Precoded Integer-Forcing Universally Achieves the MIMO Capacity to Within a Constant Gap

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***Abstract*—An open-loop single-user multiple-input multiple-output communication scheme is considered where a transmitter, equipped with multiple antennas, encodes the data into independent streams all taken from the same linear code. The coded streams are then linearly precoded using the encoding matrix of a perfect linear dispersion space-time code. At the receiver side, integer-forcing equalization is applied, followed by standard single-stream decoding. It is shown that this communication architecture achieves the capacity of any Gaussian multiple-input multiple-output channel up to a gap that depends only on the number of transmit antennas.**

I. Introduction

The Gaussian Multiple-Input Multiple-Output (MIMO) channel has been the focus of extensive research efforts over the last two decades. Here, we consider the open-loop single-user complex MIMO channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \tag{1}$$

with M transmit and N receive antennas. The input vector $\mathbf{x}$ is subject to the power constraint[1] $\mathbb{E}(\mathbf{x}^\dagger\mathbf{x}) \leq M \cdot \mathsf{SNR}$, and the additive noise $\mathbf{z}$ is a vector of i.i.d. circularly symmetric complex Gaussian entries with zero mean and unit variance. Throughout the paper we assume the channel matrix is static and perfectly known to the receiver, whereas the transmitter knows only the *white-input (WI) mutual information*[2]

$$C_{\text{WI}} = \log\det\left(\mathbf{I} + \mathsf{SNR}\mathbf{H}^\dagger\mathbf{H}\right).$$

corresponding to the channel, but *does not* know the channel coefficients.

While the theoretical performance limits of open-loop communication over a Gaussian MIMO channel are well understood, unlike for closed-loop transmission, much is still lacking when it comes to practical schemes that are able to approach these limits. Such a scheme is known for the 1×2 MISO channel where Alamouti modulation offers an optimal solution. More generally, modulation via orthogonal space-time (ST) block "codes" allows to approach C_{WI} using scalar AWGN coding and decoding in the limit of small rate, where the mutual information is governed solely through the Frobenius norm of the channel matrix.

[1]In this paper $(\mathbf{x})^\dagger$ is the conjugate transpose of $\mathbf{x}$.

[2]All logarithms in this paper are to base 2, and rates are measured in bits per channel use.

Beyond the low rate regime, the multiple degrees of freedom offered by the channel need to be utilized in order to approach capacity. For this reason, despite considerable work and progress, the problem of designing a practical open-loop scheme that simultaneously approaches C_{WI} for all channels $\mathbf{H}$ with the same white-input mutual information remains unsolved. As a consequence, less demanding benchmarks became widely accepted in the literature. First, since statistical modeling of a wireless communication link is often available, one may be content with guaranteeing good performance only for channel realizations that have a "high" probability. Further, to simplify analysis and design, the asymptotic criterion of the diversity-multiplexing tradeoff (DMT) [1] has broadly been adopted.

In [2], Tavildar and Vishwanath introduced the notion of *approximately universal ST codes* and derived a necessary and sufficient criterion for a code to be approximately universal. This criterion is closely related to the nonvanishing determinant criterion and is met by several known coding schemes [3], [4]. Roughly speaking, approximate-universality guarantees that a scheme is DMT optimal for any statistical channel model. Approximately universal schemes still suffer, however, from the asymptotic nature of the DMT criterion.

While designing a practical communication scheme that universally approaches C_{WI} is still out of reach, in the present work we take a step in this direction. Namely, a practical communication architecture that achieves the capacity of any Gaussian MIMO channel up to a *constant gap*, that depends only on the number of transmit antennas, is studied. Such traditional information-theoretic performance guarantee is substantially stronger than approximate universality. In particular, a scheme that achieves a constant gap-to-capacity is obviously also approximately universal. In the considered scheme, which we term *precoded integer-forcing*, the transmitter encodes the data into independent streams via the same linear code. The coded streams are then linearly precoded using the generating matrix of a space-time code from the class of perfect codes [3]–[5]. At the receiver side, integer-forcing (IF) equalization [6] is applied. An IF receiver attempts to decode a full-rank set of linear combinations of the transmitted streams with integer-valued coefficients. Once these equations are decoded, they can be solved for the transmitted streams.

