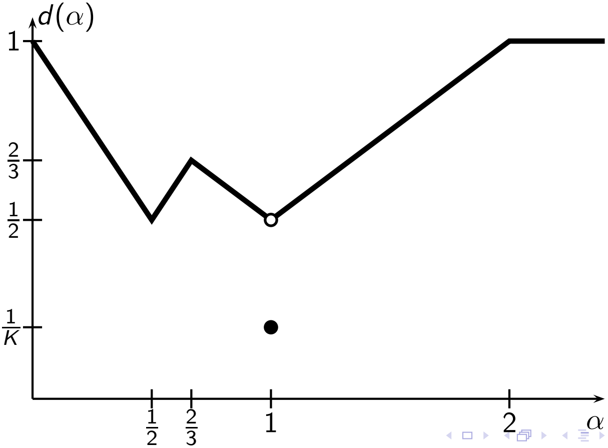
- $\text{INR} \triangleq g^2\text{SNR}$ and $\alpha \triangleq \frac{\log(\text{INR})}{\log(\text{SNR})}$.

The symmetric Gaussian K-user IC: what do we know?

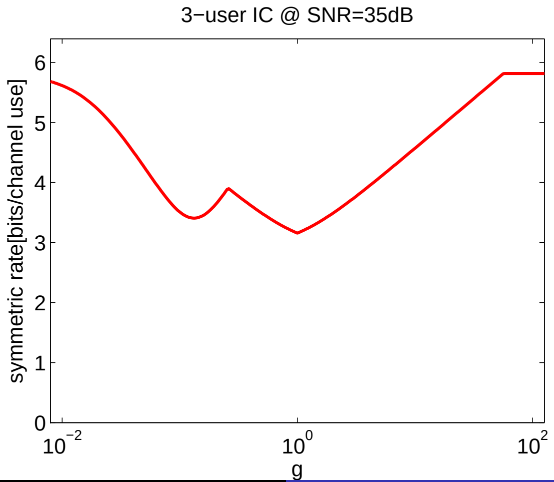
- DoF is discontinuous at the rationals (Etkin and E. Ordentlich 09, Wu *et al.* 11).
- GDoF of the symmetric K-user IC is independent of K, except for discontinuity at $\alpha = 1$ (Jafar and Vishwanath 10).



The symmetric Gaussian K-user IC: what do we know?

What about finite SNR?

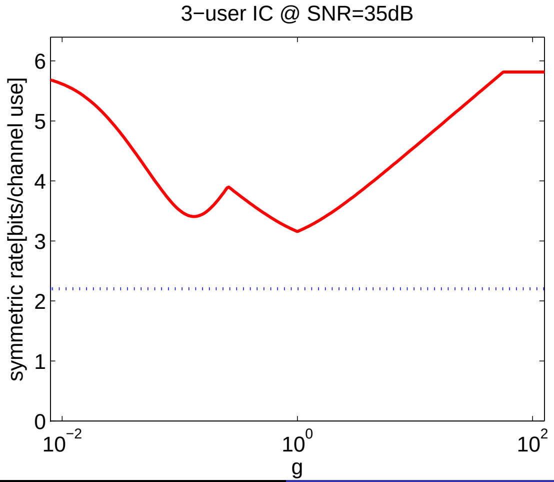
- Adding interference cannot increase capacity
  $\rightarrow$ Outer bounds for $K = 2$ remain valid for $K > 2$.



The symmetric Gaussian K-user IC: what do we know?

What about finite SNR?

- Can always use time-sharing
  $\rightarrow C_{\text{SYM}} > \frac{1}{2K}\log(1 + K\text{SNR})$.

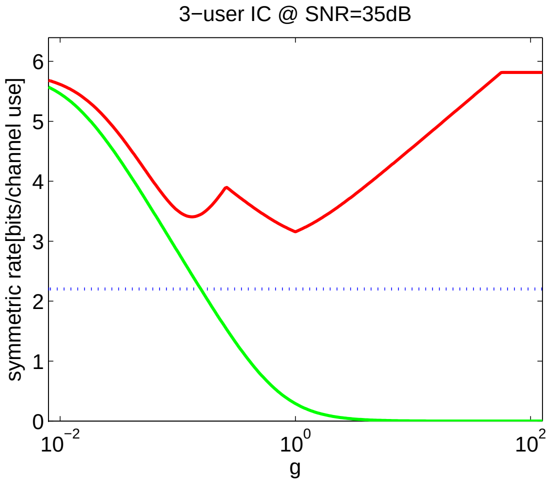


The symmetric Gaussian K-user IC: what do we know?

What about finite SNR?

- Can treat interference as noise

  $\rightarrow$ achieves the approximate capacity for noisy interference regime



The symmetric Gaussian K-user IC: what do we know?

- For the other regimes lattice codes are useful.
- Closed under addition
  $\implies K-1$ interferers folded to one effective interferer.
- Each receiver sees a K-user MAC

$$\mathbf{y}_k = \mathbf{x}_k + g \sum_{m \neq k} \mathbf{x}_m + \mathbf{z}_k,$$

The symmetric Gaussian K-user IC: what do we know?

- For the other regimes lattice codes are useful.
- Closed under addition
  $\Longrightarrow$ $K-1$ interferers folded to one effective interferer.

- Assume $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \Lambda$.
  $\Longrightarrow$ Effective 2-user MAC at each receiver

$$\mathbf{y}_k = \mathbf{x}_k + g\mathbf{x}_{\text{int},k} + \mathbf{z}_k,$$

where $\mathbf{x}_{\text{int},k} = \sum_{m \neq k} \mathbf{x}_m \in \Lambda$.

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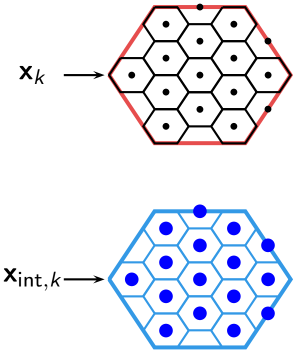
How to decode $\mathbf{x}_k$?

The symmetric Gaussian K-user IC: what do we know?

For large g, can decode sum of interferences, subtract and decode desired codeword (Sridharan *et al.* 08)



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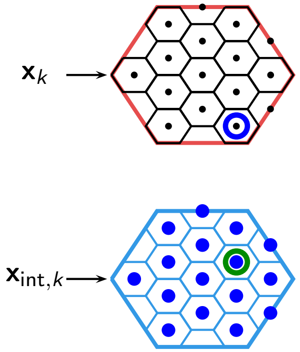

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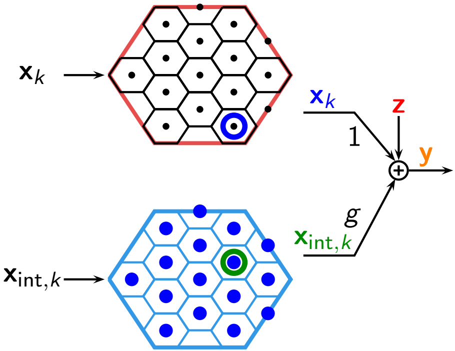

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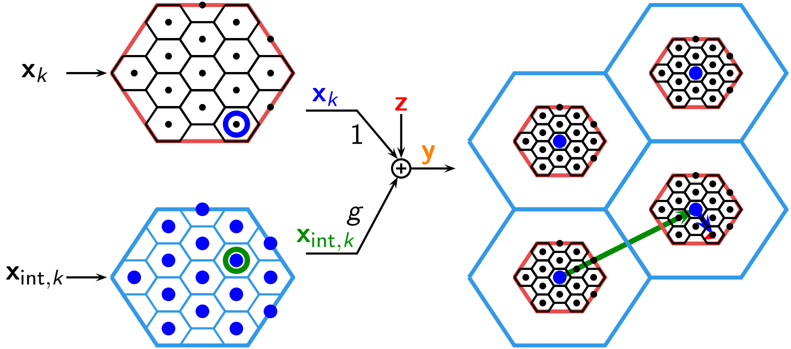

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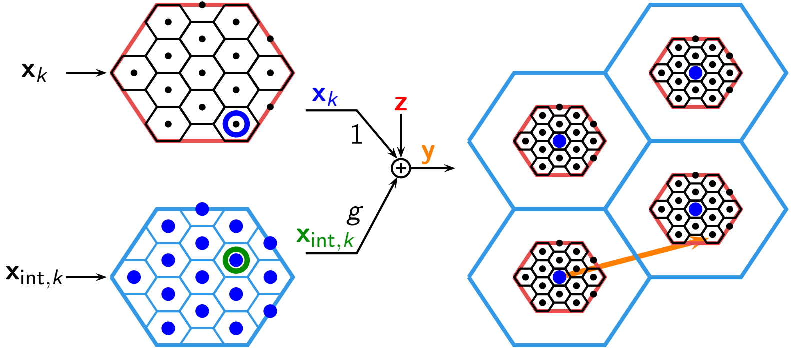