Precoded IF may be viewed as an extension of linear dis-

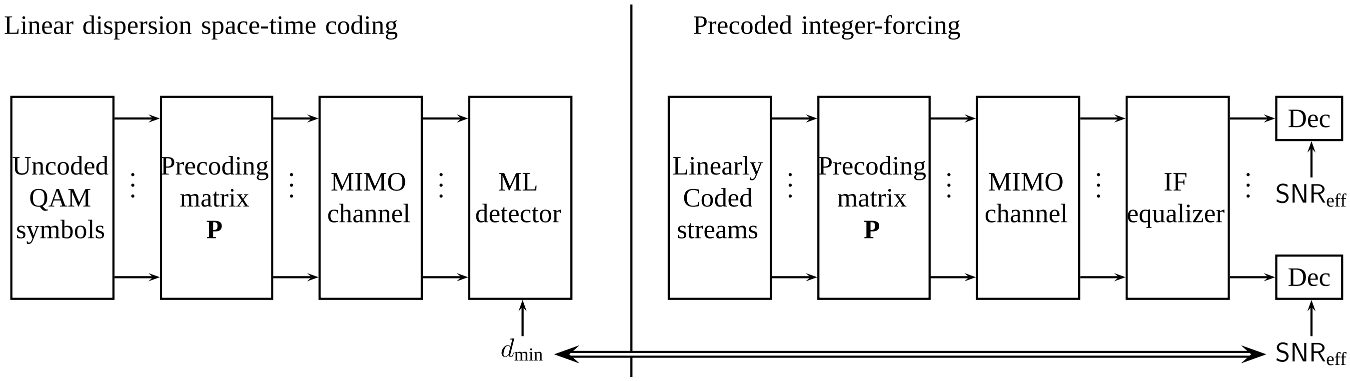

Fig. 1. An illustrative comparison between linear dispersion space-time coding and precoded integer-forcing.

persion ST "codes". In such "codes", uncoded QAM symbols are linearly modulated over space and time. This is done by linearly precoding the QAM symbols using a precoding matrix $\mathbf{P}$. For precoded IF, the same precoding matrix $\mathbf{P}$ is applied to *codewords* taken from a linear code, rather than uncoded QAM symbols. See Figure 1. The performance of linear dispersion ST "codes" is dictated by $d_{\min}$, the minimum distance in the received constellation, whereas the performance of precoded IF is determined by the effective signal-to-noise ratio $\mathsf{SNR}_{\text{eff}}$. A key result we derive is that the two quantities are closely related. Namely, minimum distance guarantees for precoded QAM symbols translate to guarantees on the effective SNR for precoded IF, when the same precoding matrix is used.

The design of precoding matrices for uncoded QAM, that guarantee an appropriate growth of $d_{\min}$ as a function of C_{WI}, has been extensively studied over the last decade. A remarkable family of such matrices are the generating matrices of *perfect* linear dispersion ST codes, which are approximately universal [3], [4]. As a consequence of the tight connection between $d_{\min}$ and $\mathsf{SNR}_{\text{eff}}$, when such precoding matrices are used for precoded IF, $\mathsf{SNR}_{\text{eff}}$ also grows appropriately with C_{WI}. Consequently, precoded IF achieves rates within a constant gap from C_{WI}, and hence also from the capacity, of any Gaussian MIMO channel.

II. Performance of the Integer-Forcing scheme

Integer-Forcing equalization is a low-complexity architecture for the MIMO channel, which was proposed by Zhan *et al*. [6]. The key idea underlying IF is to first decode integral linear combinations of the signals transmitted by all antennas, and then, after the noise is removed, invert those linear combinations to recover the individual transmitted signals. This is made possible by transmitting codewords from the *same* linear/lattice code from all M transmit antennas, leveraging the property that linear codes are closed under (modulo) linear combinations with integer-valued coefficients.

In the IF scheme, the information bits to be transmitted are partitioned into $2M$ streams. Each of the $2M$ streams is encoded by the same linear code $\mathcal{C}$, and each of the M antennas transmits two coded streams, one from its in-phase component and one from its quadrature component. Let $\mathbf{x}_m$ be the signal transmitted by the mth antenna, let $\mathbf{X} \triangleq [\mathbf{x}_1^T \ \cdots \ \mathbf{x}_M^T]^T \in \mathbb{C}^{M\times n}$ where n is the block-length of the code $\mathcal{C}$, and let the subscripts Re and Im denote the real and imaginary parts of a matrix, respectively. The channel's output can be expressed by its real-valued representation