Decode $\mathbf{x}_{\text{int},k}$

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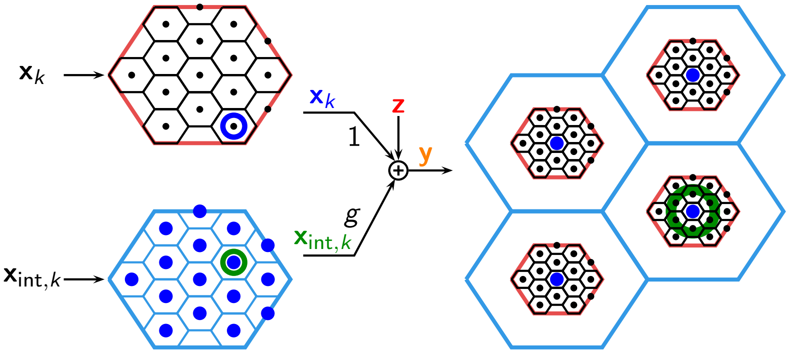

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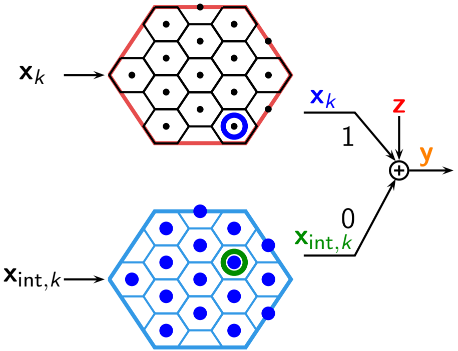

Cancel $\mathbf{x}_{\text{int},k}$

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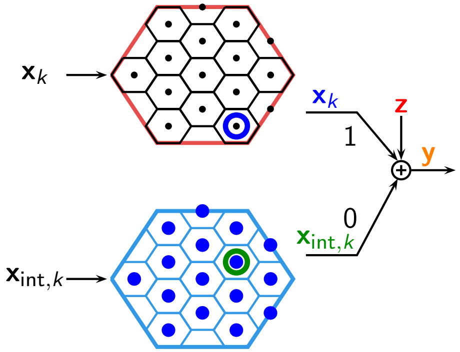

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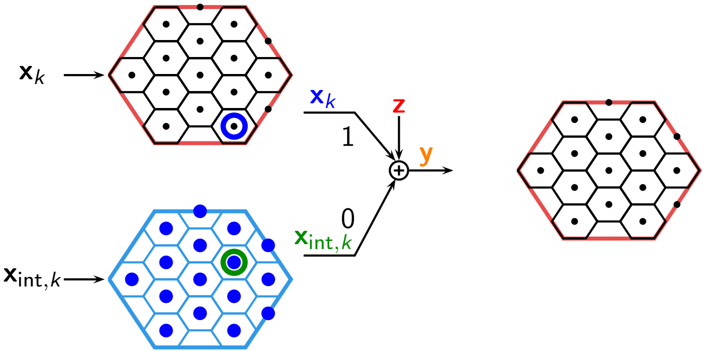

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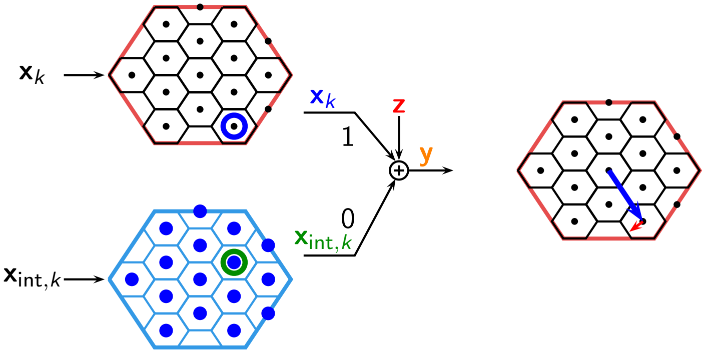

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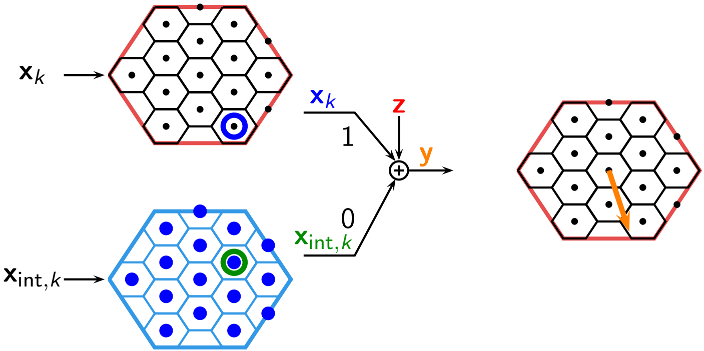

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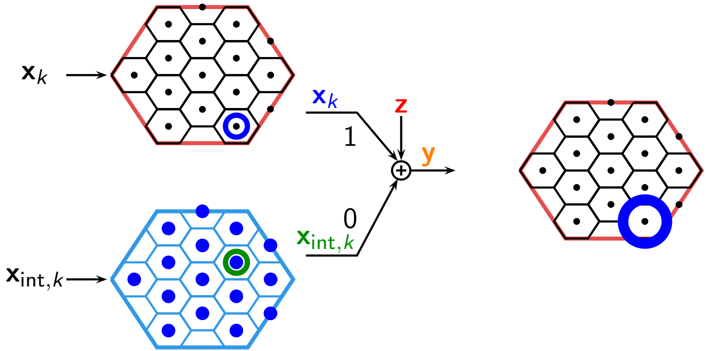

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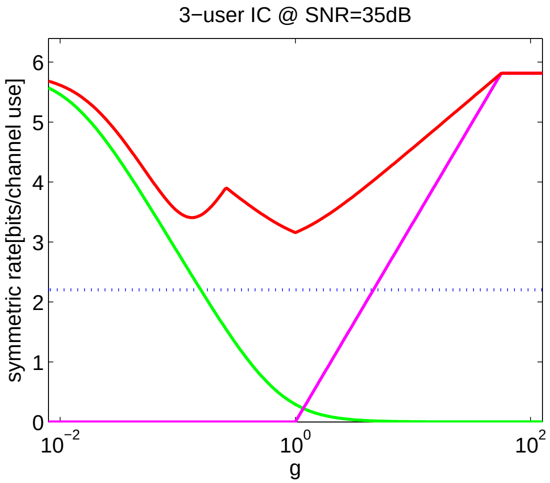
The symmetric Gaussian K-user IC: what do we know?

What about finite SNR?

- Successive decoding is optimal in the very strong interference regime.



The symmetric Gaussian K-user IC: strong interference

$$\mathbf{y}_k = \mathbf{x}_k + g\mathbf{x}_{\text{int},k} + \mathbf{z}_k, \quad \mathbf{x}_k, \mathbf{x}_{\text{int},k} \in \Lambda$$

- Assume strong interference: $g > 1$ but not $\gg 1$.
- For 2-user IC jointly decoding intended message and interference is optimal.
- For K-user IC jointly decoding $\mathbf{x}_k, \mathbf{x}_{\text{int},k}$ seems like a good idea.

Question

What rates are achievable?

The symmetric Gaussian K-user IC: strong interference

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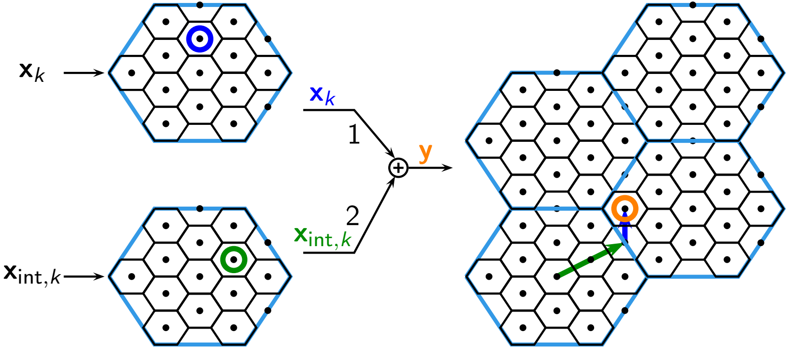
MAC capacity theorem does not hold when both transmitters use the *same* lattice codebook

$\Longrightarrow$ Need a new coding theorem.

MAC with same lattice code

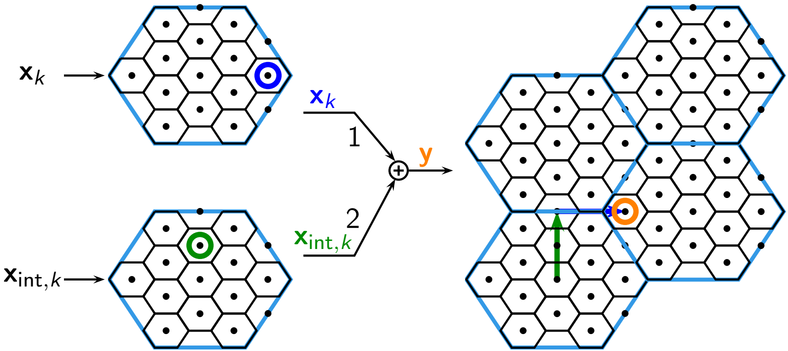
What's the problem with using the same lattice code?