$$\begin{bmatrix} \mathbf{Y}_{\mathsf{Re}} \\ \mathbf{Y}_{\mathsf{Im}} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\mathsf{Re}} & -\mathbf{H}_{\mathsf{Im}} \\ \mathbf{H}_{\mathsf{Im}} & \mathbf{H}_{\mathsf{Re}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathsf{Re}} \\ \mathbf{X}_{\mathsf{Im}} \end{bmatrix} + \begin{bmatrix} \mathbf{Z}_{\mathsf{Re}} \\ \mathbf{Z}_{\mathsf{Im}} \end{bmatrix}, \quad (2)$$

which will be written as $\tilde{\mathbf{Y}} = \tilde{\mathbf{H}}\tilde{\mathbf{X}} + \tilde{\mathbf{Z}}$ for notational compactness.

The IF receiver's goal is to decode $\mathbf{V} \triangleq \mathbf{A}\tilde{\mathbf{X}}$, for some full-rank matrix $\mathbf{A} \in \mathbb{Z}^{2M\times 2M}$ that can be chosen according to the channel coefficients. Let $\mathbf{a}_k^T$ be the kth row of $\mathbf{A}$. In order to decode the equation $\mathbf{v}_k = \mathbf{a}_k^T\tilde{\mathbf{X}}$ the receiver first performs linear MMSE estimation of it from the output $\tilde{\mathbf{Y}}$. This procedure induces $2M$ sub-channels, one for each equation

$$\tilde{\mathbf{y}}_{\text{eff},k} = \mathbf{v}_k + \mathbf{z}_{\text{eff},k}, \quad k = 1, \ldots, 2M, \quad (3)$$

where $\mathbf{z}_{\text{eff},k}$ is the estimation error of $\mathbf{v}_k$ from $\tilde{\mathbf{Y}}$ (this error can be made statistically independent of $\mathbf{v}_k$ using dithers, see [6]). The output of each sub-channel is fed to a decoder. Since the codebook $\mathcal{C}$ is linear, $\mathbf{v}_k$ is a member of the codebook (after appropriate modulo reduction), and therefore the decoding of $\mathbf{v}_k$ is done using the standard decoder associated with $\mathcal{C}$.

For IF equalization to be successful, decoding over all $2M$ sub-channels should be correct. Therefore, the worst sub-channel constitutes a bottleneck. We define the effective SNR at the kth sub-channel as

$$\mathsf{SNR}_{\text{eff},k} \triangleq \left(\mathbf{a}_k^T \left(\mathbf{I} + \mathsf{SNR}\tilde{\mathbf{H}}^T\tilde{\mathbf{H}}\right)^{-1} \mathbf{a}_k\right)^{-1}, \quad (4)$$

and the effective SNR associated with the IF scheme as

$$\mathsf{SNR}_{\text{eff}} \triangleq \min_{k=1,\ldots,2M} \mathsf{SNR}_{\text{eff},k}. \quad (5)$$

Theorem 3 in [6] states that IF equalization can achieve any rate satisfying

$$R_{\text{IF}} < M \log(\mathsf{SNR}_{\text{eff}}). \quad (6)$$

Thus, the performance of the IF scheme is dictated by $\mathsf{SNR}_{\text{eff}}$.

A. Bounding the Effective SNR for an optimal choice of $\mathbf{A}$

The matrix $\mathbf{A}$ should be chosen such as to maximize $\mathsf{SNR}_{\mathsf{eff}}$. Using (4) and (5), this criterion translates to choosing

$$\mathbf{A}^{\text{opt}} = \underset{\substack{\mathbf{A}\in\mathbb{Z}^{2M\times 2M}\\ \det(\mathbf{A})\neq 0}}{\operatorname{argmin}} \max_{k=1\ldots,2M} \mathbf{a}_k^T \left(\mathbf{I}+\mathsf{SNR}\tilde{\mathbf{H}}^T\tilde{\mathbf{H}}\right)^{-1}\mathbf{a}_k.$$