Assume there is no noise at all

MAC with same lattice code

What's the problem with using the same lattice code?



AMBIGUITY!

MAC with same lattice code: new decoder

- Decoding the two lattice points directly is difficult. Instead...

New decoder based on compute-and-forward

Decode two equations with integer coefficients and solve for desired codeword.

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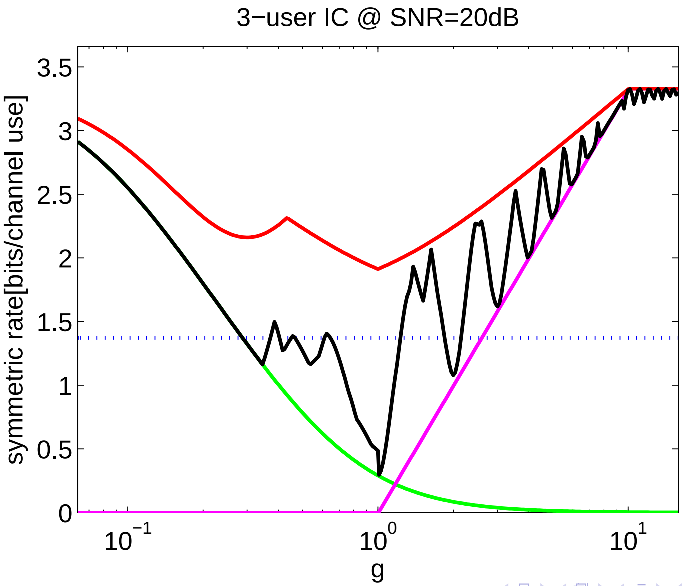
Main result

We use this approach to obtain the approximate symmetric capacity region of the K-user symmetric IC up to an outage set.

The symmetric Gaussian K-user IC: new inner bounds



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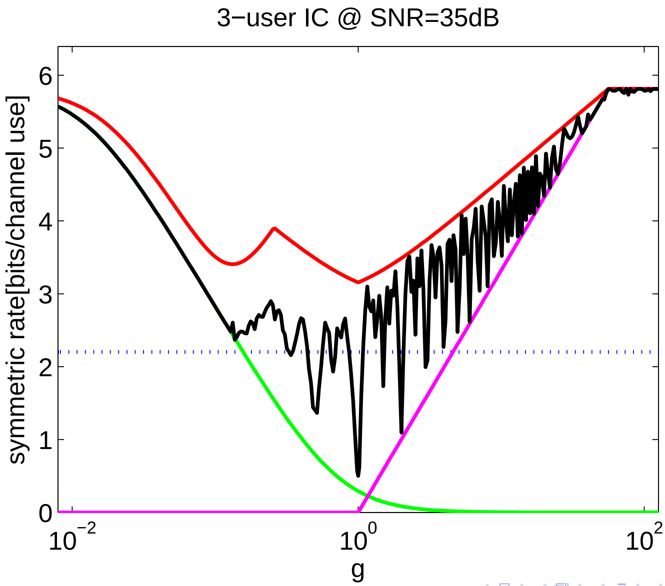

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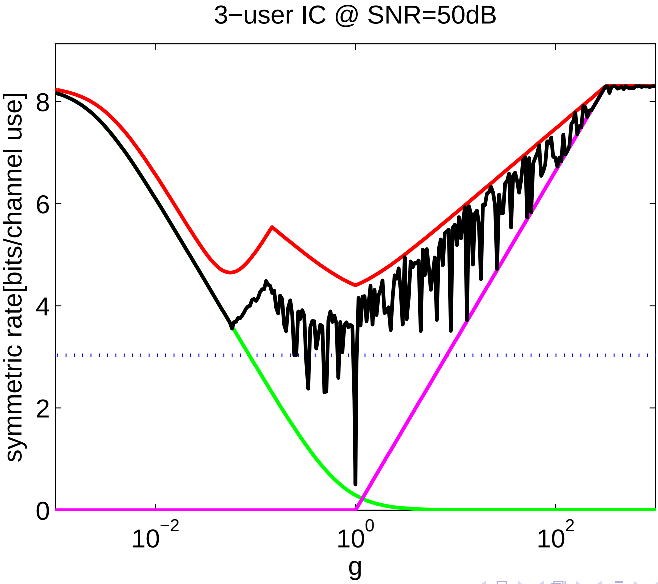

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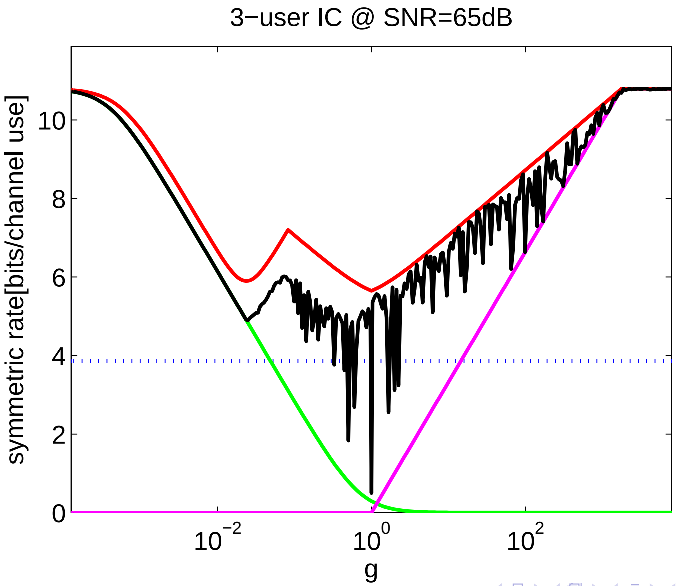

Main tool: compute-and-forward

Theorem - Nazer-Gastpar 11

For the channel $\mathbf{y} = \sum_{k=1}^{K} h_k \mathbf{x}_k + \mathbf{z}$ the equation $\sum_{k=1}^{K} a_k \mathbf{x}_k$ with $\mathbf{a} = [a_1 \ \cdots \ a_K] \in \mathbb{Z}^K$ can be decoded reliably as long as the rates of all users satisfy

$$R < \frac{1}{2} \log \left(\frac{\text{SNR}}{\text{SNR} \|\beta \mathbf{h} - \mathbf{a}\|^2 + \beta^2} \right)$$

for some $\beta \in \mathbb{R}$.

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Use one channel output to decode two equations

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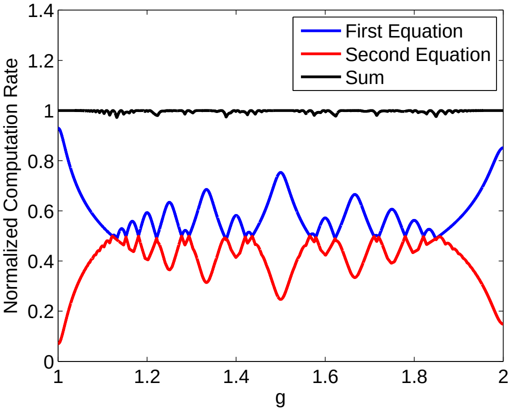
Main tool: compute-and-forward

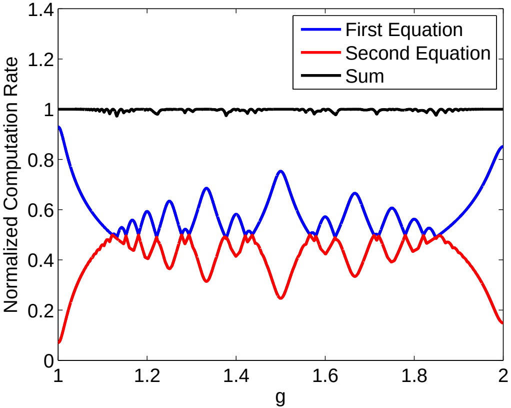
- Decoding two equations is not very effective when channel gains are close to integers.
- This causes the notches in the achievable rate region.
- Fortunately, this rarely happens...



Main tool: compute-and-forward

PROMO

To hear more about this come to "The Compute-and-Forward Transform" tomorrow at 15:20.



Compute-and-forward for the symmetric K-user IC

Transmit	Equations Decoded by Receivers	
$\mathbf{x}_1$	$a_{11}\mathbf{x}_1 + a_{12}\sum_{\ell \neq 1} \mathbf{x}_\ell$	$a_{21}\mathbf{x}_1 + a_{22}\sum_{\ell \neq 1} \mathbf{x}_\ell$
$\mathbf{x}_2$	$a_{11}\mathbf{x}_2 + a_{12}\sum_{\ell \neq 2} \mathbf{x}_\ell$	$a_{21}\mathbf{x}_2 + a_{22}\sum_{\ell \neq 2} \mathbf{x}_\ell$
$\vdots$	$\vdots$	$\vdots$
$\mathbf{x}_K$	$a_{11}\mathbf{x}_K + a_{12}\sum_{\ell \neq 1} \mathbf{x}_\ell$	$a_{21}\mathbf{x}_K + a_{22}\sum_{\ell \neq 1} \mathbf{x}_\ell$

- From one real equation decode two linearly independent equations with integer coefficients.
- Corresponding computation rates are $R_{\text{comp},1}$, $R_{\text{comp},2}$.