The matrix $\left(\mathbf{I}+\mathsf{SNR}\tilde{\mathbf{H}}^T\tilde{\mathbf{H}}\right)^{-1}$ is symmetric and positive definite, and therefore it admits a Cholesky decomposition

$$\left(\mathbf{I}+\mathsf{SNR}\tilde{\mathbf{H}}^T\tilde{\mathbf{H}}\right)^{-1} = \mathbf{L}\mathbf{L}^T, \tag{7}$$

where $\mathbf{L}$ is a lower triangular matrix with strictly positive diagonal entries. With this notation the optimization criterion becomes

$$\mathbf{A}^{\text{opt}} = \underset{\substack{\mathbf{A}\in\mathbb{Z}^{2M\times 2M}\\ \det(\mathbf{A})\neq 0}}{\operatorname{argmin}} \max_{k=1\ldots,2M} \|\mathbf{L}^T\mathbf{a}_k\|^2.$$

Denote by $\Lambda(\mathbf{L}^T)$ the $2M$ dimensional lattice spanned by the matrix $\mathbf{L}^T$, i.e.,

$$\Lambda(\mathbf{L}^T) \triangleq \left\{\mathbf{L}^T\mathbf{a} \ : \ \mathbf{a}\in\mathbb{Z}^{2M}\right\}.$$

It follows that $\mathbf{A}^{\text{opt}}$ should consist of the set of $2M$ linearly independent integer-valued vectors that result in the shortest set of linearly independent lattice vectors in $\Lambda(\mathbf{L}^T)$. A theorem by Banaszczyk [7] can be used to connect the length of the $2M$th shortest lattice vector in $\Lambda(\mathbf{L}^T)$ with the shortest vector in the lattice $\Lambda(\mathbf{L}^{-1})$ spanned by $\mathbf{L}^{-1}$; see [8] for more details. Using this relation, we obtain the following theorem which lower bounds $\mathsf{SNR}_{\mathsf{eff}}$.

Theorem 1: Consider the complex MIMO channel $\mathbf{y}=\mathbf{H}\mathbf{x}+\mathbf{z}$ with M transmit antennas and N receive antennas, power constraint $\mathbb{E}(\mathbf{x}^\dagger\mathbf{x})\leq M\cdot\mathsf{SNR}$, and additive noise $\mathbf{z}$ with i.i.d. circularly symmetric complex Gaussian entries with zero mean and unit variance. The effective signal-to-noise ratio when integer-forcing equalization is applied is lower bounded by

$$\mathsf{SNR}_{\mathsf{eff}} > \frac{1}{4M^2}\min_{\mathbf{a}\in\mathbb{Z}^M+i\mathbb{Z}^M\setminus\mathbf{0}} \mathbf{a}^\dagger\left(\mathbf{I}+\mathsf{SNR}\mathbf{H}^\dagger\mathbf{H}\right)\mathbf{a}. \tag{8}$$

Proof: See [8]. ∎

B. Relation between the effective SNR and the minimum distance for uncoded QAM

A basic communication scheme for the MIMO channel is transmitting independent uncoded QAM symbols from each antenna. In this case, the error probability strongly depends on the *minimum distance* at the receiver. For a positive integer L, we define

$$d_{\mathsf{min}}(\mathbf{H},L)\triangleq \min_{\mathbf{a}\in\text{QAM}^M(L)\setminus\mathbf{0}}\|\mathbf{H}\mathbf{a}\|, \tag{9}$$

where

$$\begin{aligned}\text{QAM}(L)\triangleq\ &\{-L,-L+1,\ldots,L-1,L\}\\ &+i\{-L,-L+1,\ldots,L-1,L\},\end{aligned} \tag{10}$$

and $\text{QAM}^M(L)$ is an M-dimensional vector whose components all belong to $\text{QAM}(L)$. Note that if L is an even integer, $d_{\mathsf{min}}(\mathbf{H},L)$ is the minimum distance at the receiver when each antenna transmits symbols from a $\text{QAM}(L/2)$ constellation. This is true since

$$\min_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\text{QAM}^M(L/2)\\ \mathbf{x}_1\neq\mathbf{x}_2}}\|\mathbf{H}\mathbf{x}_1-\mathbf{H}\mathbf{x}_2\| = \min_{\mathbf{x}\in\text{QAM}^M(L)\setminus\mathbf{0}}\|\mathbf{H}\mathbf{x}\|.$$

In the IF scheme *there is no assumption* that QAM symbols are transmitted. Rather, each antenna transmits codewords taken from a linear codebook. Nevertheless, we show that the performance of the IF receiver over the channel $\mathbf{H}$ can be tightly related to those of a *hypothetical* uncoded QAM system over the same channel. See Figure 1. Namely, $\mathsf{SNR}_{\mathsf{eff}}$ is closely related to $d_{\mathsf{min}}(\mathbf{H},L)$. This relation is formalized in the next key lemma, which is a simple consequence of Theorem 1.