Approx. symmetric capacity: strong interference regime

$$C_{\text{SYM}} \geq R_{\text{comp},2}$$

- $R_{\text{comp},2}$ is the solution to an integer-least squares optimization problem.
- Inner bound can be found numerically and plotted.

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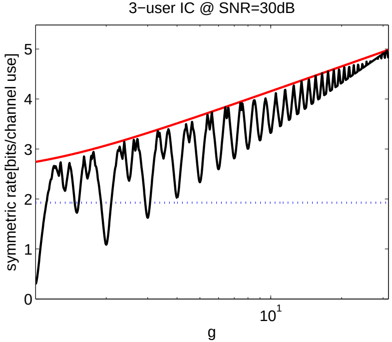

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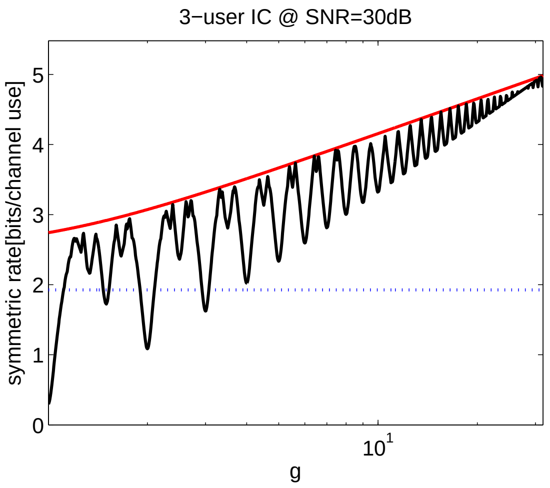
Question

For $c > 0$ bits, what is the fraction of channel gains g for which

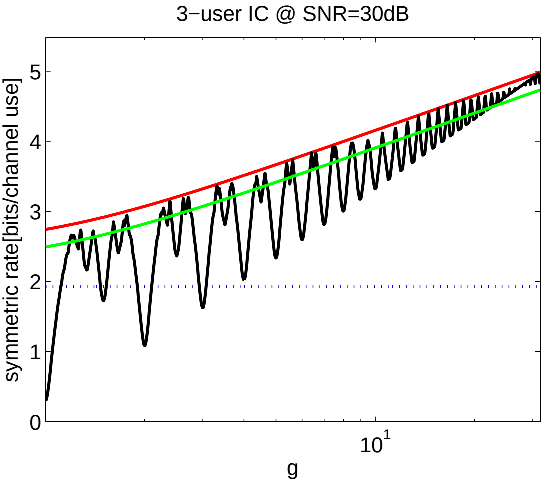
$$\text{outer bound} - \text{inner bound} > c \text{ bits?}$$