Lemma 1 (Relation between $\mathsf{SNR}_{\mathsf{eff}}$ *and* d_{min}*):* Consider the complex MIMO channel $\mathbf{y}=\mathbf{H}\mathbf{x}+\mathbf{z}$ with M transmit antennas and N receive antennas, power constraint $\mathbb{E}(\mathbf{x}^\dagger\mathbf{x})\leq M\cdot\mathsf{SNR}$, and additive noise $\mathbf{z}$ with i.i.d. circularly symmetric complex Gaussian entries with zero mean and unit variance. The effective signal-to-noise ratio when integer-forcing equalization is applied is lower bounded by

$$\mathsf{SNR}_{\mathsf{eff}} > \frac{1}{4M^2}\min_{L=1,2,\ldots}\left(L^2+\mathsf{SNR}d^2_{\mathsf{min}}(\mathbf{H},L)\right),$$

where $d^2_{\mathsf{min}}(\mathbf{H},L)$ is defined in (9).

Proof: The bound from Theorem 1 can be written as

$$\mathsf{SNR}_{\mathsf{eff}} > \frac{1}{4M^2}\min_{\mathbf{a}\in\mathbb{Z}^M+i\mathbb{Z}^M\setminus\mathbf{0}}\|\mathbf{a}\|^2+\mathsf{SNR}\|\mathbf{H}\mathbf{a}\|^2. \tag{11}$$

Let

$$\rho(\mathbf{a})\triangleq\max_{m=1,\ldots,M}\max\left(a_{m_{\mathsf{Re}}},a_{m_{\mathsf{Im}}}\right),$$

i.e., $\rho(\mathbf{a})$ is the maximum absolute value of all real and imaginary components of $\mathbf{a}$. With this notation, (11) is equivalent to

$$\begin{aligned}\mathsf{SNR}_{\mathsf{eff}} &> \frac{1}{4M^2}\min_{L=1,2,\ldots}\min_{\substack{\mathbf{a}\in\mathbb{Z}^M+i\mathbb{Z}^M\setminus\mathbf{0}\\ \rho(\mathbf{a})=L}}\|\mathbf{a}\|^2+\mathsf{SNR}\|\mathbf{H}\mathbf{a}\|^2\\ &\geq \frac{1}{4M^2}\min_{L=1,2,\ldots}\left(L^2+\mathsf{SNR}d^2_{\mathsf{min}}(\mathbf{H},L)\right),\end{aligned}$$

as desired. ∎

III. Precoded Integer-Forcing

Clearly, there are instances of MIMO channels for which the lower bound (8) on $\mathsf{SNR}_{\mathsf{eff}}$ does not increase with the WI mutual information. For example, consider a channel $\mathbf{H}$ where one of the NM entries equals h whereas all other gains are zero. For such a channel $C_{\mathsf{WI}}=\log(1+|h|^2\mathsf{SNR})$, yet $\mathsf{SNR}_{\mathsf{eff}}=1$ (and the bound (8) only gives $\mathsf{SNR}_{\mathsf{eff}}>1/(4M^2)$).

Thus, it is evident that IF equalization alone can perform arbitrarily far from C_{WI}.

This problem can be overcome by transmitting linear combinations of multiple streams from each antenna. More precisely, instead of transmitting $2M$ linearly coded streams, one from the in-phase component and one from the quadrature component of each antenna, over n channel uses, $2MT$ linearly coded streams are precoded by a unitary matrix and transmitted over nT channel uses.

Precoded IF, i.e., combining IF equalization with linear precoding, was proposed by Domanovitz *et al*. [9]. The idea is to transform the $N \times M$ complex MIMO channel (1) into an aggregate $NT \times MT$ complex MIMO channel and then apply IF equalization to the aggregate channel. The transformation is done using a unitary precoding matrix $\mathbf{P} \in \mathbb{C}^{MT\times MT}$. Specifically, let $\bar{\mathbf{x}} \in \mathbb{C}^{MT\times 1}$ be the input vector to the aggregate channel. This vector is multiplied by $\mathbf{P}$ to form the vector $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}} \in \mathbb{C}^{MT\times 1}$ which is transmitted over the channel (1) during T consecutive channel uses. Let

$$\mathcal{H} = \mathbf{I}_T \otimes \mathbf{H} = \begin{bmatrix} \mathbf{H} & 0 & \cdots & 0 \\ 0 & \mathbf{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H} \end{bmatrix}, \tag{12}$$

where $\otimes$ denotes the Kronecker product. The output of the aggregate channel is obtained by stacking T consecutive outputs of the channel (1) one below the other and is given by

$$\bar{\mathbf{y}} = \mathcal{H}\mathbf{P}\bar{\mathbf{x}} + \bar{\mathbf{z}} = \bar{\mathbf{H}}\bar{\mathbf{x}} + \bar{\mathbf{z}}, \tag{13}$$

where $\bar{\mathbf{H}} \triangleq \mathcal{H}\mathbf{P} = (\mathbf{I}_T \otimes \mathbf{H})\mathbf{P} \in \mathbb{C}^{NT\times MT}$ is the aggregate channel matrix, and $\bar{\mathbf{z}} \in \mathbb{C}^{NT\times 1}$ is a vector of i.i.d. circularly symmetric complex Gaussian entries. See Figure 5 in [8].

A remaining major challenge is how to choose the precoding matrix $\mathbf{P}$ (recall that an open-loop scenario is considered, and hence, the choice of $\mathbf{P}$ cannot depend on $\mathbf{H}$). As observed in Section II, the performance of the IF equalizer is dictated by $\mathsf{SNR}_{\text{eff}}$. Thus, in order to obtain achievable rates that are comparable to the WI mutual information, $\mathsf{SNR}_{\text{eff}}$ must scale appropriately with C_{WI}. The precoding matrix $\mathbf{P}$ should therefore be chosen such as to guarantee this property for all channel matrices with the same WI mutual information.

Lemma 1 indicates that for the aggregate channel $\mathsf{SNR}_{\text{eff}}$ increases with $d_{\min}(\bar{\mathbf{H}}, L)$, where

$$d_{\min}(\bar{\mathbf{H}}, L) = \min_{\mathbf{a}\in\text{QAM}^{MT}(L)\setminus\mathbf{0}} \|\mathcal{H}\mathbf{P}\mathbf{a}\|. \tag{14}$$

Thus, the precoding matrix $\mathbf{P}$ should be chosen such as to guarantee that $d_{\min}(\bar{\mathbf{H}}, L)$ increases appropriately with C_{WI}. This boils down to the problem of designing precoding matrices for transmitting QAM symbols over an unknown MIMO channel with the aim of maximizing the received minimum distance. This problem was extensively studied during the past decade, under the framework of linear dispersion space-time coding, and unitary precoding matrices that satisfy the aforementioned criterion were found. Therefore, the same matrices that proved so useful for space-time coding are also useful for precoded integer-forcing. A major difference, however, between the two is that while for linear dispersion space-time coding the precoding matrix $\mathbf{P}$ is applied to QAM symbols, in precoded integer-forcing it is applied to *coded streams*. This in turn, yields an achievable rate characterization which is not available using linear dispersion space-time coding. In particular, very different asymptotics can be analyzed. Rather than fixing the block length and taking SNR to infinity, as usually done in the space-time coding literature, here, we fix the channel and take the *block length* to infinity, as in the traditional information-theoretic framework.

The aim of the next section is to lower bound $d_{\min}(\bar{\mathbf{H}}, L)$ as a function of C_{WI} for precoding matrices $\mathbf{P}$ that generate perfect linear dispersion space-time codes. This lower bound will be instrumental in proving that precoded IF universally attains the MIMO capacity to within a constant gap.

IV. Linear Dispersion Space-Time Codes

An $M \times T$ linear dispersion ST code $\mathcal{C}^{\text{ST}}$ over the constellation $\mathcal{S}$ is the collection of all matrices $\mathbf{X} \in \mathbb{C}^{M\times T}$ that can be uniquely decomposed as