Outage set



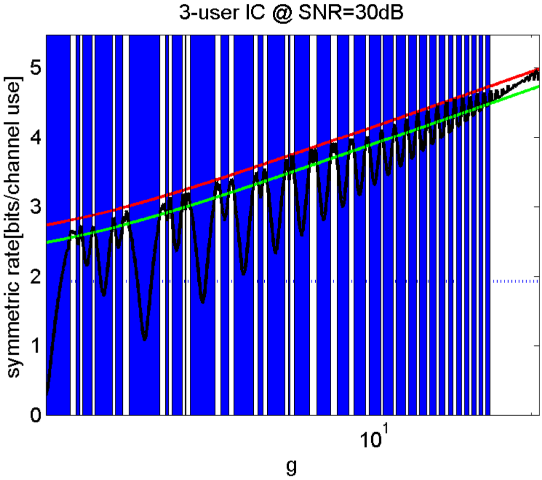
Strong interference regime

Outage set



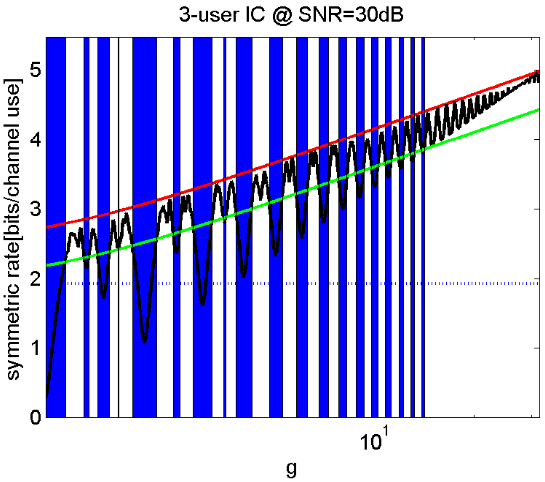
$c = 0.25$ bits

Outage set



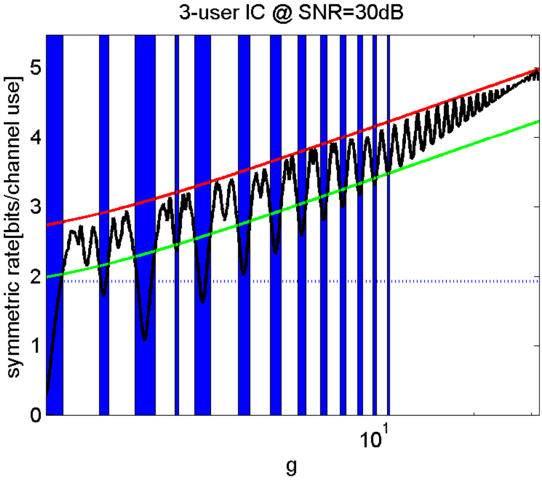
- 48% outage for $c = 0.25$ bits

Outage set



- 22% outage for $c = 0.5$ bits

Outage set



- 11% outage for $c = 0.75$ bits

Approx. symmetric capacity: strong interference regime

Theorem - inner bound for the strong interference regime

The symmetric capacity of the symmetric Gaussian K-user IC is lower bounded by

$$C_{\mathsf{SYM}} \geq \frac{1}{4}\log^{+}(\mathsf{INR}) - \frac{c}{2} - 3$$

for all values of $1 \leq g^2 < \mathsf{SNR}$ except for an outage set whose measure is a fraction of 2^{-c} of the interval $1 \leq |g| < \sqrt{\mathsf{SNR}}$, for any $c > 0$.

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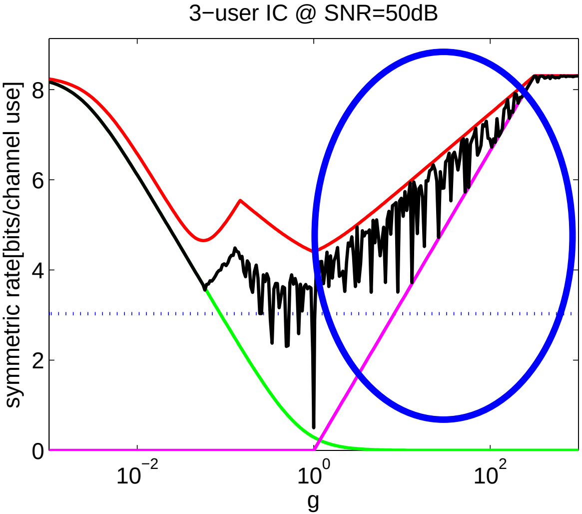
- Outage set approach appeared first in Niesen and Maddah-Ali 11 (next talk)
- The outage set phenomena seems inherent to the problem (Etkin and E. Ordentlich 09).

Weak interference regime: Lattice Han-Kobayshi

- Similar approach works for the weak interference regime.
- Just choose public and private codewords from lattice codebooks.
- Decoding is done using compute-and-forward.
- Achievable rate is the solution to integer least-squares optimization problem.
- Can be shown to be within a constant gap from outer bound (except for an outage set).

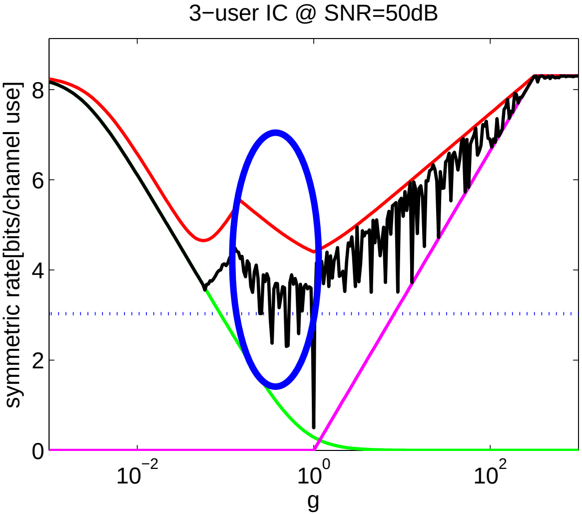
Summary: new inner bounds

- New inner bound for strong interference regime.
  - Constant gap from outer bound except for outage set.



Summary: new inner bounds

- New inner bound for moderately weak interference regime.
  - Constant gap from outer bound except for outage set.



Summary: new inner bounds

- New inner bound for weak interference regime.
  - Constant gap from outer bound for all channel gains.