$$\mathbf{X} = \sum_{k=1}^{K} s_k \mathbf{F}_k, \;\; s_k \in \mathcal{S},$$

where $\mathcal{S}$ is some constellation and the matrices $\mathbf{F}_k \in \mathbb{C}^{M\times T}$ are fixed and independent of the constellation symbols s_k. Denoting by $\text{vec}(\mathbf{X})$ the vector obtained by stacking the columns of $\mathbf{X}$ one below the other, and letting $\mathbf{s} = [s_1 \;\cdots\; s_K]^T$ gives $\text{vec}(\mathbf{X}) = \mathbf{P}\mathbf{s}$, where $\mathbf{P} = [\text{vec}(\mathbf{F}_1) \;\text{vec}(\mathbf{F}_2) \;\cdots\; \text{vec}(\mathbf{F}_K)]$ is the code's $MT \times K$ *generating matrix*. A linear dispersion ST code is *full-rate* if $K = MT$. In the sequel, linear dispersion ST codes over a QAM(L) constellation, defined in (10), will play a key role. The linear dispersion ST code obtained by using the infinite constellation QAM$(\infty) = \mathbb{Z}+i\mathbb{Z}$ is referred to as $\mathcal{C}^{\text{ST}}_{\infty}$, and, after vectorization, is in fact a complex lattice with generating matrix $\mathbf{P}$. Since the QAM(L) constellation is a subset of $\mathbb{Z}+i\mathbb{Z}$ it follows that for any finite L the QAM(L) based code $\mathcal{C}^{\text{ST}}$ is a subset of $\mathcal{C}^{\text{ST}}_{\infty}$.

An important class of linear dispersion ST codes is that of *perfect codes* [3], [4] which is defined next.

Definition 1: An $M \times M$ linear dispersion ST code over a QAM constellation is called *perfect* if

1) It is full-rate;
2) It satisfies the nonvanishing determinant criterion

$$\delta_{\min}(\mathcal{C}^{\text{ST}}_{\infty}) \triangleq \min_{\mathbf{0}\neq\mathbf{X}\in\mathcal{C}^{\text{ST}}_{\infty}} |\det(\mathbf{X})|^2 > 0;$$

3) The code's generating matrix is unitary, i.e., $\mathbf{P}^{\dagger}\mathbf{P} = \mathbf{I}$.

In [3], perfect linear dispersion ST codes were found for $M = 2, 3, 4$ and 6, whereas in [4] perfect QAM based linear dispersion ST codes were obtained for any positive integer M. Thus, for any positive integer M, there exist codes that satisfy the requirements of Definition 1.

The approximate universality of an ST code over the MIMO channel was studied in [2]. This property refers to an ST code being optimal in terms of DMT regardless of the fading statistics of $\mathbf{H}$. A sufficient and necessary condition for an ST code to be approximately universal was derived in [2]. This condition is closely related to the nonvanishing determinant criterion and is satisfied by perfect linear dispersion ST codes. The next Theorem exploits the approximate universality of perfect linear dispersion ST codes in order to lower bound $d_{\min}(\bar{\mathbf{H}}, L)$ for the aggregate channel obtained using the generating matrix $\mathbf{P}$ of such a code as the precoding matrix. The notation $[x]^+ \triangleq \max(x, 0)$ is used.

Theorem 2: Let $\mathbf{P} \in \mathbb{C}^{M^2 \times M^2}$ be a generating matrix of a perfect $M \times M$ QAM based linear dispersion ST code $\mathcal{C}_\infty^{\text{ST}}$ with $\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}}) = \min_{\mathbf{0} \neq \mathbf{X} \in \mathcal{C}_\infty^{\text{ST}}} |\det(\mathbf{X})|^2 > 0$. Then, for all channel matrices $\mathbf{H}$ with corresponding white input mutual information $C_{\text{WI}} = \log\det(\mathbf{I} + \mathsf{SNR}\mathbf{H}^\dagger\mathbf{H})$

$$\mathsf{SNR}d_{\min}^2(\bar{\mathbf{H}}, L) \geq \left[\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})^{\frac{1}{M}} 2^{\frac{C_{\text{WI}}}{M}} - 2M^2L^2\right]^+,$$

where $\bar{\mathbf{H}} = \mathcal{H}\mathbf{P} = (\mathbf{I}_M \otimes \mathbf{H})\,\mathbf{P}$.

Proof: The proof closely follows that of [2, Theorem 3.1], and can be found in [8]. ∎

V. Main Result

The next theorem shows that precoded IF can achieve the WI mutual information of any Gaussian MIMO channel to within a constant gap that depends only on the number of transmit antennas. Since the closed-loop capacity (with optimal input covariance matrix) differs from C_{WI} in no more than $M \log M$ bits, precoded IF also attains the closed-loop MIMO capacity to within a constant gap.

Theorem 3: Let $\mathbf{P} \in \mathbb{C}^{M^2 \times M^2}$ be a generating matrix of a perfect $M \times M$ QAM based linear dispersion ST code $\mathcal{C}_\infty^{\text{ST}}$ with $\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}}) = \min_{\mathbf{0} \neq \mathbf{X} \in \mathcal{C}_\infty^{\text{ST}}} |\det(\mathbf{X})|^2 > 0$. For all channel matrices $\mathbf{H}$ with M transmit antennas and an arbitrary number of receive antennas, precoded integer-forcing with the precoding matrix $\mathbf{P}$ achieves any rate satisfying

$$R_{\text{P-IF}} < C_{\text{WI}} - \Gamma\left(\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}}), M\right),$$

where $C_{\text{WI}} = \log\det(\mathbf{I} + \mathsf{SNR}\mathbf{H}^\dagger\mathbf{H})$, and

$$\Gamma\left(\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}}), M\right) \triangleq \log\frac{1}{\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})} + 3M\log(2M^2) \quad (15)$$

Proof: In precoded IF, the matrix $\mathbf{P}$ is used as a precoding matrix that transforms the original $N \times M$ MIMO channel (1) to the aggregate $NM \times M^2$ MIMO channel $\bar{\mathbf{y}} = \bar{\mathbf{H}}\bar{\mathbf{x}} + \bar{\mathbf{z}}$, as described in Section III, and then IF equalization is applied to the aggregate channel. By (6), IF equalization can achieve any rate satisfying $R_{\text{IF,aggregate}} < M^2 \log(\mathsf{SNR}_{\text{eff}})$. Applying Lemma 1 to the aggregate $NM \times M^2$ channel matrix $\bar{\mathbf{H}} = \mathcal{H}\mathbf{P}$ gives

$$\mathsf{SNR}_{\text{eff}} > \frac{1}{4M^4} \min_{L=1,2,\ldots}\left(L^2 + \mathsf{SNR}d_{\text{min}}^2(\bar{\mathbf{H}}, L)\right).$$

Using Theorem 2, this is bounded by

$$\begin{aligned}\mathsf{SNR}_{\text{eff}} &> \frac{1}{4M^4}\min_{L\in\mathbb{N}}\left(L^2 + \left[\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})^{\frac{1}{M}} 2^{\frac{C_{\text{WI}}}{M}} - 2M^2L^2\right]^+\right) \\ &\geq \frac{1}{4M^4}\min_{L\in\mathbb{N}}\left(L^2 + \left[\frac{\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})^{\frac{1}{M}} 2^{\frac{C_{\text{WI}}}{M}}}{2M^2} - L^2\right]^+\right) \\ &\geq \frac{1}{8M^6}\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})^{\frac{1}{M}} 2^{\frac{C_{\text{WI}}}{M}}\end{aligned}$$

It follows that any rate satisfying

$$\begin{aligned}R_{\text{IF,aggregate}} &< M^2\log\left(\frac{1}{8M^6}\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})^{\frac{1}{M}} 2^{\frac{C_{\text{WI}}}{M}}\right) \\ &= MC_{\text{WI}} - M\log\frac{1}{\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}})} - M^2\log(8M^6)\end{aligned}$$

is achievable over the aggregate channel. Since each channel use of the aggregate channel corresponds to M channel uses of the original channel, the communication rate should be normalized by a factor of $1/M$. Thus, $R_{\text{P-IF}} = R_{\text{IF,aggregate}}/M$, and the theorem follows. ∎

Example 1: The golden-code [5] is a QAM-based perfect 2×2 linear dispersion ST code, with $\delta_{\text{min}}(\mathcal{C}_\infty^{\text{ST}}) = 1/5$. Thus, for a MIMO channel with $M = 2$ transmit antennas its generating matrix $\mathbf{P} \in \mathbb{C}^{4\times 4}$ can be used for precoded integer-forcing. Theorem 3 implies that with this choice of $\mathbf{P}$, precoded integer-forcing achieves C_{WI} to within a gap of $\Gamma(1/5, 2) = 20.32$ bits, which translates to a gap of 5.08 bits per real dimension. While these constants may seem quite large, one has to keep in mind that this is a worst-case bound, whereas for the typical case, under common statistical assumptions such as Rayleigh fading, the gap-to-capacity obtained by precoded IF is considerably smaller, as seen in the numerical results of [9].

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